Uniformity of Uniform Convergence on the Family of Sets

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Electronically published on July 29, 2016

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ABSTRACT. We prove that for every Tychonoff space X and any uniform cub-space $(Y, \mathcal{U})$, the topology on $C(X, Y)$, induced by the uniformity $\hat{\mathcal{U}}|\lambda$ of uniform convergence on the saturation family λ, coincides with the set-open topology on $C(X, Y)$. For every Y^2-compact space X and any uniform space $(Y, \mathcal{U})$, the topology on $C(X, Y)$, induced by the uniformity $\hat{\mathcal{U}}$ of uniform convergence, coincides with the Y-compact-open topology on $C(X, Y)$ and depends only on the topology induced on Y by the uniformity $\mathcal{U}$. In particular, for every pseudocompact space X and any metrizable topological vector space Y with uniform $\mathcal{U}$, the topology on $C(X, Y)$, induced by the uniformity $\hat{\mathcal{U}}$ of uniform convergence, coincides with the C-compact-open topology on $C(X, Y)$ (with the compact-open topology on $C(\nu X, Y)$ where νX is the Hewitt real-compactification of X) and depends only on the topology induced on Y by the uniformity $\mathcal{U}$. It is also shown that in the class of closed-homogeneous complete uniform spaces Y, a necessary condition for coincidence of topologies is Y-compactness of the elements of the family λ.

1. INTRODUCTION

Let X be a Hausdorff space and let $(Y, \mathcal{U})$ be a uniform space. We shall denote by $C(X, Y)$ the set of all continuous mappings of the space X to the space Y, where Y is equipped with the topology induced by $\mathcal{U}$. For every

$V \in \mathcal{U}$, denote by $\hat{V}$ the entourage of the diagonal $\Delta \subset C(X,Y) \times C(X,Y)$ defined by the formula

$$\hat{V} = \{(f,g) : (f(x), g(x)) \in V \text{ for every } x \in X\}.$$

The uniformity on the set $C(X,Y)$ generated by this family is called the *uniformity of uniform convergence induced by $\mathcal{U}$* and will be denoted by $\hat{\mathcal{U}}$. For two uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ on Y which induce the same topology, the topologies on $C(X,Y)$ induced by $\hat{\mathcal{U}}_1$ and $\hat{\mathcal{U}}_2$ can be different ([3, Example 4.2.14]). It turns out, however, that for a compact space X, the topology on $C(X,Y)$ is independent of the choice of a particular uniformity $\mathcal{U}$ on the space Y because the topology induced by $\hat{\mathcal{U}}$ coincides with the compact-open topology on $C(X,Y)$.

2. Preliminaries

Let X and Y be Tychonoff spaces.

For a fixed natural number n, a subset A of X is said to be Y^n-*compact* provided $f(A)$ is compact for any $f \in C(X, Y^n)$. Note that any Y^{n+1}-compact subset of X is a Y^n-compact subset of X.

For example, if space Y is a metrizable topological vector space, then a Y-compact subset A of X is a C-compact subset of X and, moreover, A is a Y^n-compact subset of X for any $n \in \mathbb{N}$ (and even Y^ω-compact) [5]. Recall that a subset A of space X is a C-compact subset of X provided that a set $f(A)$ is compact for every $f \in C(X, \mathbb{R})$.

Definition 2.1. A space Y will be called a *cub-space* if, for any $x \in Y \times Y$, there are a continuous map f from $Y \times Y$ to Y and a point $y \in Y$ such that $f^{-1}(y) = x$.

Note that if Y is a cub-space, then for any $x \in Y^n$ where $n > 1$, there are a continuous map f from Y^n to Y and a point $y \in Y$ such that $f^{-1}(y) = x$. For example, let $n = 3$ and $x \in Y^3 = Y^2 \times Y$ where $x = (x_1, x_2, x_3)$. For a point $z = (x_1, x_2)$, there are a continuous map h_1 from $Y \times Y$ to Y and a point $y \in Y$ such that $h_1^{-1}(y) = z$. For a point $p = (y, x_3)$, there are a continuous map h_2 from $Y \times Y$ to Y and a point $s \in Y$ such that $h_2^{-1}(s) = p$. So a map $f = h_2 \circ (h_1, id_Y)$ is a continuous map from Y^3 to Y such that $f^{-1}(s) = x$. We now proceed by induction.

