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TOPOLOGY PROCEEDINGS

Volume 50, 2017

Pages 49–57

http://topology.nipissingu.ca/tp/

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Electronically published on June 30, 2016

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

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TOPOLOGY PROCEEDINGS
Volume 50 (2017)
Pages 49-57

E-Published on June 30, 2016

PERIODIC POINTS OF SOLENOIDAL AUTOMORPHISMS

SHARAN GOPAL AND C. R. E. RAJA

ABSTRACT. We give a characterization of the sets of periodic points of toral automorphisms. Then we describe the one-dimensional solenoids as the quotients of the (additive) group of adeles and characterize the sets of periodic points of automorphisms on these solenoids. We also determine the sets of periodic points for automorphisms on a full solenoid.

1. Introduction

A *dynamical system* is by definition a pair (X, f), where X is a topological space and f is a continuous map of X. A point $x \in X$ is said to be *periodic* if there is an $n \in \mathbb{N}$ such that $f^n(x) = x$; any such n is called a *period* of x and the least among them is called the *least period* of x. A well-studied problem on the periodicity is the characterization of sets of least periods and periodic points of a family of dynamical systems. To put formally, we seek the following. If $\mathcal{F}$ is a family of maps on a space X, then give a characterization of the collections $\{Per(f) : f \in \mathcal{F}\}$ and $\{P(f) : f \in \mathcal{F}\}$, where $Per(f) = \{n \in \mathbb{N} : f$ has a periodic point of least period $n\}$ and $P(f) = \{x \in X : x$ is a periodic point of $f\}$. The papers [1], [5], [6], [9], [13], [15] give such characterizations for various families, and for a nice survey on the characterization of the sets of least periods, see [8]. On the other hand, [12] gives the number of periodic points of any given period for some continuous homomorphisms of a solenoid.

2010 *Mathematics Subject Classification.* Primary 54H20; Secondary 22C05.
Key words and phrases. adeles, periodic points, solenoid, torus.
The first author was supported by NBHM, India.
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In this article, we consider the underlying space X to be a compact group and the map f to be an automorphism of X; by an automorphism, we mean a continuous automorphism.

The *n-dimensional torus* $\mathbb{T}^n$ is considered as the quotient topological group $\mathbb{R}^n/\mathbb{Z}^n$. A *toral automorphism* is a continuous group automorphism of $\mathbb{T}^n$. For $A \in GL(n,\mathbb{Z})$, the map T_A, defined as $T_A(x) = \pi(Ax)$, where $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is the canonical projection, is a toral automorphism. In fact, every toral automorphism T arises as T_A for some such matrix (see [11]).

In [15], the sets of periodic points are characterized for the 2-dimensional toral automorphisms. Here, we extend this to toral automorphisms of any arbitrary dimension (see Theorem 2.1).

A bigger class of compact groups that includes tori is solenoids. A compact connected finite-dimensional abelian group is called a *solenoid*. Equivalently, a topological group Σ is a solenoid if and only if its Pontryagin dual $\widehat{\Sigma}$ is (isomorphic to) a subgroup of the discrete additive group $\mathbb{Q}^n$ and contains $\mathbb{Z}^n$; i.e., $\mathbb{Z}^n \leq \widehat{\Sigma} \leq \mathbb{Q}^n$ (see [14]). In particular, when $\widehat{\Sigma} = \mathbb{Z}^n$, then Σ is an n-dimensional torus and when $\widehat{\Sigma} = \mathbb{Q}^n$, we say that Σ is a *full (n-dimensional) solenoid*. Here, we describe 1-dimensional solenoids as quotients of the adele group $\mathbb{Q}_{\mathbb{A}}$, based on a description of subgroups of $\mathbb{Q}$. We give a characterization of the sets of periodic points for automorphisms of 1-dimensional solenoids (see Theorem 3.2) and a characterization of the sets of periodic points for automorphisms of any full solenoid (see Theorem 3.5).

An automorphism α of a compact group G is called *ergodic* if the only α-invariant subsets of G are those of full measure or zero measure (with respect to the Haar measure on G). Since the dual of an automorphism α of a solenoid is an automorphism of a subgroup of $\mathbb{Q}^n$ for some n, it is given by an invertible matrix in $M_n(\mathbb{Q})$ (which we will again denote by α). It can be proved that α is ergodic if and only if α has no eigenvalue of absolute value 1 (see [3]).

