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by

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ORDERABILITY OF PRODUCTS

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Abstract. We prove that for non-discrete spaces X and Y,

(1) if the product space $X \times Y$ is suborderable, then both X and Y are hereditarily paracompact and there is a unique regular infinite cardinal κ such that for every $z \in X \cup Y$, the cofinality from left (right) of z is either $0, 1$ or κ;
(2) if X and Y are subspaces of an ordinal, then the converse implication of (1) is also true.

1. Introduction

Recently, a kind of orderability of X^2 is known to be related to selection theory; see [5] and [3]. In this paper, we see the results in the abstract.

Spaces mean regular topological spaces. Let $<$ be a linear order on a set X. The usual order topology is denoted by $\lambda(<)$, that is, the topology generated by

$$\{(a, \rightarrow) : a \in X\} \cup \{(\leftarrow, b) : b \in X\}$$

as a subbase, where $(a, \rightarrow) = \{x \in X : a < x\}$, $(a, b) = \{x \in X : a < x < b\}$, etc. If necessary, we write $<_X$ and $(a, b)_X$ instead of $<$ and (a, b), respectively. A *linearly ordered topological space (LOTS)* X means the triple $\langle X, <, \lambda(<)\rangle$. As usual, we consider an ordinal α as the set of smaller ordinals and as a LOTS with the order $\in$ (we identify it with $<$). Similarly, a *generalized ordered space (GO-space)* means the triple $\langle X, <, \tau\rangle$ where τ is a topology on X with $\lambda(<) \subset \tau$ which has a base consisting of convex sets, where a subset A is *convex* if $(a, b) \subset A$ whenever $a, b \in A$ with $a < b$.

A topological space $\langle X, \tau\rangle$, where τ is a topology on X, is said to be *orderable* if $\tau = \lambda(<)$ for some linear order $<$ on X. Also, a topological

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space $\langle X, \tau \rangle$ is said to be *suborderable* if it is a subspace of some orderable space. It is well known that orderable spaces are hereditarily normal. Also, it is well known that

(1) if $\langle L, <_L, \lambda(<_L) \rangle$ is a LOTS and $X \subset L$, then $\langle X, <_L \restriction X, \lambda(<_L) \restriction X \rangle$ is a GO-space, where $<_L \restriction X$ is the restricted order of $<_L$ to X and $\lambda(<_L) \restriction X$ is the subspace topology of $\lambda(<_L)$ on X, i.e., $\{U \cap X : U \in \lambda(<_L)\}$. On the other hand,
(2) if $\langle X, <_X, \tau \rangle$ is a GO-space, then there is a LOTS $\langle L, <_L, \lambda(<_L) \rangle$ with $X \subset L$ such that $\langle X, \tau \rangle$ is a dense subspace of $\langle L, \lambda(<_L) \rangle$ and $<_X = <_L \restriction X$; therefore, $\langle X, \tau \rangle$ is suborderable. Obviously, a suborderable space is a GO-space with some linear order. Moreover,
(3) if $\langle X, <_X, \lambda(<_X) \rangle$ is a LOTS, there is a LOTS $\langle L, <_L, \lambda(<_L) \rangle$ with $X \subset L$ and $<_X = <_L \restriction X$ such that the space $\langle L, \lambda(<_L) \rangle$ is compact and contains $\langle X, \lambda(<_X) \rangle$ as a dense subspace. Therefore, by (2) and (3), we have
(4) if $\langle X, <_X, \tau \rangle$ is a GO-space, then there is a compact LOTS $\langle L, <_L, \lambda(<_L) \rangle$ with $X \subset L$ and $<_X = <_L \restriction X$ such that the compact space $\langle L, \lambda(<_L) \rangle$ contains $\langle X, \tau \rangle$ as a dense subspace. So we say a GO space $\langle X, <_X, \tau \rangle$ has a linearly ordered compactification $\langle L, <_L, \lambda(<_L) \rangle$ or more simply, a GO-space X has a linearly ordered compactification L.

Note that a compact LOTS $\langle L, <_L, \lambda(<_L) \rangle$ has the largest element $\max L$ and the smallest element $\min L$. Also note that if X is a convex subset of a LOTS $\langle L, <_L, \lambda(<_L) \rangle$, then the subspace topology $\lambda(<_L) \restriction X$ coincides with the order topology $\lambda(< \restriction X)$ on X. For more details, see [10] and [8]. Usually, if there is no confusion, we do not distinguish the symbols $<_X$ and $<_L$, and simply write $<$.

In general, a GO-space can have many linearly ordered compactifications. But it is known that a GO-space X has a linearly ordered compactification lX such that, for every linearly ordered compactification cX of X, there is a continuous function $f : cX \to lX$ with $f(x) = x$ for every $x \in X$; see [9]. Observe that by the definition, lX is unique up to order isomorphisms and is said to be the minimal linearly ordered compactification of X and is characterized as follows.

