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REALIZING FINITE TOPOLOGIES BY T-CLOSED EQUIVALENCE DECOMPOSITIONS

JAMES MAISSEN

Abstract. The set-valued function T is a well-established tool that aids in the classification of metric and Hausdorff continua. I answer in full a question by David Bellamy on which finite T_0 connected topologies can be realized as the T-closed equivalences of continua.

1. Introduction

At the 49th Spring Topology and Dynamics Conference, David P. Bellamy posed the following question:

> Given a finite connected T_0 space $\hat{X}$, is there a continuum X such that the T-closed equivalence decomposition of X is topologically equal to $\hat{X}$?

In this paper, the question is answered in the affirmative for all finite connected T_0 topologies.

2. Terms and Notation

In this paper, the term "continuum" will mean a non-degenerate compact, connected, Hausdorff space even though the continua actually constructed herein will all be metric continua. A continuum is *indecomposable* if it cannot be expressed as the union of two proper subcontinua. Let $\mathbb{N}$ denote the strictly positive integers. Given a compact space X, denote the hyperspace of compact subsets of X by 2^X and the power set

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of X by $\mathbb{P}(X)$. Let $\mathcal{K}$ represent the *buckethandle continuum* [2] defined as $\mathcal{K} := \varprojlim\{[0,1], \Lambda\}$ where Λ is the tent map given by

$$\Lambda(x) := \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2; \\ 2 - 2x & \text{if } 1/2 < x \leq 1. \end{cases}$$

The buckethandle continuum is also known as a horseshoe in dynamics or as the B-J-K (Brouwer-Janiszewski-Knaster) continuum.

Definition 2.1 (Aposyndetic [6]). Let M be a continuum and let x and y be distinct points of M. If M contains a subcontinuum H and an open set U such that $\{x\} \subset U \subset H \subset M \setminus \{y\}$, then M is said to be *aposyndetic at x with respect to y*. If, for every $y \in M \setminus \{x\}$, M is aposyndetic at x with respect to y, then M is said to be *aposyndetic at x*.

Definition 2.2 (Semilocally connected [7]). A continuum M is said to be *semilocally connected at a point p* of M provided that, for every open set U containing p, there exists an open set V containing p such that $V \subset U$ and $M \setminus V$ have a finite number of components.

Definition 2.3 (Connected im kleinen [5]). A topological space X is *connected im kleinen at a point x* if, for every open set $U \subset X$ with $x \in U$, there is an open V with $x \in V \subset U$ such that, for every $y \in V$, there exists $C_y \subset U$ such that C_y is connected and $\{x, y\} \subset C_y$.

3. Background

Definition 3.1 (The set function T [1], [4], [6]). Given a continuum M, define $T : \mathbb{P}(M) \to \mathbb{P}(M)$ such that $M \setminus T(A) := \{y \in M|$ there exists a subcontinuum $W \subset M \setminus A$, and an open set $Q \subset W$ such that $y \in Q \subset W \subset M \setminus A\}$. Thus, for each singleton $p \in M$, the image $T(p) := T(\{p\})$ is the set of all points $y \in M$ such that M is not aposyndetic at y with respect to p.

Theorem 3.2 (Jones [6], Davis [4]). *For any continuum M and for any subset $A \subset M$, the set $T(A)$ is closed in M.*

In light of Theorem 3.2, the set function T on a continuum M can be seen as $T : \mathbb{P}(M) \to 2^M$. This set function T has been very useful in seeing properties of continua.

Theorem 3.3 (Jones [6]). *Given a compact continuum M, M is semilocally connected at a point $p \in M$ if and only if $T(p) = \{p\}$.*

Theorem 3.4 (Davis [3]). *Given a compact continuum M, M is connected im kleinen at a point $p \in M$ if and only if for every closed $A \subset M$, $p \in A$ if and only if $p \in T(A)$.*

The following two definitions and the question were presented by David P. Bellamy in his talk *Some problems on T-closed subsets of continua* at the 49$^{\text{th}}$ Spring Topology and Dynamics Conference 2015.

Definition 3.5 (T-closed sets (Bellamy))**.** A closed subset A of a continuum M is said to be *T-closed* if and only if $T(A) = A$.

From Theorem 3.2 the requirement in the preceding definition that the set A be closed is merely for emphasis, and it is likewise the case in the definition below.

Definition 3.6 (T-equivalence (Bellamy))**.** Let M be a continuum. Define the equivalence relation $\sim$ such that $x \sim y$ implies that for every closed $A \subset M$ with $T(A) = A$, we have that $x \in A$ if and only if $y \in A$. That is, two points are equivalent if and only if the collections of T-closed sets containing them are identical.

With these definitions and this context in mind, the following question was posed.

Question 3.7 (Bellamy)**.** Given a finite connected T_0 space $\hat{X}$, is there a continuum X such that the T-closed equivalence decomposition of X is topologically equal to $\hat{X}$?