2. Toral Automorphisms

In this section, we give a characterization of the sets of periodic points of toral automorphisms. If $T : \mathbb{T}^n \to \mathbb{T}^n$ is a toral automorphism, then, since $\mathbb{Q}^n/\mathbb{Z}^n$ is a torsion group and contains only a finite number of elements of any given order, we get that $\mathbb{Q}^n/\mathbb{Z}^n \subset P(T)$. We use a result that a closed subgroup of $\mathbb{T}^n$ is topologically isomorphic to $\mathbb{T}^m \times D$ for some $m \leq n$ and a finite discrete group D (see [14]). A subgroup of $\mathbb{T}^n$, which itself is isomorphic to a torus, is called a *subtorus* of $\mathbb{T}^n$.

Theorem 2.1. *If T is an automorphism on $\mathbb{T}^n$, then the set $P(T)$ of periodic points of T is given by $P(T) = M + \frac{\mathbb{Q}^n}{\mathbb{Z}^n}$, where M is a subtorus of $\mathbb{T}^n$. Further, T is ergodic if and only if $M = (0)$.*

Conversely, if M is any subtorus of $\mathbb{T}^n$, then there is an automorphism T on $\mathbb{T}^n$ such that $P(T) = M + \frac{\mathbb{Q}^n}{\mathbb{Z}^n}$.

Proof. We view the torus $\mathbb{T}^n$ as $\frac{\mathbb{R}^n}{\mathbb{Z}^n}$. Let T be a toral automorphism and A be the (integer) matrix that represents T. Suppose T is ergodic. If $T^k(x + \mathbb{Z}^n) = x + \mathbb{Z}^n$, then $(A^k - I)x = m$ for some $m \in \mathbb{Z}^n$. If $det(A^k - I) = 0$, then there is an eigenvalue of A which is a root of unity, contradicting the ergodicity of T. Thus, $det(A^k - I) \neq 0$ and then $x = (A^k - I)^{-1}m \in \mathbb{Q}^n$. Thus, $P(T) = \frac{\mathbb{Q}^n}{\mathbb{Z}^n}$.

If T is not ergodic, then A has an eigenvalue that is a root of unity; so, there exist $v(\neq 0) \in \mathbb{R}^n$ and $k \in \mathbb{N}$ such that $A^k(v) = v$. Let $W = \{x \in \mathbb{R}^n : A^k(x) = x\}$. Then $\pi(W)$ is contained in the set $P_k := \{y \in \mathbb{T}^n : T^k(y) = y\}$, where $\pi : \mathbb{R}^n \to \mathbb{T}^n$ is the canonical map. Since W is a non-trivial subspace of $\mathbb{R}^n$, P_k is a closed subgroup of $\mathbb{T}^n$ with positive dimension and hence contains a non-trivial subtorus M'.

It is clear that T^k is the identity on M'. If M'' is a subtorus on which $T^{k'}$ is the identity for some $k' \in \mathbb{N}$, then $T^{kk'}$ is the identity on the subtorus $M = M' + M''$. Repeating this argument and since $\mathbb{T}^n$ is finite dimensional, we get a subtorus M that is maximal with respect to the property that T^l is the identity on M for some $l \in \mathbb{N}$.

Since $\frac{\mathbb{Q}^n}{\mathbb{Z}^n} \subset P(T)$, $M + \frac{\mathbb{Q}^n}{\mathbb{Z}^n} \subset P(T)$. Now, let $x \in P(T)$ and say $T^m(x) = x$. The closed subgroup $\overline{< M, x >}$ generated by M and x is of the form $M_0 \times F$, where M_0 is a subtorus of $\mathbb{T}^n$ and F is a finite group (see [14]). It follows that $M \subset M_0$ and T^{lm} is the identity on M_0 and by maximality of M, we have $M = M_0$. So $\overline{< M, x >}/M$ is a finite group and hence $mx \in M$ for some $m \in \mathbb{N}$. Since M is a divisible group, there exists $y \in M$ such that $my = mx$ and hence $x - y \in \frac{\mathbb{Q}^n}{\mathbb{Z}^n}$. Thus, $x = y + (x - y) \in M + \frac{\mathbb{Q}^n}{\mathbb{Z}^n}$.