Lemma 1.1 ([9, Lemma 2.1])**.** *A linearly ordered compactification cX of a GO-space X is minimal if and only if $(a, b)_{cX} \neq \emptyset$ for every $a, b \in cX \setminus X$ with $a < b$.*

2. Results

Let $\{X_\alpha : \alpha \in \Lambda\}$ be a pairwise disjoint collection of spaces. Then $\bigoplus_{\alpha\in\Lambda} X_\alpha$ denotes the topological sum of X_α's, i.e., the space $\bigcup_{\alpha\in\Lambda} X_\alpha$ with the topology generated by $\bigcup_{\alpha\in\Lambda} \tau_\alpha$ as a base, where τ_α is the topology on X_α. Note that the subspace $\{0\}\cup(1,2)$ of the real line is suborderable but not orderable. This means that the topological sum of orderable spaces need not be orderable. On the other hand, the infinite discrete space $D(\kappa)$ of cardinality κ is orderable because the LOTS $\kappa \times \mathbb{Z}$ with the lexicographic order is homeomorphic to $D(\kappa)$, where $\mathbb{Z}$ is the set of integers.

Let S be a subset of an ordinal α. $\mathrm{Lim}_\alpha(S)$ denotes the set $\{\beta \in \alpha : \sup(S \cap \beta) = \beta\}$, i.e., the set of all cluster points of S in α. If the contexts are clear, we simply write $\mathrm{Lim}(S)$. Obviously, if S is closed in α, then $\mathrm{Lim}(S) \subset S$. $\mathrm{Succ}(S)$ denotes the set $S \setminus \mathrm{Lim}(S)$, i.e., the set of all isolated points of S.

A subset S of a regular uncountable cardinal κ is *stationary* if it intersects with all closed unbounded (club) sets C of κ, where a subset C of κ is unbounded if, for every $\alpha < \kappa$, there is $\beta \in C$ with $\alpha \le \beta$. Note that if S is unbounded in κ, then $\mathrm{Lim}(S)$ is club in κ.

Lemma 2.1. *Let S be a stationary set in a regular uncountable cardinal κ and let X be a non-discrete space of cardinality $< \kappa$. Then the subspace $X \times S$ of $X \times \kappa$ is not hereditarily normal.*

Proof. Let x be a non-isolated point of X and let $Y = (X \setminus \{x\}) \times S \cup \{x\} \times \mathrm{Succ}(S)$. Then it is routine to check that $F_0 = \{x\} \times \mathrm{Succ}(S)$ and $F_1 = (X \setminus \{x\}) \times (S \cap \mathrm{Lim}(S))$ are disjoint closed sets in Y which cannot be separated by disjoint open sets. $\square$

Lemma 2.2. *Let κ and λ be regular infinite cardinals with $\kappa \ne \lambda$. Then the subspace $(\mathrm{Succ}(\kappa)\cup\{\kappa\}) \times (\mathrm{Succ}(\lambda)\cup\{\lambda\})$ of $(\kappa+1)\times(\lambda+1)$ is not suborderable.*

Proof. Let $X = \mathrm{Succ}(\kappa) \cup \{\kappa\}$ and $Y = \mathrm{Succ}(\lambda) \cup \{\lambda\}$ and assume that $X \times Y$ is suborderable. Denote the product topology of $X \times Y$ by τ. Fix a linearly ordered set $\langle L, <_L\rangle$ such that $X \times Y \subset L$ and $\lambda(<_L)\restriction X \times Y = \tau$, where $\lambda(<_L)$ denotes the order topology on L. Denote the restricted order $<_L\restriction X \times Y$ on $X \times Y$ by $<$. We may assume $\omega \le \kappa < \lambda$. Let $F_0 = \{\kappa\} \times \mathrm{Succ}(\lambda)$ and $F_1 = \mathrm{Succ}(\kappa) \times \{\lambda\}$. Put

$$F_0^- = \{\beta \in \mathrm{Succ}(\lambda) : \langle\kappa,\beta\rangle < \langle\kappa,\lambda\rangle\},$$

$$F_0^+ = \{\beta \in \mathrm{Succ}(\lambda) : \langle\kappa,\lambda\rangle < \langle\kappa,\beta\rangle\},$$

$$F_1^- = \{\alpha \in \mathrm{Succ}(\kappa) : \langle\alpha,\lambda\rangle < \langle\kappa,\lambda\rangle\},$$

$$F_1^+ = \{\alpha \in \mathrm{Succ}(\kappa) : \langle \kappa, \lambda \rangle < \langle \alpha, \lambda \rangle\}.$$

Note $F_0 = \{\kappa\} \times (F_0^- \cup F_0^+)$ and $F_1 = (F_1^- \cup F_1^+) \times \{\lambda\}$.

CLAIM 1. $|F_1^-| < \kappa$ or $|F_1^+| < \kappa$.

Proof. Assume that both F_1^- and F_1^+ have cardinality κ. For every $\alpha \in F_1^-$, since $(\leftarrow, \langle \kappa, \lambda \rangle)_L \cap X \times Y$ is a τ-neighborhood of $\langle \alpha, \lambda \rangle$ in $X \times Y$, there is $g(\alpha) < \lambda$ such that $\{\alpha\} \times (g(\alpha), \lambda] \cap X \times Y \subset (\leftarrow, \langle \kappa, \lambda \rangle)_L \cap X \times Y$, where $(\leftarrow, \langle \kappa, \lambda \rangle)_L$ denotes the interval in L and $(g(\alpha), \lambda]$ denotes the usual interval in $\lambda + 1$. Similarly, for every $\alpha \in F_1^+$, we can find $g(\alpha) < \lambda$ such that $\{\alpha\} \times (g(\alpha), \lambda] \cap X \times Y \subset (\langle \kappa, \lambda \rangle, \rightarrow)_L \cap X \times Y$.