One can see that if Y is a Tychonoff space with countable pseudocharacter containing a nontrivial path or if Y is a zero-dimensional space with countable pseudocharacter containing a nontrivial convergent sequence, then Y is a cub-space. To wit, if Y is a Tychonoff space with countable pseudocharacter, then each point in $Y \times Y$ is a zero set. For a point $(x, y) \in Y \times Y$, there is a continuous function $f : Y \times Y \mapsto \mathbb{I} = [0, 1]$

such that $f^{-1}(0) = (x, y)$. Since Y contains a nontrivial path, $\mathbb{I}$ is homeomorphic to a subspace of Y. Analogously, if Y is a zero-dimensional space with countable pseudocharacter containing a nontrivial convergent sequence, then each point in $Y \times Y$ is a zero set. For a point $(x, y) \in Y \times Y$, there is a continuous function $f : Y \times Y \mapsto \mathbb{A}$ where $\mathbb{A}$ is a nontrivial convergent sequence such that $f^{-1}(0) = (x, y)$. Since Y contains a nontrivial convergent sequence, $\mathbb{A}$ is homeomorphic to a subspace of Y.

For example, a Tychonoff space with G_δ-diagonal containing a nontrivial path and a zero-dimensional space with G_δ-diagonal containing a nontrivial convergent sequence are cub-spaces.

In [2], a space M_1 with the following properties is constructed: M_1 is a metric continuum; if Z is a subcontinuum of M_1 and $f : Z \mapsto M_1$ is a continuous mapping, then either f is constant or $f(x) = x$ for all $x \in X$. It follows that M_1 is not a cub-space.

Proposition 2.2. *Let Y be a cub-space. Then any Y-compact subset of X is a Y^n-compact subset of X.*

Proof. Let A be a Y-compact subset X and $g \in C(X, Y^n)$. Suppose that there is $z \in \overline{g(A)} \setminus g(A)$. So there are a continuous map f from Y^n to Y and a point $y \in Y$ such that $f^{-1}(y) = z$. It follows that $f(g(A))$ is not a compact subset of Y, which contradicts the Y-compactness of A. $\square$

A subset A of X is said to be a Y-*zero set* provided $A = f^{-1}(Z)$ for some zero set Z of Y and $f \in C(X, Y)$. For example, if Y is the real numbers space $\mathbb{R}$, then any zero set subset of X is an $\mathbb{R}$ zero set of X.

Proposition 2.3. *Let X and Y be topological spaces, A be a Y^2-compact subset of X, and B be a Y-zero set such that $B \bigcap A \neq \emptyset$. Then $B \bigcap A$ is a Y-compact subset of X.*

Proof. Let $g \in C(X, Y)$. We fix a continuous mapping h of Y into $\mathbb{R}$ with $Z = h^{-1}(0)$ and $f \in C(X, Y)$ such that $B = f^{-1}(Z)$. Let f_1 be the diagonal product of the mappings g and f; that is, $f_1(x) = (g(x), f(x)) \in Y \times Y$. The set $S = f_1(B \bigcap A) = f_1(A) \bigcap (Y \times Z)$ is closed in $Y \times Y$, and it follows that S is compact.

Let π be a natural projection of $Y \times Y$ onto Y, associating with every point its first coordinate. Then, clearly, $g = \pi \circ f_1$ and $g(B \bigcap A) = \pi(S)$.

Since π is continuous and S is compact, we conclude that $g(B \bigcap A)$ is also compact. $\square$

Proposition 2.4. *Let X be a topological space, Y be a cub-space, A be a Y-compact subset of X, and B be a Y-zero set such that $B \bigcap A \neq \emptyset$. Then $B \bigcap A$ is a Y-compact subset of X.*

3. Uniformity of Uniform Convergence on Y-Compact Sets

Recall that a family λ of non-empty subsets of a topological space (X, τ) is called a π-*network* for X if, for any nonempty open set $U \in \tau$, there exists $A \in \lambda$ such that $A \subseteq U$.

For a Hausdorff space X, a π-network λ for X, and a uniform space $(Y, \mathcal{U})$, we shall denote by $\hat{\mathcal{U}}|\lambda$ the uniformity on $C(X, Y)$ generated by the base consisting of all finite intersections of the sets of the form

$$\hat{V}|A = \{(f, g) : (f(x), g(x)) \in V \text{ for every } x \in A\},$$

where $V \in \mathcal{U}$, $A \in \lambda$.

The uniformity $\hat{\mathcal{U}}|\lambda$ will be called the *uniformity of uniform convergence on family λ induced by $\mathcal{U}$*.