Conversely, if M is any subtorus of $\mathbb{T}^n$, then $\mathbb{T}^n = M \oplus L$ for some subtorus L. If $M = \mathbb{T}^n$, then take T to be the identity on $\mathbb{T}^n$. If $dim(L) \geq 2$, then define an automorphism T on $\mathbb{T}^n$ such that M and L are T-invariant, $T|_M$ is the identity, and $T|_L$ is ergodic. Then $P(T) = M + \frac{Q^n}{\mathbb{Z}^n}$. If $dim(L) = 1$, then let $\phi : L \to M$ be an injective homomorphism; such a map exists because $dim(M) \geq dim(L)$. Define T on $\mathbb{T}^n$ as $T(x) = (m + \phi(l)) + l$ for every $x \in \mathbb{T}^n$, where $x = m + l$ for $m \in M$ and $l \in L$. If $x = m + l$ is a periodic point of T with period k, then $m + k\phi(l) = m$ which implies that $k\phi(l)$ $(= \phi(kl))$ is zero in M. Since ϕ is injective, $l \in \frac{\mathbb{Q}^n}{\mathbb{Z}^n}$. Thus, $P(T) = M + \frac{\mathbb{Q}^n}{\mathbb{Z}^n}$. $\square$

3. Solenoidal Automorphisms

This section deals with the solenoidal automorphisms. We first give a description of one-dimensional solenoids using the adeles and then characterize the sets of periodic points of the family of automorphisms on a one-dimensional solenoid. Then we consider the same problem of characterization for higher dimensional solenoids. The techniques used for the characterization in the case of one-dimensional solenoids cannot be used for higher dimensional ones. However, Theorem 3.5 addresses this problem for full solenoids of any dimension. Nevertheless, the general problem is unanswered.

3.1. One-dimensional solenoids.

We give here a characterization of the family $\{P(\alpha) : \alpha$ is an automorphism of a 1-dimensional solenoid$\}$. By definition, a 1-dimensional solenoid Σ is a compact group such that $\mathbb{Z} \leq \widehat{\Sigma} \leq \mathbb{Q}$. We first give a description of the subgroups of $\mathbb{Q}$ as given in [2]. Let P be the set of all prime numbers and let $M \subset \mathbb{Q}$. For any $x \in M$ and $p \in P$, the *p-height* of x, $h_p(x)$ is defined to be the largest non-negative integer n, if it exists, such that $\frac{x}{p^n} \in M$; otherwise, define $h_p(x) = \infty$. Thus, $(h_p(x))$ is a sequence (where p runs through the primes in the natural order) with values in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ called the *height sequence* of x. Two height sequences (u_p) and (v_p) are said to be *equivalent* if $u_p = v_p$ for all but finitely many primes and $u_p = \infty \;\Leftrightarrow\; v_p = \infty$ for any p. Given a subgroup M of $\mathbb{Q}$, there is a unique height sequence (up to equivalence) associated to all elements of M. Thus, we can associate a height sequence, unique up to equivalence, to any given subgroup of $\mathbb{Q}$; moreover, two subgroups of $\mathbb{Q}$ are isomorphic if and only if their associated height sequences are equivalent.

We denote by $\mathbb{Q}_p$ and $\mathbb{Z}_p$ the field of p-adic numbers and the ring of p-adic integers, respectively. The multiplicative group $\{x \in \mathbb{Z}_p : |x|_p = 1\}$ is denoted by $\mathbb{Z}_p^*$. By convention, we denote $\mathbb{R}$ by $\mathbb{Q}_\infty$. The *ring of adeles* $\mathbb{Q}_{\mathbb{A}}$ is defined as the restricted product $\mathbb{R} \times \prod_{p \in P} \mathbb{Q}_p$ with respect to $\mathbb{Z}_p$; i.e., for any $(a_\infty, a_2, a_3, ...) \in \mathbb{Q}_{\mathbb{A}}$, $a_p \in \mathbb{Z}_p$ for all but finitely many p. Here, we will consider only the additive group structure of $\mathbb{Q}_{\mathbb{A}}$. Since every rational number has p-adic norm equal to 1 for all but finitely many p, we have a diagonal inclusion $\delta : \mathbb{Q} \to \mathbb{Q}_{\mathbb{A}}$ given by $(\delta(r))_p = r$ for every $p \leq \infty$ and for every $r \in \mathbb{Q}$. Now, for any $a = (a_p) \in \mathbb{Q}_{\mathbb{A}}$, we can associate a character ψ_a of $\mathbb{Q}$ as $\psi_a(r) = e^{-2\pi i r a_\infty} \prod_{p<\infty} e^{2\pi i \{r a_p\}_p}$, where $\{x\}_p$ is the p-adic fractional part of x. The map $\psi : \mathbb{Q}_{\mathbb{A}} \to \widehat{\mathbb{Q}}$, given by $a \mapsto \psi_a$, is a surjective homomorphism and $\widehat{\mathbb{Q}}$ is isomorphic to $\frac{\mathbb{Q}_{\mathbb{A}}}{\delta(\mathbb{Q})}$ (see [7]). If