Put $\beta_0 = \sup\{g(\alpha) : \alpha \in F_1^- \cup F_1^+\}$. Then by $\kappa < \lambda$, we have $\beta_0 < \lambda$. Pick $\beta \in (\beta_0, \lambda) \cap \mathrm{Succ}(\lambda)$. We may assume $\beta \in F_0^-$; then $\langle \kappa, \beta \rangle <_L \langle \kappa, \lambda \rangle$. On the other hand, by $|F_1^+| = \kappa$ and $F_1^+ \times \{\beta\} \subset (\langle \kappa, \lambda \rangle, \rightarrow)_L$, we have $\langle \kappa, \beta \rangle \in \mathrm{Cl}_\tau F_1^+ \times \{\beta\} \subset [\langle \kappa, \lambda \rangle, \rightarrow)_L$. Therefore, $\langle \kappa, \lambda \rangle \leq_L \langle \kappa, \beta \rangle$, a contradiction.

Now we may assume $|F_1^+| < \kappa$, then $|F_1^-| = \kappa$ and $\langle \kappa, \lambda \rangle \in \mathrm{Cl}_\tau F_1^- \times \{\lambda\} \subset (\leftarrow, \langle \kappa, \lambda \rangle]_L$.

CLAIM 2. $|F_0^+| = \lambda$.

Proof. Assume $|F_0^+| < \lambda$; then $|F_0^-| = \lambda$. Therefore, we have $\langle \kappa, \lambda \rangle \in \mathrm{Cl}_\tau \{\kappa\} \times F_0^- \subset (\leftarrow, \langle \kappa, \lambda \rangle]_L$. For every $\beta \in F_0^-$, since $(\langle \kappa, \beta \rangle, \rightarrow)_L \cap X \times Y$ is a τ-neighborhood of $\langle \kappa, \lambda \rangle$ and $\langle \kappa, \lambda \rangle \in \mathrm{Cl}_\tau F_1^- \times \{\lambda\}$, there is $\alpha(\beta) \in F_1^-$ such that $\langle \kappa, \beta \rangle <_L \langle \alpha(\beta), \lambda \rangle$. Since $\kappa < \lambda$, there are $\alpha_0 \in F_1^-$ and $F \subset F_0^-$ of size λ such that $\alpha(\beta) = \alpha_0$ for each $\beta \in F$. Note $\langle \alpha_0, \lambda \rangle <_L \langle \kappa, \lambda \rangle$. Then $\{\kappa\} \times F \subset (\leftarrow, \langle \alpha_0, \lambda \rangle)_L$; therefore, $\mathrm{Cl}_\tau \{\kappa\} \times F \subset (\leftarrow, \langle \alpha_0, \lambda \rangle]_L$. On the other hand, it follows from $|F| = \lambda$ that $\langle \kappa, \lambda \rangle \in \mathrm{Cl}_\tau \{\kappa\} \times F$; thus, $\langle \kappa, \lambda \rangle \leq_L \langle \alpha_0, \lambda \rangle$, a contradiction.

Now for each $\beta \in F_0^+$, it follows from $\langle \kappa, \lambda \rangle <_L \langle \kappa, \beta \rangle$ that there is $f(\beta) < \kappa$ such that

$$(*) \ ((\mathrm{Succ}(\kappa) \cup \{\kappa\} \cap (f(\beta), \kappa]) \times \{\beta\} \subset (\langle \kappa, \lambda \rangle, \rightarrow)_L.$$

By $\kappa < \lambda$, there are $\alpha_0 < \kappa$ and $F \subset F_0^+$ of cardinality λ such that $f(\beta) = \alpha_0$ for every $\beta \in F$.

Since $|F_1^-| = \kappa$, one can pick $\alpha \in F_1^-$ with $\alpha_0 < \alpha$. Then $\langle \alpha, \lambda \rangle <_L \langle \kappa, \lambda \rangle$. On the other hand, by $(*)$, we have $\{\alpha\} \times F \subset (\langle \kappa, \lambda \rangle, \rightarrow)_L$; therefore, $\langle \alpha, \lambda \rangle \in \mathrm{Cl}_\tau \{\alpha\} \times F \subset [\langle \kappa, \lambda \rangle, \rightarrow)_L$, a contradiction. □

Definition 2.3. Let κ be a regular infinite cardinal, let $\mathcal{X} = \{X_\alpha : \alpha \in \Lambda\}$ a pairwise disjoint collection of non-empty spaces, and let x_0 be a point with $x_0 \notin \bigcup_{\alpha \in \Lambda} X_\alpha$, where $\Lambda \subset \kappa$. Put $X = (\bigcup_{\alpha \in \Lambda} X_\alpha) \cup \{x_0\}$ and equip the topology τ generated by

$$(\bigcup_{\alpha \in \Lambda} \tau_\alpha) \cup \{(\bigcup_{\alpha \in \Lambda \cap (\gamma, \kappa)} X_\alpha) \cup \{x_0\} : \gamma < \kappa\}$$

as a base, where τ_α is the topology on X_α. We call this topological space $\langle X, \tau\rangle$ a 1-*point extension of the topological sum* $\bigoplus_{\alpha\in\Lambda} X_\alpha$ *with the* κ*-limit point* x_0 and denote it by $X(\mathcal{X}, x_0)$.