It is the goal of the paper to demonstrate by construction the following theorem completely answering this question.

Theorem 3.8. *Given any connected T_0 topology $\mathcal{T}$ on a non-empty finite set $\hat{X}$, there is a continuum X such that the T-closed equivalence decomposition of X is topologically equal to $\hat{X}$.*

4. Building Blocks

Note that, for any indecomposable continuum, the only T-closed set is the entire space itself. For the construction, the buckethandle continuum $\mathcal{K}$ will be used as our basic atomic unit, but it is a fairly arbitrary choice amongst indecomposable continua.

To start, consider a space $X = \mathcal{K}_1 \cup \mathcal{K}_2$ where each $\mathcal{K}_i \cong \mathcal{K}$ and $\mathcal{K}_1 \cap \mathcal{K}_2 = \{x\}$ for some point $x \in X$. Now observe that $T(x) = X$, but for any $y \in \mathcal{K}_i \setminus \{x\}$, the image $T(y) = \mathcal{K}_i \subsetneq X$. Now the set $T(y)$ is not T-closed since $x \in T(y)$ implies that $T(T(y)) = X \neq T(y)$.

Construct a space C formed by an infinite chain of buckethandles $\mathcal{K}_i$. To wit, pick two distinct points $x, y \in \mathcal{K}$ and label $x_i, y_i \in \mathcal{K}_i$ so that they are the points in $\mathcal{K}_i$ corresponding to x and y. Thus, $\mathcal{K}_i \cap \mathcal{K}_j \neq \emptyset$ implies $|i - j| \leq 1$ and $\mathcal{K}_n \cap \mathcal{K}_{n+1} = \{y_n\} = \{x_{n+1}\}$. The idea is essentially illustrated in Figure 1.

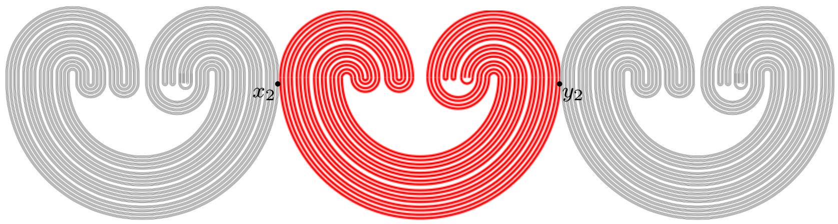

FIGURE 1. A chain of three buckethandles $\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$
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Compactify the space C to $\bar{C}$ by having the chain of continua of C limit down to another buckethandle $\mathcal{K}_\omega$. Observe that $\bar{C}$ has exactly two T-closed sets, namely the remainder $\bar{C} \setminus C$ and the entire space $\bar{C}$. When considering the T-closed decomposition of $\bar{C}$, it has exactly two points $\{a, b\}$ where point a corresponds with C and point b corresponds with $\bar{C} \setminus C$. The quotient topology on $\{a, b\}$ is Sierpinski space. The singleton set $\{a\}$ is open, while the singleton set $\{b\}$ is not. This technique of chaining infinitely many copies of a continuum such that it limits down to another continuum will be the key component of the construction. Constructions analogous to the one that created the set C and the continuum $\bar{C}$ will be utilized in the next sections.

5. Constructing a Continuum Corresponding to a Given Finite Connected T_0 Space

To illustrate the general construction in the following section, consider the finite connected T_0 space with four elements $\{a_1, a_2, a_3, a_4\}$ endowed with the topology $\mathcal{T} = \big\{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_3, a_4\}\big\}$. Let $\{B_i\}_{i=1}^4$ be the connected basis for $\mathcal{T}$ defined as follows: $B_1 := \{a_1\}$, $B_2 := \{a_2\}$, $B_3 := \{a_1, a_2, a_3\}$, and $B_4 := \{a_1, a_2, a_3, a_4\}$.

Again take $\mathcal{K}$ as the buckethandle continuum defined earlier and begin with four distinct copies of $\mathcal{K}$ denoted by $X_1, X_2, X_3,$ and X_4. As in the prior section, fix two distinct points $x, y \in \mathcal{K}$ denoting their images in X_i by x^i and y^i.

The first non-singleton basic open set in $\{B_i\}_{i=1}^4$ is $B_3 = \{a_1, a_2, a_3\}$. Now B_3 can be uniquely written as a disjoint union of basic open sets (occurring before B_3 in the given order) together with the singleton $\{a_3\}$ by $B_3 = B_1 \cup B_2 \cup \{a_3\}$. Denote by $\bar{C}^1$ and $\bar{C}^2$ two copies of the basic building block $\bar{C}$ described in the prior section. For $\bar{C}^1$, identify the initial indecomposable continuum in this first copy of $\bar{C}$ (denoted by $\mathcal{K}_1^1$)

with the buckethandle X_1 and identify $\mathcal{K}^1_\omega$ with X_3. Likewise within $\bar{C}^2$, identify its initial indecomposable continuum $\mathcal{K}^2_1$ with the buckethandle X_2 and identify $\mathcal{K}^2_\omega$ with X_3.