$c \in \delta(\mathbb{Q})$, then $\psi_c(r) = 1$ for every $r \in \mathbb{Q}$; thus, if $(a_p), (b_p) \in \mathbb{Q}_{\mathbb{A}}$ such that there is an $s \in \mathbb{Q}$ with $a_p = s + b_p$ for every $p \leq \infty$, then $\psi_{(a_p)} = \psi_{(b_p)}$. Since every 1-dimensional solenoid should be a quotient of $\widehat{\mathbb{Q}}$, it can be realized as a quotient of $\mathbb{Q}_{\mathbb{A}}$. The following proposition describes this quotient explicitly.

Proposition 3.1. *Let Σ be a one-dimensional solenoid. Let $D = \{p \in P : h_p(x) \neq 0$ for some $x \in \widehat{\Sigma}\}$, $n_p := sup\{h_p(x) : x \in \widehat{\Sigma} \cap \mathbb{Z}_p^*\}$, and $D_\infty = \{p \in D : n_p = \infty\}$. Then $\Sigma = \frac{\mathbb{Q}_{\mathbb{A}}}{\delta(\mathbb{Q})+L}$, where $L = \prod_{p\leq\infty} U_p$ and*

$$U_p = \begin{cases} (0) & \text{if } p \in D_\infty \cup \{\infty\} \\ p^{n_p}\mathbb{Z}_p & \text{if } p \in D \setminus D_\infty \\ \mathbb{Z}_p & \text{if } p \notin D. \end{cases}$$

Proof. Since Σ is a 1-dimensional solenoid, $\widehat{\Sigma} < \mathbb{Q}$ and $\widehat{\widehat{\Sigma}}(= \Sigma) = \frac{\widehat{\mathbb{Q}}}{V}$, where V is the annihilator of $\widehat{\Sigma}$ (see [14]). Let $\pi : \widehat{\mathbb{Q}} \to \Sigma$ be the canonical projection. Then the map $\psi' = \pi \circ \psi : \mathbb{Q}_{\mathbb{A}} \to \Sigma$ is a surjective homomorphism, where $\psi : \mathbb{Q}_{\mathbb{A}} \to \widehat{\mathbb{Q}}$ is given by $a \mapsto \psi_a$ and $\psi_a(r) = e^{-2\pi i r a_\infty} \prod_{p<\infty} e^{2\pi i \{r a_p\}_p}$ for every $r \in \mathbb{Q}$. We now claim that $Ker(\psi') = \delta(\mathbb{Q}) + L$.

Before proving the claim, we show that for every $(l_p) \in \mathbb{R} \times \prod_{p<\infty} \mathbb{Z}_p$ and $r \in \widehat{\Sigma}$, $\psi_{(l_p)}(r) = e^{-2\pi i r l_\infty} \prod_{p\in D} e^{2\pi i\{r l_p\}_p}$. Let $r = \frac{m}{n}$, $(m, n) = 1$, and p be a prime not in D. If $p|n$, then $\frac{m}{p} \in \widehat{\Sigma}$ and thus $h_p(m) \neq 0$, which is a contradiction to the fact that $p \notin D$. So, $p \notin D \Rightarrow |r|_p \leq 1$ and thus $\prod_{p\notin D} e^{2\pi i\{r l_p\}_p} = 1$. Hence, $\psi_{(l_p)}(r) = e^{-2\pi i r l_\infty} \prod_{p\in D} e^{2\pi i\{r l_p\}_p}$.