In the definition above, note that

- for every $\alpha \in \Lambda$, X_α is clopen in X. Thus, the topological sum $\bigoplus_{\alpha\in\Lambda} X_\alpha$ is a subspace of X;
- x_0 has a neighborhood base of cardinality $\leq \kappa$;
- Λ is unbounded in κ if and only if x_0 is a non-isolated point of X.

Now let C be a club set in a regular infinite cardinal κ and $\alpha < \kappa$. Let

$$\alpha_C^- = \sup(C \cap \alpha) \text{ and } \alpha_C^+ = \min\{\beta \in C : \alpha < \beta\},$$

where $\sup \emptyset = -1$. If contexts are clear, then we write simply α^- and α^+. Note that $\alpha \in \mathrm{Succ}(C)$ if and only if $\alpha^- < \alpha$ and that $\alpha < \alpha^+$ for every $\alpha \in C$.

Lemma 2.4. *Let* κ *be a regular infinite cardinal, let* $\mathcal{X} = \{X_\alpha : \alpha \in \Lambda\}$ *be a pairwise disjoint collection of non-empty suborderable spaces with* $\Lambda \subset \mathrm{Succ}(C)$ *for some club set* C *of* κ*, and let* $x_0 \notin \bigcup_{\alpha\in\Lambda} X_\alpha$*. Then the 1-point extension* $X(\mathcal{X}, x_0)$ *of* $\bigoplus_{\alpha\in\Lambda} X_\alpha$ *with the* κ*-limit point* x_0 *is suborderable.*

Proof. For every $\alpha \in \Lambda$, pick a compact LOTS $\langle L_\alpha, <_\alpha, \lambda(<_\alpha)\rangle$ such that $\langle L_\alpha, \lambda(<_\alpha)\rangle$ contains $\langle X, \tau_\alpha\rangle$ as a dense subspace, where τ_α denotes the topology on X_α. For every $\alpha \in C \setminus \Lambda$, let $L_\alpha = \{l_\alpha\}$ be a one point set with the trivial order $<_\alpha$. By taking an isomorphic compact LOTS, we may assume that $\{L_\alpha : \alpha \in C\}$ is pairwise disjoint with $x_0 \notin \bigcup_{\alpha\in C} L_\alpha$. Let $L = (\bigcup_{\alpha\in C} L_\alpha) \cup \{x_0\}$ and define a linear order $<_L$ on L as follows:

- for every $x \in \bigcup_{\alpha\in C} L_\alpha$, $x <_L x_0$; that is, $x_0 = \max L$;
- if $x, y \in L_\alpha$ for some $\alpha \in C$, then $x <_L y$ is defined by $x <_\alpha y$;
- if $x \in L_\alpha$ and $y \in L_\alpha$ with $\alpha, \beta \in C$ and $\alpha \neq \beta$, then $x <_L y$ is defined by $\alpha < \beta$.

Then obviously $<_L\restriction L_\alpha$ coincides with $<_\alpha$ for every $\alpha \in C$.

Claim 1. For every $\alpha \in \mathrm{Succ}(C)$, L_α is open in $\langle L, \lambda(<_L)\rangle$.

Proof. It follows from $L_\alpha = (\max L_{\alpha^-}, \min L_{\alpha^+})_L$ that L_α is open in $\langle L, \lambda(<_L)\rangle$.

Claim 2. For every $\alpha \in C$, $\langle L_\alpha, \lambda(<_\alpha)\rangle$ is a convex closed subspace of $\langle L, \lambda(<_L)\rangle$.

Proof. Since L_α is represented as $L_\alpha = [\min L_\alpha, \max L_\alpha]_L$, it is closed and convex. Therefore, $\lambda(<_L) \restriction L_\alpha = \lambda(<_L\restriction L_\alpha) = \lambda(<_\alpha)$.

Since $\lambda(<_\alpha) \restriction X_\alpha = \tau_\alpha$ for each $\alpha \in \Lambda$, by Claim 2, we have the following.

Claim 3. For every $\alpha \in \Lambda$, $\langle X_\alpha, \tau_\alpha \rangle$ is a subspace of $\langle L, \lambda(<_L) \rangle$.

To finish the proof of the lemma, it suffices to see the following.

Claim 4. $\tau = \lambda(<_L) \restriction X$, where τ denotes the topology of $X = X(\mathcal{X}, x_0)$.

Proof. First, we prove $\tau \subset \lambda(<_L) \restriction X$. Let $\mathcal{B}$ be the base $(\bigcup_{\alpha \in \Lambda} \tau_\alpha) \cup \{(\bigcup_{\alpha \in \Lambda \cap (\gamma, \kappa)} X_\alpha) \cup \{x_0\} : \gamma < \kappa\}$ of τ. It suffices to see $\mathcal{B} \subset \lambda(<_L) \restriction X$. Let $U \in \mathcal{B}$.

Case 1. $U \in \tau_\alpha$ for some $\alpha \in \Lambda$.

By Claim 3, there is $V \in \lambda(<_L)$ with $V \cap X_\alpha = U$. By Claim 1, we have $X_\alpha = X \cap L_\alpha \in \lambda(<_L) \restriction X$. Therefore, $U = V \cap X_\alpha = (V \cap X) \cap X_\alpha \in \lambda(<_L) \restriction X$ holds.