Recall that all sets of the form $[F, U] := \{f \in C(X, Y) : f(F) \subseteq U\}$, where $F \in \lambda$ and U is an open subset of Y, form a subbase of the set-open (λ-open) topology on $C(X, Y)$.

We use the following notations for various topological spaces on the set $C(X, Y)$:

      $C_{\hat{\mathcal{U}}|\lambda}(X, Y)$ for the topology induced by $\hat{\mathcal{U}}|\lambda$,

      $C_\lambda(X, Y)$ for the λ-open topology.

Let y be a point of a uniform space $(Y, \mathcal{U})$ and let $V \in \mathcal{U}$. Recall that the set $B(y, V) = \{z \in Y : (y, z) \in V\}$ is called the *ball with center y and radius V* or, briefly, the *V-ball about y*. For a set $A \subset Y$ and a $V \in \mathcal{U}$, by the V-ball about A we mean the set $B(A, V) = \bigcup_{y \in A} B(y, V)$.

Lemma 3.1 ([3, Lemma 8.2.5])**.** *If $\mathcal{U}$ is a uniformity on a space X, then, for every compact set $Z \subseteq X$ and any open set G containing Z, there exists a $V \in \mathcal{U}$ such that $B(Z, V) \subseteq G$.*

A family λ will be called *hereditary with respect to the Y-zero subsets* of X if any nonempty $A \bigcap B \in \lambda$ where $A \in \lambda$ and B is a Y-zero set of X.

Definition 3.2. A π-network λ for X, hereditary with respect to the Y-zero-subsets of X and consisting of Y-compact subsets, will be called a *saturation family*.

Theorem 3.3. *For every Tychonoff space X and any uniform cub-space $(Y, \mathcal{U})$, the topology on $C(X, Y)$ induced by the uniformity $\hat{\mathcal{U}}|\lambda$ of uniform convergence on the saturation family λ coincides with the λ-open topology on $C(X, Y)$, where Y has the topology induced by $\mathcal{U}$.*

Proof. Denote by τ_1 the topology on $C(X, Y)$ induced by the uniformity $\hat{\mathcal{U}}|\lambda$ and by τ_2 the λ-open topology. First, we shall prove that $\tau_2 \subseteq \tau_1$. Clearly, it suffices to show that all sets $[A, U]$, where $A \in \lambda$ and U is an open subset of Y, belong to τ_1. Consider an $A \in \lambda$, an open set $U \subseteq Y$, and an $f \in [A, U]$. Since A is a Y-compact subset of X, $f(A)$ is a compact subspace of U. Applying Lemma 3.1, take a $V \in \mathcal{U}$ such that $B(f(A), V) \subseteq U$. Since $B(f, \hat{V}|A) \subseteq [A, U]$ and since f is an arbitrary element of $[A, U]$, we have $[A, U] \in \tau_1$.

We shall now prove that $\tau_1 \subseteq \tau_2$. Clearly, it suffices to show that for any $A \in \lambda$, $V \in \mathcal{U}$, and $f \in C(X, Y)$, there exist Y-compact subsets $A_1, ..., A_k \in \lambda$ and open subsets $U_1, ..., U_k$ of Y such that

$$f \in \bigcap_{i=1}^{k} [A_i, U_i] \subset B(f, \hat{V}|A).$$

By [3, Corollary 8.1.12], there exists an entourage $W \in \mathcal{U}$ of the diagonal $\Delta \subset Y \times Y$ which is closed with respect to the topology induced by $\mathcal{U}$ on $Y \times Y$ and satisfies the inclusion $3W \subset V$. It follows from the compactness of $f(A)$ that there exists a finite set $\{x_1, ..., x_k\} \subset A$ such that $f(A) \subseteq \bigcup_{i=1}^{k} B(f(x_i, W))$. Note that $f(A) \subset \bigcup_{i=1}^{k} U_i$ where $U_i = IntB(f(x_i), 2W)$. Observe that from the closedness of W in $Y \times Y$ follows the closedness of balls $B(f(x_i), W)$ in Y and the compactness of the sets $f(A) \bigcap B(f(x_i), W)$. Let Z_i be a zero set of Y such that $f(A) \bigcap B(f(x_i), W) \subseteq Z_i \subseteq U_i$. By Proposition 2.4, the set $A_i = f^{-1}(Z_i) \bigcap A$ is Y-compact. Note that $A_i \in \lambda$ because the family λ is hereditarily Y-compact.