Now, if $y = (y_p) \in \delta(\mathbb{Q}) + L$, then, for every $p \leq \infty$, $y_p = s + l_p$ for some $s \in \mathbb{Q}$ and $l_p \in U_p$. Then, for any $r \in \widehat{\Sigma}$, $\psi_y(r) = \psi_{(l_p)}(r) = \prod_{p\in D\setminus D_\infty} e^{2\pi i\{r l_p\}_p}$. Fix some $p \in D \setminus D_\infty$ and $r \in \widehat{\Sigma}$. Say $r = p^k s$, where $s = \frac{m}{n} \in \mathbb{Z}_p^*$; then $|r|_p = p^{-k}$. If $k \geq 0$, then $r \in \mathbb{Z}_p$ and thus $r l_p \in \mathbb{Z}_p$. Suppose $k < 0$. Then $nr = \frac{m}{p^{-k}} \in \widehat{\Sigma}$ and thus $h_p(m) \geq -k$. Since $m \in \mathbb{Z}_p^*$, $-k \leq n_p$. Thus, $|r l_p|_p = p^{-k}|l_p|_p \leq p^{n_p}|l_p|_p \leq 1$ and hence $r l_p \in \mathbb{Z}_p$. Thus, $\psi_y(r) = 1$ for every $r \in \widehat{\Sigma}$; i.e., $\psi_y \in V$. Hence, $y \in Ker(\psi')$.

Conversely, let $y \in Ker(\psi')$. Then $\psi'(y) \in V$; i.e., $\psi_y(r) = 1$ for every $r \in \widehat{\Sigma}$. Since $\mathbb{Q}_{\mathbb{A}} = \delta(\mathbb{Q}) + ([0, 1) \times \prod_{p<\infty} \mathbb{Z}_p)$ (see [16]), y can be written as $y = (s + l_p)$ for some $s \in \mathbb{Q}$, $l_\infty \in [0, 1)$, and $l_p \in \mathbb{Z}_p$. Then $\psi_y(r) = \psi_{(l_p)}(r) = 1$ for every $r \in \widehat{\Sigma}$. Choosing $r = 1$, we get $\psi_{(l_p)}(1) = e^{-2\pi i l_\infty}$ and thus $l_\infty = 0$. Hence, $\psi_{(l_p)}(r) = \prod_{p\in D} e^{2\pi i\{r l_p\}_p}$ for every $r \in \widehat{\Sigma}$.

Let $p \in D \setminus D_\infty$. By the definition of n_p, there exists $x \in \widehat{\Sigma} \cap \mathbb{Z}_p^*$ with $h_p(x) = n_p$. Let $r = \frac{x}{p^{n_p}}$ and $x = \frac{m}{n}$. Then $nr = \frac{m}{p^{n_p}} \in \widehat{\Sigma}$; further, $|nr|_q \leq 1$ for every $q \neq p$ and $|nr|_p = p^{n_p}$. Thus, $\psi_{(l_p)}(nr) = e^{2\pi i \{nrl_p\}_p}$.

But $\psi_{(l_p)}(nr) = 1 \Rightarrow \{nrl_p\}_p = 0$
$\Rightarrow nrl_p \in \mathbb{Z}_p$
$\Rightarrow |l_p|_p \leq \frac{1}{|nr|_p} = \frac{1}{p^{n_p}}$
$\Rightarrow l_p \in p^{n_p}\mathbb{Z}_p$.

Finally, let $p \in D_\infty$. Since $n_p = \infty$, for any $l \in \mathbb{N}$, there exists $x \in \widehat{\Sigma} \cap \mathbb{Z}_p^*$ such that $h_p(x) > l$; i.e., $\frac{x}{p^{l+1}} \in \widehat{\Sigma}$. Again, say $x = \frac{m}{n}$ and $r = \frac{x}{p^{l+1}}$; then $nr = \frac{m}{p^{l+1}} \in \widehat{\Sigma}$ and $|nr|_q \leq 1$ for every $q \neq p$, whereas $|nr|_p = p^{l+1}$. Thus, $\psi_{(l_p)}(nr) = e^{2\pi i \{nrl_p\}_p}$.

So $\psi_{(l_p)}(nr) = 1 \Rightarrow \{nrl_p\}_p = 0$
$\Rightarrow nrl_p \in \mathbb{Z}_p$
$\Rightarrow |l_p|_p \leq \frac{1}{|nr|_p} = \frac{1}{p^{l+1}}$.