Case 2. $U = (\bigcup_{\alpha \in \Lambda \cap (\gamma, \kappa)} X_\alpha) \cup \{x_0\}$ for some $\gamma < \kappa$.

Let $\alpha_0 = \min(\Lambda \cap (\gamma, \kappa))$. Then we have $\alpha_0 \in \Lambda \subset \mathrm{Succ}(C)$ and $U = ((\bigcup_{\alpha \in (\alpha_0^-, \kappa) \cap C} L_\alpha) \cup \{x_0\}) \cap X = (\max L_{\alpha_0^-}, x_0]_L \cap X \in \lambda(<_L) \restriction X$.

Next, we show $\tau \supset \lambda(<_L) \restriction X$. Let $z \in L$. It suffices to see the following two facts.

Fact 1. $(\leftarrow, z)_L \cap X \in \tau$.

If $z = x_0$, $(\leftarrow, z)_L \cap X = \bigcup_{\alpha \in \Lambda} X_\alpha \in \tau$ holds. So we may assume $z \neq x_0$. Take $\alpha \in C$ with $z \in L_\alpha$. If $\alpha \notin \Lambda$, then $(\leftarrow, z)_L \cap X = \bigcup_{\beta \in \Lambda \cap \alpha} X_\beta \in \tau$. If $\alpha \in \Lambda$, then by Claim 3, we have $(\leftarrow, z)_L \cap X_\alpha \in \tau_\alpha \subset \tau$; therefore, $(\leftarrow, z)_L \cap X = (\bigcup_{\beta \in \Lambda \cap \alpha} X_\beta) \cup ((\leftarrow, z)_L \cap X_\alpha) \in \tau$.

Fact 2. $(z, \rightarrow)_L \cap X \in \tau$.

If $z = x_0$, $(z, \rightarrow)_L \cap X = \emptyset \in \tau$. So we may assume $z \neq x_0$. Take $\alpha \in C$ with $z \in L_\alpha$. If $\alpha \notin \Lambda$, then $(z, \rightarrow)_L \cap X = (\bigcup_{\beta \in \Lambda \cap (\alpha, \kappa)} X_\beta) \cup \{x_0\} \in \tau$. If $\alpha \in \Lambda$, then by Claim 3, we have $(z, \rightarrow)_L \cap X_\alpha \in \tau_\alpha \subset \tau$; therefore, $(z, \rightarrow)_L \cap X = (\bigcup_{\beta \in \Lambda \cap (\alpha, \kappa)} X_\beta) \cup ((z, \rightarrow)_L \cap X_\alpha) \in \tau$. □

The following corollary is well known by different approaches.

Corollary 2.5. *If $\mathcal{X} = \{X_\alpha : \alpha \in \Lambda\}$ is a pairwise disjoint collection of non-empty suborderable spaces, then the topological sum $\bigoplus_{\alpha \in \Lambda} X_\alpha$ is also suborderable.*

Proof. We may assume that all X_α's are non-empty. Take a suitably large regular infinite cardinal κ with $|\Lambda| \leq \kappa$ and we may assume $\Lambda \subset \mathrm{Succ}(\kappa)$. By Lemma 2.4, $X(\mathcal{X}, x_0)$ is suborderable for some x_0. Therefore, the subspace $\bigoplus_{\alpha \in \Lambda} X_\alpha$ of $X(\mathcal{X}, x_0)$ is suborderable. □

Corollary 2.5 shows the following corollary.

Corollary 2.6. *If X is a suborderable space and Y is a discrete space, then $X \times Y$ is suborderable.*

Therefore, when we discuss suborderability of $X \times Y$, we may assume that both X and Y are non-discrete. Additionally, note that if X is an orderable space and Y is a discrete space, then $X \times Y$ is orderable.

Corollary 2.7. *Let κ be a regular infinite cardinal. Then $X = (\mathrm{Succ}(\kappa) \cup \{\kappa\})^2$ is suborderable.*

Proof. For every $\alpha \in \mathrm{Succ}(\kappa)$, let

$$X_\alpha = (\{\alpha\} \times [\alpha, \kappa] \cap X) \bigoplus ((\alpha, \kappa] \times \{\alpha\} \cap X);$$

moreover, let

$$\mathcal{X} = \{X_\alpha : \alpha \in \mathrm{Succ}(\kappa)\}.$$

Then obviously $\mathcal{X}$ is a pairwise disjoint collection of suborderable spaces. One can check that both topologies of X and $X(\mathcal{X}, \langle \kappa, \kappa \rangle)$ coincide by carefully comparing both neighborhood bases at $\langle \kappa, \kappa \rangle$. Lemma 1.5 above shows that X is suborderable. $\square$

In particular, $(\omega + 1)^2$ is suborderable [7].

Lemma 2.8 ([1, Problem 3.12.3(a)])**.** *Let $\langle L, <, \lambda(<) \rangle$ be a LOTS. Then the following are equivalent.*

(1) *The space $\langle L, \lambda(<) \rangle$ is compact.*
(2) *For every subset A of L, A has the least upper bound $\sup_L A$ in $\langle L, < \rangle$.*
(3) *For every subset A of L, A has the greatest lower bound $\inf_L A$ in $\langle L, < \rangle$.*

Note that $\sup \emptyset = \min L$ and $\inf \emptyset = \max L$ whenever L is a compact LOTS.