We have $f \in \bigcap_{i=1}^{k} [A_i, U_i]$. If $g \in \bigcap_{i=1}^{k} [A_i, U_i]$, then for any $x \in A$ there is A_i such that $x \in A_i$ and we have $g(x) \in B(f(x_i), 2W)$ and $f(x) \in B(f(x_i), W)$. It follows that $(f(x), g(x)) \in 3W \subset V$ for any $x \in A$, so that $g \in B(f, \hat{V}|A)$. $\qquad\square$

The Y-compact-open topology on $C(X, Y)$ is the topology generated by the base consisting of sets $\bigcap_{i=1}^{k} [A_i, U_i]$, where A_i is a Y-compact subset of X and U_i is an open subset of Y for $i = 1, ..., k$.

Corollary 3.4. *For every Tychonoff space X and any space $(Y, \mathcal{U})$ with G_δ-diagonal containing a nontrivial path, the topology on $C(X, Y)$, induced by the uniformity $\hat{\mathcal{U}}|\lambda$ of uniform convergence on the saturation family λ, coincides with the λ-open topology on $C(X, Y)$, where Y has the topology induced by $\mathcal{U}$.*

Corollary 3.5. *For every Tychonoff space X and a zero-dimensional space $(Y, \mathcal{U})$ with G_δ-diagonal, the topology on $C(X, Y)$, induced by the uniformity $\hat{\mathcal{U}}|\lambda$ of uniform convergence on the saturation family λ, coincides with the λ-open topology on $C(X, Y)$, where Y has the topology induced by $\mathcal{U}$.*

Corollary 3.6. *For every Y^2-compact space X and any uniform space $(Y, \mathcal{U})$, the topology on $C(X, Y)$, induced by the uniformity $\hat{\mathcal{U}}$ of uniform convergence, coincides with the Y-compact-open topology on $C(X, Y)$ and depends only on the topology induced on Y by the uniformity $\mathcal{U}$.*

Note that $\mathbb{R}$-compactness (C-compactness) of a space X is equivalent to pseudocompactness of X.

Corollary 3.7. *For every pseudocompact space X and any metrizable topological vector space Y with uniform $\mathcal{U}$, the topology on $C(X, Y)$, induced by the uniformity $\hat{\mathcal{U}}$ of uniform convergence, coincides with the C-compact-open topology on $C(X, Y)$ and depends only on the topology induced on Y by the uniformity $\mathcal{U}$.*

Note that for a pseudocompact space X the space $C(X, Y)$ with the C-compact-open topology is linearly homeomorphic to $C(\nu X, Y)$ with the compact-open topology where νX is the Hewitt realcompactification of X.

4. Closed-Homogeneous Spaces

Recall that a space X is *strongly locally homogeneous* (*SLH*) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f : X \mapsto X$ which is supported on B (that is, f is the identity outside B) and moves x to y. The well-known homogeneous continua are SLH: the Hilbert cube, the universal Menger continua, and manifolds without boundaries.

A topological space X is said to be closed homogeneous provided that for any $x, y \in X$ and any K closed subset of $X \setminus \{x, y\}$, there is a homeomorphism $f : X \mapsto X$ which is supported on $X \setminus K$ (that is, f is the identity on K) and moves x to y.

It is well known that a zero-dimensional homogeneous space is closed homogeneous. Observe that a closed-homogeneous space is SLH. Note that there exists an SLH space X which is not closed homogeneous (see [4]). In fact, if we take $X = \mathbb{R} \setminus \{0\}$ and $\beta = \{\{x\} : x < 0\} \bigcup \{(a, b) : 0 < a < b\}$, then the topological space $(X, \tau(\beta))$, generated by the base β, is an SLH metrizable space which is not closed homogeneous.

5. Uniformity of Uniform Convergence on Y-Closed Totally Bounded Sets

Recall that if Y is uniformized by a uniformity $\hat{U}$, a subset A of X is said to be *Y-totally bounded* when $f(A)$ is totally bounded for any $f \in C(X,Y)$ (see [1]).

A subset A of X is said to be *Y-closed totally bounded* if $f(A)$ is closed totally bounded for any $f \in C(X,Y)$.

Theorem 5.1. *Let X be a Tychonoff space, Y be a uniform closed-homogeneous space, and $C_{\hat{U}|\lambda}(X,Y) = C_\lambda(X,Y)$. Then the family λ consists of Y-closed totally bounded sets.*

Proof. Suppose that there is $A \in \lambda$ which is not a Y-totally bounded set. Then there is $f \in C(X,Y)$ such that $f(A)$ is not totally bounded. Let $B(f,\hat{V}|A)$ be an open neighborhood of f in the topological space $C_{\hat{U}|\lambda}(X,Y)$.