Since l can be arbitrarily large, we have $|l_p|_p = 0$, and thus $l_p = 0$. $\square$

As noted, the dual of an automorphism α of a 1-dimensional solenoid Σ is defined by multiplication with a non-zero rational number, which will be denoted again by α. Also, α is ergodic on Σ if and only if $\alpha \notin \{-1, 1\}$. So, it is enough to consider only the ergodic automorphisms, because for the other two automorphisms (i.e., when $\alpha \in \{-1, 1\}$), we have $P(\alpha) = \Sigma$.

Theorem 3.2. *Let Σ, L, D, and D_∞ be defined as in the above proposition. If α is an ergodic automorphism of Σ, then $P(\alpha) = \frac{\delta(\mathbb{Q}) + \prod' \mathbb{Q}_p}{\delta(\mathbb{Q}) + L}$, where $\prod' \mathbb{Q}_p := \{x \in \mathbb{Q}_\mathbb{A} : x_p = 0$ for every $p \in D_\infty \cup \{\infty\}$ and $x_p \in p^{n_p}\mathbb{Z}_p$ for all but finitely many p in $D \setminus D_\infty\}$.*

Proof. Let $\pi(x) \in \Sigma$ be α-periodic with period n, where $\pi : \mathbb{Q}_\mathbb{A} \to \frac{\mathbb{Q}_\mathbb{A}}{\delta(\mathbb{Q}) + L}$ is the canonical projection. Then $(\alpha^n - 1)x \in \delta(\mathbb{Q}) + L$; say $(\alpha^n - 1)x_p = r + l_p$, where $r \in \mathbb{Q}$ and $l_p \in U_p$. Thus, $(x_p) = (\frac{r}{\alpha^n - 1}) + (\frac{l_p}{\alpha^n - 1}) \in \delta(\mathbb{Q}) + \prod' \mathbb{Q}_p$.

Conversely, let $x \in \delta(\mathbb{Q}) + \prod' \mathbb{Q}_p$; say $x_p = r + l_p$, for every $p \leq \infty$. Then, for any $n \in \mathbb{N}$, $(\alpha^n - 1)x_p = (\alpha^n - 1)r + (\alpha^n - 1)l_p$. If $p \in D_\infty \cup \{\infty\}$, then $(\alpha^n - 1)l_p = 0$. So, it suffices to show that there is an $n \in \mathbb{N}$ such that $(\alpha^n - 1)l_p \in U_p$ for every $p \notin D_\infty \cup \{\infty\}$. Now, since $(l_p) \in \mathbb{Q}_\mathbb{A}$, there is a finite set F of primes such that $l_p \in \mathbb{Z}_p$ for every prime $p \notin F$. Let $F \setminus D = \{p_1, p_2, ...p_k\}$. Again, since $(l_p) \in \prod' \mathbb{Q}_p$, there is a finite set $G \subset D \setminus D_\infty$ such that $l_p \in p^{n_p}\mathbb{Z}_p$ for every $p \in D \setminus G$. Let $G = \{p_{k+1}, p_{k+2}, ..., p_{k+l}\}$. Noting that $n_p = 0$ for every prime $p \notin D$, we can see that $l_p \in p^{n_p}\mathbb{Z}_p$ for every prime $p \notin \{p_1, p_2, ...p_{k+l}\}$.

Now, let $\alpha = \frac{\alpha_1}{\alpha_2}$ where $\alpha_i \in \mathbb{Z}$ for $i = 1, 2$. If $p|\alpha_2$, then $p \in D_\infty$ because $\alpha^j \in \widehat{\Sigma}$ for every $j \in \mathbb{N}$; thus, $|\alpha|_p \leq 1$ for every prime $p \notin D_\infty$. Since α is an automorphism, we can show in a similar way that $|\alpha^{-1}|_p \leq 1$ for any $p \notin D_\infty$. Hence, $\alpha \in \mathbb{Z}_p^*$, in particular, for $p \in \{p_1, p_2, ...p_{k+l}\}$. Also, for every $p \notin D_\infty$, $|\alpha^n - 1|_p \leq 1$ for every $n \in \mathbb{N}$. So, for any $p \notin D_\infty$, $|(\alpha^n - 1)l_p|_p \leq |l_p|_p$ and, in addition, if $p \notin \{p_1, p_2, ..., p_{k+l}\}$, then $(\alpha^n - 1)l_p \in p^{n_p}\mathbb{Z}_p$. Thus, it suffices to show that there exists an $n \in \mathbb{N}$ such that $(\alpha^n - 1)l_p \in p^{n_p}\mathbb{Z}_p$ for every $p \in \{p_1, p_2, ..., p_{k+l}\}$.