Definition 2.9. Let L be a compact LOTS and $x \in L$. A subset $A \subset (\leftarrow, x)_L$ is said to be 0-*unbounded* for x in L if, for every $y < x$, there is $a \in A$ with $y \leq a$. Similarly, for a subset $A \subset (x, \rightarrow)_L$, 1-*unbounded* for x is defined. Now the 0-*cofinality* $0\text{-}\mathrm{cf}_L\, x$ *of x in L* is defined by

$$0\text{-}\mathrm{cf}_L\, x = \min \{|A| : A \text{ is 0-unbounded for } x \text{ in } L\}.$$

Analogously, $1\text{-}\mathrm{cf}_L\, x$ is defined. If there is no confusion, we write simply $0\text{-}\mathrm{cf}\, x$ and $1\text{-}\mathrm{cf}\, x$. Observe that

- if x is the smallest element of L, then $0\text{-}\mathrm{cf}\, x = 0$;
- if x has the immediate predecessor in L, then $0\text{-}\mathrm{cf}\, x = 1$;
- otherwise, $0\text{-}\mathrm{cf}\, x$ is a regular infinite cardinal.

Moreover, note

- $\omega \leq$ 0-cf x if and only if $\sup_L(\leftarrow, x)_L = x$ if and only if $x \in \mathrm{Cl}_L(\leftarrow, x)_L$.

If 0-cf $x = \kappa$, then we can define a strictly increasing function $c : \kappa \to L$ which is continuous with its range $c[\kappa]$ 0-unbounded for x. We call such a function c a 0-*normal function* for x in L. The reader should note that these methods in a compact LOTS extend the usual methods in ordinal numbers.

Observe that in the notation above, for every closed set F of κ, $c[F]$ is also closed in $(\leftarrow, x)_L$. Therefore, c is an embedding such that $c[\kappa]$ is closed in $(\leftarrow, x)$ and 0-unbounded for x. Note that there can be many 0-normal functions for x in L.

Also note that if cX and $c'X$ are two linearly ordered compactifications of a GO-space X, then i-$\mathrm{cf}_{cX}\, x$ coincides with i-$\mathrm{cf}_{c'X}\, x$ for every $x \in X$ and $i \in 2 = \{0, 1\}$. In our discussion, we apply these methods for $L = lX$ with a GO-space X and consider 0-$\mathrm{cf}_{lX}\, x$ or 1-$\mathrm{cf}_{lX}\, x$ for $x \in lX$. In particular, if X is a subspace of an ordinal, say $X \subset [0, \gamma]$, with the usual order, then we can check using Lemma 1.1 that $lX = \mathrm{Cl}_{[0,\gamma]}X$. Moreover, in this case, for every $x \in lX$, obviously 1-cf x is 0 or 1; furthermore, we can easily check that 0-cf x is equal to cf x in the usual sense whenever $x \in \mathrm{Lim}(X)$. Let X be a GO-space, $x \in X$, and $\kappa =$ 0-cf $x \geq \omega$ and fix a 0-normal function $c : \kappa \to lX$. Inductively, one can take a strictly increasing sequence $\{x(\alpha) : \alpha < \kappa\} \subset (\leftarrow, x)_{lX} \cap X$ with $\sup(\{c(\beta) : \beta \leq \alpha\} \cup \{x(\beta) : \beta < \alpha\}) < x(\alpha)$. Then, obviously, $\{x(\alpha) : \alpha < \kappa\} \cup \{x\}$ is homeomorphic to $\mathrm{Succ}(\kappa) \cup \{\kappa\}$. Similarly, whenever X is a subspace of an ordinal and $\alpha \in X \cap \mathrm{Lim}(X)$, one can fix a strictly increasing sequence $\{\alpha(\gamma) : \gamma < \kappa\} \subset X$ which is cofinal in α such that $\{\alpha(\gamma) : \gamma < \kappa\} \cup \{\alpha\}$ is homeomorphic to $\mathrm{Succ}(\kappa) \cup \{\kappa\}$, where $\kappa = \mathrm{cf}\, \alpha$.

R. Engelking and D. Lutzer [2] proved that a suborderable space is paracompact if and only if it does not have a closed subspace which is homeomorphic to a stationary set in a regular uncountable cardinal. Therefore, we have the following lemma.

Lemma 2.10 ([2]). *A suborderable space is hereditarily paracompact if and only if it does not have a subspace which is homeomorphic to a stationary set in a regular uncountable cardinal.*

Now we are prepared to find properties implied by the suborderability of product spaces. Note that if the product space $X \times Y$ is suborderable, then both X and Y are suborderable. Therefore, we may assume that X and Y are GO-spaces under the assumption that $X \times Y$ is suborderable.