Since $C_{\hat{U}|\lambda}(X,Y) = C_\lambda(X,Y)$, there is an open set $\bigcap_{i=1}^{k}[A_i, U_i]$ in the topological space $C_\lambda(X,Y)$ such that $f \in \bigcap_{i=1}^{k}[A_i, U_i] \subseteq B(f,\hat{V}|A)$. Consider a subset M of $f(A)$ such that

1. M is not totally bounded;
2. either $M \subset U_i$ or $\overline{f(A) \bigcap U_i} \bigcap M = \emptyset$ for every $i = 1,...,k$.

Let $W = \bigcap U_i$ where U_i is such that $M \subseteq U_i$. Let $y_1, y_2 \in W$ such that $y_1 \in M$ and $(y_1, y_2) \notin V$. Since Y is a closed-homogeneous space there is a homeomorphism $h : Y \mapsto Y$ which is supported on W (that is, h is the identity on $X \setminus W$) and moves y_1 to y_2. Consider a continuous map $g = h \circ f$. Note that $g \in \bigcap_{i=1}^{k}[A_i, U_i]$. It is clear that if $x \in f^{-1}(y_1) \bigcap A$, then $(f(x), g(x)) \notin V$ and $g \notin B(f,\hat{V}|A)$. This contradicts our assumption. So a set $f(A)$ is a totally bounded subset of space Y and A is a Y-totally bounded set.

Suppose that $f(A)$ is not closed. Then we have a point $y \in \overline{f(A)} \setminus f(A)$. Let $S = Y \setminus \{y\}$ and $[A, S]$ be an open set of space $C_\lambda(X,Y)$. Then there exists an open set $B(f,\hat{V}|B)$ of space $C_{\hat{U}|\lambda}(X,Y)$ such that $f \in B(f,\hat{V}|B) \subseteq [A, S]$. Let z be a point of $IntB(y, W)$ where $2W \subseteq V$ such that $f^{-1}(z) \bigcap A \neq \emptyset$. Since Y is a closed-homogeneous space, there is a homeomorphism $p : Y \mapsto Y$ which is supported on $IntB(y, W)$ and moves z to y. Consider a continuous map $q = p \circ f$. It is clear that if $x \in f^{-1}(IntB(y, W)) \bigcap B$, then $(f(x), q(x)) \in 2W \subseteq V$ and if $x \in f^{-1}(z) \bigcap A$, then $q(x) = y$. Thus, $q \in B(f,\hat{V}|B)$ and $q \notin [A, S]$. This

contradicts our assumption. We have that A is a Y-closed totally bounded subset of X. $\square$

Theorem 5.2. *Let X be a Tychonoff space, Y be a closed-homogeneous complete uniform space, and $C_{\hat{\mathcal{U}}|\lambda}(X,Y) = C_{\lambda}(X,Y)$. Then the family λ consists of Y-compact sets.*

Proof. It suffices to note that a closed totally bounded subset of a complete uniform space is a compact set. $\square$

Corollary 5.3. *Let X be a Tychonoff space, Y be a zero-dimensional homogeneous complete uniform space, and $C_{\hat{\mathcal{U}}|\lambda}(X,Y) = C_{\lambda}(X,Y)$. Then the family λ consists of Y-compact sets.*

Example 5.4. If Z is the Sorgenfrey line and $C_{\hat{\mathcal{U}}|\lambda}(Z,Z) = C_{\lambda}(Z,Z)$, then the family λ consists of compact sets. On the other hand, since Z is a cub-space, we get that, for any Hausdorff space X, the topology on $C(X,Z)$, induced by the uniformity $\hat{\mathcal{U}}|\lambda$ of uniform convergence on the saturation compact family λ, coincides with the λ-open topology on $C(X,Z)$.

References

[1] Anna Di Concilio and Som Naimpally, *Proximal set-open topologies*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **3** (2000), no. 1, 173–191.

[2] H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. **60** (1967), 241–249.

[3] Ryszard Engelking, *General Topology*. Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[4] Ali Ahmad Fora, *New and old types of homogeneity*, Turkish J. Math. **24** (2000), no. 4, 335–344.

[5] A. V. Osipov, *The set-open topology*, Topology Proc. **37** (2011), 205–217.

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