Since $\alpha \in \mathbb{Z}_p^*$ for $p \in \{p_1, p_2, ...p_{k+l}\}$, we have $\bar{\alpha} = (\alpha, \alpha, ...\alpha) \in \prod_{i=1}^{k+l} \mathbb{Z}_{p_i}^*$, which is a compact group. The set $\{\bar{\alpha}^n : n \in \mathbb{N}\}$ is a semi-group and thus its closure in $\prod_{i=1}^{k+l} \mathbb{Z}_{p_i}^*$ is a closed semigroup; so $\overline{\{\bar{\alpha}^n : n \in \mathbb{N}\}}$ is a subgroup of $\prod_{i=1}^{k+l} \mathbb{Z}_{p_i}^*$ (see [17]).

Thus, $(1, 1, ..., 1) \in \overline{\{\bar{\alpha}^n : n \in \mathbb{N}\}}$

$\Rightarrow$ there exists a sequence (n_i) such that $(\bar{\alpha}^{n_i}) \to (1, 1, ..., 1)$

$\Rightarrow (\alpha^{n_i}) \to 1$ in $\mathbb{Z}_{p_m}^*$ for each $1 \leq m \leq k + l$

$\Rightarrow (|\alpha^{n_i} - 1|_{p_m}) \to 0$ for each $1 \leq m \leq k + l$.

Now, given $l_{p_m} \in \mathbb{Q}_{p_m}$ for each $1 \leq m \leq k + l$, choose N such that $|\alpha^N - 1|_p < \frac{1}{p^{n_p}|l_p|_p}$ for each $p \in \{p_1, p_2, ..., p_{k+l}\}$. Thus, $(\alpha^N - 1)l_p \in p^{n_p}\mathbb{Z}_p$ for each $p \in \{p_1, p_2, ..., p_{k+l}\}$. $\square$

Remark 3.3. It follows that $\{P(\alpha) : \alpha \in Aut(\Sigma)\} = \{\Sigma, \frac{\delta(\mathbb{Q})+\prod' \mathbb{Q}_p}{\delta(\mathbb{Q})+L}\}$.

As a consequence, we obtain the following result that is contained in Theorem 3.1 in [12].

Corollary 3.4. *If α is an ergodic automorphism of a 1-dimensional solenoid Σ, then $|P_n(\alpha)| = \prod_{p\notin D_\infty} |\alpha^n - 1|_p^{-1}$, where $P_n(\alpha) := \{x \in \Sigma : \alpha^n(x) = x\}$ and $|P_n(\alpha)|$ is its cardinality.*

Proof. Let $x \in \mathbb{Q}_\mathbb{A}$ such that $\pi(x) \in P_n(\alpha)$, where $\pi : \mathbb{Q}_\mathbb{A} \to \Sigma$ $(= \frac{\mathbb{Q}_\mathbb{A}}{\delta(\mathbb{Q})+L})$ is the canonical projection. Then we have $x = (r + l_p)$, where $r \in \mathbb{Q}$, $(l_p) \in \prod' \mathbb{Q}_p$, and $(\alpha^n - 1)l_p \in U_p$. Thus, $l_p = 0$ for every $p \in D_\infty \cup \{\infty\}$ and $|(\alpha^n - 1)l_p|_p \leq \frac{1}{p^{n_p}}$; i.e., $|l_p|_p \leq \frac{1}{p^{n_p}|\alpha^n-1|_p}$ for every $p \notin D_\infty$.