Theorem 2.11. *Let X and Y be non-discrete GO-spaces. If the product space $X \times Y$ is suborderable, then*

(1) *X and Y are hereditarily paracompact;*
(2) *there is a unique regular infinite cardinal κ such that, for every $z \in X \cup Y$ and $i \in 2$, i-$\mathrm{cf}\, z$ is 0, 1, or κ, where i-$\mathrm{cf}\, z$ means i-$\mathrm{cf}_{lX}\, z$ (i-$\mathrm{cf}_{lY}\, z$) whenever $z \in X$ ($z \in Y$, respectively);*
(3) *X or Y are hereditarily disconnected.*

Proof. Assume that $X \times Y$ is suborderable. Fix a linearly ordered set $\langle L, <_L \rangle$ such that $X \times Y$ is a subspace of $\langle L, \lambda(<_L) \rangle$.

(1) We will see that Y is hereditarily paracompact (the case for X is similar). Assume not; then by Lemma 2.10, there is a subspace which is homeomorphic to a stationary set S in a regular uncountable cardinal in κ. Since X is non-discrete, there is $i \in 2$ and $x \in X$ with $\lambda = i\text{-}\,\mathrm{cf}_{lX}\, x \geq \omega$. As mentioned above, X has a subspace which is homeomorphic to $\mathrm{Succ}(\lambda) \cup \{\lambda\}$.

Case 1. $\lambda < \kappa$.

By Lemma 2.1, the hereditarily normal space $X \times Y$ has a non-hereditarily normal subspace, a contradiction.

Case 2. $\kappa \leq \lambda$.

Since S is stationary, we can take $\alpha \in S \cap \mathrm{Lim}(S)$. Set $\mu = \mathrm{cf}\, \alpha$; then $\mu < \lambda$. As mentioned above, S has a subspace which is homeomorphic to $\mathrm{Succ}(\mu) \cup \{\mu\}$. Then the suborderable space $X \times Y$ contains a subspace which is homeomorphic to $(\mathrm{Succ}(\lambda) \cup \{\lambda\}) \times (\mathrm{Succ}(\mu) \cup \{\mu\})$. This contradicts Lemma 2.2.

(2) Assume that (2) does not hold. Since both X and Y are non-discrete, there are $x \in X$, $y \in Y$, and $i, j \in 2$ with i-$\mathrm{cf}\, x \geq \omega$, j-$\mathrm{cf}\, y \geq \omega$, and i-$\mathrm{cf}\, x \neq j$-$\mathrm{cf}\, y$. Set $\kappa = i$-$\mathrm{cf}\, x$ and $\lambda = j$-$\mathrm{cf}\, y$. Then the suborderable space $X \times Y$ contains a subspace which is homeomorphic to $(\mathrm{Succ}(\kappa) \cup \{\kappa\}) \times (\mathrm{Succ}(\lambda) \cup \{\lambda\})$. This contradicts Lemma 2.2.

(3) Recall that a space is *hereditarily disconnected* if every non-empty connected subset is a one-point set. Assume neither X nor Y is hereditarily disconnected. Then there are connected subsets C and D of X and Y, respectively, with $2 \leq |C|$ and $2 \leq |D|$. Fix $x_0, x_1 \in C$ and $y_0, y_1 \in D$ with $x_0 \neq x_1$ and $y_0 \neq y_1$, respectively. We may assume $\langle x_0, y_0 \rangle <_L \langle x_0, y_1 \rangle <_L \langle x_1, y_1 \rangle$; otherwise, change the indices. Then $\langle x_1, y_0 \rangle \in C \times \{y_0\} \cap \{x_1\} \times D$; moreover, both $C \times \{y_0\}$ and $\{x_1\} \times D$ are connected. Therefore, $C \times \{y_0\} \cup \{x_1\} \times D$ is a connected subset of $X \times Y \setminus \{\langle x_0, y_1 \rangle\}$ containing the points $\langle x_0, y_0 \rangle$ and $\langle x_1, y_1 \rangle$. On the other hand, the disjoint open sets $(\leftarrow, \langle x_0, y_1 \rangle)_L \cap X \times Y$ and $(\langle x_0, y_1 \rangle, \rightarrow)_L \cap X \times Y$ separate the connected set $C \times \{y_0\} \cup \{x_1\} \times D$, a contradiction. $\square$

Whenever X and Y are subspaces of an ordinal, then the converse implication of the theorem above is also true.

Theorem 2.12. *Let X and Y be non-discrete subspaces of an ordinal. Then the product space $X \times Y$ is suborderable, if*

(1) *X and Y are hereditarily paracompact, and*
(2) *there is a unique regular infinite cardinal κ such that, for every $z \in X \cup Y$ and $i \in 2$, $\mathrm{cf}\, z$ is either $0, 1$ or κ; equivalently, for every $z \in (X \cap \mathrm{Lim}(X)) \cup (Y \cap \mathrm{Lim}(Y))$, $\mathrm{cf}\, z = \kappa$.*

Proof. Note that every subspace of an ordinal is hereditarily disconnected. We may assume $X \cup Y \subset [0, \gamma]$ for some ordinal γ. It suffices to see that by induction on $\alpha \leq \gamma$, $(X \cap [0, \alpha]) \times Y$ is suborderable (because $\alpha = \gamma$ finishes the proof). Assume that $\alpha \leq \gamma$ and for every $\alpha' < \alpha$, $(X \cap [0, \alpha']) \times Y$ is suborderable.

Case 1. $\alpha \notin \mathrm{Lim}(X)$.

In this case, let $\alpha' = \sup(X \cap \alpha)$. By $\alpha' < \alpha$, since $(X \cap [0, \alpha]) \times Y$ is homeomorphic to $(X \cap [0, \alpha']) \times Y \bigoplus (X \cap \{\alpha\}) \times Y$, it is suborderable by the assumption.

Case 2. $\alpha \in \mathrm{Lim}(X)$.

Set $\lambda = \mathrm{cf}\, \alpha$ and fix a normal function $c : \lambda \to \alpha$ for α; that is, it is a strictly increasing continuous cofinal function into α, where $c(-1) = -1$. Since λ is homeomorphic to $c[\lambda]$, by Lemma 2.10, $c^{-1}[X]$ is non-stationary in λ whenever λ is uncountable.

Subcase 2.1. $\alpha \notin X$.

When $\lambda = \omega$, $(X \cap [0, \alpha]) \times Y$ is homeomorphic to $\bigoplus_{n \in \omega}(X \cap (c(n-1), c(n)]) \times Y$. When $\omega < \lambda$, taking a club set C in λ with $C \cap c^{-1}[X] = \emptyset$, $(X \cap [0, \alpha]) \times Y$ is homeomorphic to $\bigoplus_{\delta \in \mathrm{Succ}(C)}(X \cap (c(\delta^-), c(\delta)]) \times Y$. In either case, $(X \cap [0, \alpha]) \times Y$ is suborderable by the inductive assumption.

Subcase 2.2. $\alpha \in X$.

By assumption (2), we have $\lambda = \kappa$. We will see by induction on $\beta \leq \gamma$ that $(X \cap [0, \alpha]) \times (Y \cap [0, \beta])$ is suborderable (then $\beta = \gamma$ finishes this subcase). Assume that $\beta \leq \gamma$ and for every $\beta' < \beta$, $(X \cap [0, \alpha]) \times (Y \cap [0, \beta'])$ is suborderable. It suffices to check the case $\beta \in Y \cap \mathrm{Lim}(Y)$, because other cases are similar to Case 1 and Subcase 2.1. By assumption (2), we have $\mathrm{cf}\, \beta = \kappa$. Let $d : \kappa \to \beta$ be a normal function for β. When $\kappa = \omega$, let $C = \omega$. When $\kappa > \omega$, by Lemma 2.10, take a club set C of κ with $C \cap (c^{-1}[X] \cup d^{-1}[Y]) = \emptyset$. For every $\delta \in \mathrm{Succ}(C)$, let $Z_\delta =$

$$(X \cap (c(\delta^-), \alpha]) \times (Y \cap (d(\delta^-), d(\delta)]) \bigoplus (X \cap (c(\delta^-), c(\delta)]) \times (Y \cap (d(\delta), \beta]).$$

By the inductive assumption, Z_δ is suborderable. Put $\Lambda = \{\delta \in \mathrm{Succ}(C) : Z_\delta \neq \emptyset\}$ and $\mathcal{Z} = \{Z_\delta : \delta \in \Lambda\}$. Note that $\mathcal{Z}$ is pairwise disjoint. It is easy to see that $(X \cap [0,\alpha]) \times (Y \cap [0,\beta]) = (\bigcup_{\delta\in\Lambda} Z_\delta) \cup \{\langle\alpha,\beta\rangle\}$ and the product topology coincides with topology of the 1-point extension of $\bigoplus_{\delta\in\Lambda} Z_\delta$ with the κ-limit point $\langle\alpha,\beta\rangle$. It follows from Lemma 2.4 that $(X \cap [0,\alpha]) \times (Y \cap [0,\beta])$ is suborderable. $\square$

Note that the product of two subspaces of an ordinal is scattered (= every subspace has an isolated point), and that scattered suborderable spaces are orderable [11]. Thus, in Theorem 2.12, "suborderable" is replaced by "orderable."

Example 2.13. The square $\mathbb{S}^2$ of the Sorgenfrey line $\mathbb{S}$ with the usual order satisfies (1), (2), and (3) with $X = Y = \mathbb{S}$ in Theorem 2.11. But $\mathbb{S}^2$ is not suborderable.

It is well known that $\mathbb{S}$ is hereditarily paracompact and hereditarily disconnected. Since $\mathbb{S}^2$ is not normal, it is not suborderable. We check (2). We may assume $\mathbb{S} = (0,1)$ with the usual order and the topology induced by $\{(a,\rightarrow) : a \in (0,1)\} \cup \{(\leftarrow,b] : b \in (0,1)\}$, where $(0,1)$ denotes the unit open interval. Then using Lemma 1.1 and Lemma 2.8, it is easy to check $l\mathbb{S} = [0,1] \times \{0\} \cup (0,1) \times \{1\}$ with the lexicographic order identifying $\mathbb{S}$ with $(0,1) \times \{0\}$. Then, for every $x \in l\mathbb{S}$ and $i \in 2$, $i\text{-}\operatorname{cf} x$ is either 0, 1, or ω.

Question 2.14. For non-discrete suborderable spaces X and Y, characterize suborderability of $X \times Y$.

Concerning monotonical normality, the following are known.

- If $X \times Y$ is monotonically normal and if Y contains a countable set with a limit point, then X is stratifiable [6].
- If X^2 is monotonically normal, then X is hereditarily paracompact and X^n is monotonically normal for each finite n [4].

So we also ask the following question.

Question 2.15. Characterize suborderable spaces X and Y for which $X \times Y$ is monotonically normal.

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