It follows from the proof of Theorem 3.2 that for every prime $p \notin D_\infty$, $|\alpha^n - 1|_p \leq 1$. Therefore, as $\alpha^n - 1 \in \mathbb{Q}$, there are at most finitely many primes outside D_∞, say $p_1,\ p_2, ..., p_k$, such that $|\alpha^n - 1|_p = 1$ for every $p \notin D_\infty \cup \{p_1,\ p_2, ..., p_k\}$ and $|\alpha^n - 1|_p < 1$ for every $p \in \{p_1,\ p_2, ..., p_k\}$. For each $1 \leq i \leq k$, let $|\alpha^n - 1|_{p_i} = p_i^{-k_i}$ for some $k_i > 0$. Then $l_p \in p^{n_p}\mathbb{Z}_p$ for every $p \notin D_\infty \cup \{p_1,\ p_2, ..., p_k\}$ and $l_p \in p^{n_p - k_i}\mathbb{Z}_p$ for $p \in \{p_1,\ p_2, ..., p_k\}$. Hence, $P_n(\alpha) = \frac{\delta(\mathbb{Q})+\prod V_p}{\delta(\mathbb{Q})+L}$, where $V_p = p^{n_p - k_i}\mathbb{Z}_p$ for $p \in \{p_1,\ p_2, ..., p_k\}$, and for the rest of the primes, $V_p = U_p$. Thus, $|P_n(\alpha)| = \prod_{i=1}^{k} |\frac{V_{p_i}}{U_{p_i}}| = \prod_{i=1}^{k} p^{k_i} = \prod_{i=1}^{k} |\alpha^n - 1|_{p_i}^{-1} = \prod_{p \notin D_\infty} |\alpha^n - 1|_p^{-1}$. □

3.2. Higher dimensional solenoids.

The above characterizations depend upon the description of the subgroups of $\mathbb{Q}$ using the notion of p-heights. However, no such description is available for the subgroups of $\mathbb{Q}^n$ for $n > 1$. In fact, Alexander S. Kechris [10] implies that there is probably "no reasonably simple classification" of these groups. However, for a full solenoid, we have the following description of the sets of periodic points. Recall that Σ is called a full solenoid if $\widehat{\Sigma} \simeq \mathbb{Q}^d$ for some $d \geq 0$.

Theorem 3.5. *Let α be an automorphism of a full solenoid Σ. Then $P(\alpha) = F(\alpha^m)$ for some $m \geq 1$ and $P(\alpha)$ is a full solenoidal subgroup of Σ; that is, the dual of $P(\alpha)$ is a $\mathbb{Q}$-vector subspace. In particular, α is ergodic if and only if $P(\alpha)$ is trivial.*

The proof depends on the following lemma.

Lemma 3.6. *Let α be an automorphism of a full solenoid Σ. Then $F(\alpha) = \{x \in \Sigma \mid \alpha(x) = x\}$ is a full solenoidal subgroup of Σ.*

Proof. Let $\widehat{\Sigma} = \mathbb{Q}^d$ for some $d \geq 1$. For any $r \in \mathbb{Q}$, let $m_r \colon \Sigma \to \Sigma$ be the automorphism whose dual $\widehat{m_r} \colon \mathbb{Q}^d \to \mathbb{Q}^d$ is the multiplication by the rational number r. Then $\widehat{\alpha}\widehat{m_r} = \widehat{m_r}\widehat{\alpha}$; hence, $\alpha m_r = m_r \alpha$. This implies that $F(\alpha)$ is m_r-invariant and hence the dual of $F(\alpha)$ is invariant under multiplication by any rational. Thus, the dual of $F(\alpha)$ is a $\mathbb{Q}$-vector subspace of $\mathbb{Q}^d$. □

Proof of Theorem 3.5. It follows from Lemma 3.6 that $F(\alpha^i)$ is a full solenoidal subgroup of Σ. Since Σ is finite-dimensional, there is an $m \geq 1$ such that $\dim(F(\alpha^m)) \geq \dim(F(\alpha^i))$ for any $i \geq 1$. Now, for any $i \geq 1$, there is a $k \geq 1$ such that $F(\alpha^k)$ contains both $F(\alpha^i)$ and $F(\alpha^m)$. This implies that $\dim(F(\alpha^k)) \geq \dim(F(\alpha^m))$ and hence $\dim(F(\alpha^k)) = \dim(F(\alpha^m))$. This implies that since $F(\alpha^k)$ and $F(\alpha^m)$ are (full) solenoids and $F(\alpha^k)$ contains $F(\alpha^m)$, $F(\alpha^m) = F(\alpha^k)$. Therefore, $F(\alpha^m)$ contains all $F(\alpha^i)$. Thus, $F(\alpha^m) = P(\alpha)$. □

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