The next and last remaining non-singleton basic open set in $\{B_i\}_{i=1}^4$ is $B_4 = \{a_1, a_2, a_3, a_4\}$. The only way to write B_4 as a disjoint union of other (prior) basic open sets and a singleton is by $B_4 = B_3 \cup \{a_4\}$. Within the basic open set B_3, take the point with the largest index, namely $a_3 \in B_3$. Take yet another copy of the basic building block $\bar{C}$ (denoted $\bar{C}^3$) from the prior section. Identify the initial indecomposable continuum $\mathcal{K}^3_1$ with X_3 and the copy of $\mathcal{K}^3_\omega$ with X_4.

Let X be the continuum comprised of the buckethandles X_1, X_2, X_3, and X_4 together with $\bar{C}^1, \bar{C}^2$, and $\bar{C}^3$ joined as described above. Define the open set $A_1 := X_1 \cup C^1$, the open set $A_2 := X_2 \cup C^2$, the neither-open-nor-closed set $A_3 := X_3 \cup C^3$, and the closed set $A_4 := X_4$. Please note that it is C^k and not $\bar{C}^k$ in the decomposition above; hence, $X = A_1 \cup A_2 \cup A_3 \cup A_4$ is a pairwise disjoint union.

Consider $p \in X_1 \setminus \{y^1\}$. Since $\mathcal{K}$ is indecomposable, $T(p) = X_1$, but $T(X_1) = X_1 \cup \mathcal{K}^1_2$. Likewise, $T(X_1 \cup \mathcal{K}^1_2) = X_1 \cup \mathcal{K}^1_2 \cup \mathcal{K}^1_3$ and so on. Furthermore, $T(\mathcal{K}^1_j) = \mathcal{K}^1_{j-1} \cup \mathcal{K}^1_j \cup \mathcal{K}^1_{j+1}$ for any $j > 1$. In general, for any $p_1 \in A_1$, not only is $\{p_1\}$ not T-closed, but the only T-closed sets containing p_1 will also contain all of A_1. Likewise, for any $p_2 \in A_2$, the only T-closed sets containing p_2 must also contain all of A_2. Also note that neither A_1 nor A_2 can be T-closed as neither is a closed set in X.

Consider $p \in X_3 \setminus \{y^3\}$. Now $T(p) = X_3$, but just as before $T(X_3) \neq X_3$; rather $T(X_3) = X_3 \cup \mathcal{K}^3_2$ and so on. Thus, just as before for any $p_3 \in A_3$, the only T-closed sets containing p_3 must also contain all of A_3. Finally, for any $p \in X_4$, $T(p) = X_4 = T(X_4)$ making X_4 a T-closed set (noting $X_4 \cap X \setminus X_4 = \emptyset$). Likewise, the only T-closed sets containing a point of $A_4 = X_4$ must also contain all of A_4.

From the above, it is clear that for any T-closed set V that whenever $A_i \cap V \neq \emptyset$, it follows that $A_i \subset V$. There are a total of four T-closed sets of X. There are exactly two T-closed sets containing A_1, namely the subcontinuum $A_1 \cup A_3 \cup A_4$ and all of X itself. Likewise, there are exactly two T-closed sets containing A_2, namely $A_2 \cup A_3 \cup A_4$ and X. There are three T-closed sets containing A_3, namely $A_1 \cup A_3 \cup A_4$, $A_2 \cup A_3 \cup A_4$, and X. Finally, all four T-closed sets contain A_4: the three listed above and the previously observed T-closed set, that is, $A_4 = X_4$ itself.

The decomposition by T-closed sets identifies A_i with a_i for $i = 1, 2, 3, 4$. The sets A_1 and A_2 are open, and thus $\{a_1\}$ and $\{a_2\}$ are open. The set A_3 is neither open nor closed, but A_4 is closed, so $X \setminus A_4$ is open, and thus $\{a_1, a_2, a_3\}$ is open. By definition, the whole space X is open, so the whole space $\{a_1, a_2, a_3, a_4\}$ is open. Thus, the T-closed decomposition of X is the four-point space with the given connected T_0 topology $\mathcal{T}$ as desired.

6. General Construction

For the trivial case dealing with the singleton space $\{a_1\}$, take the indecomposable buckethandle continuum $\mathcal{K}$. As explained earlier, $\mathcal{K}$ has only one T-closed set ($\mathcal{K}$ itself), and thus the T-closed decomposition of $\mathcal{K}$ is topologically equal to the singleton space $\{a_1\}$.

Begin the construction for connected T_0 topologies with a finite number $n > 1$ points using n distinct copies of the buckethandle $\mathcal{K}$ denoted by X_i. In other words, for each $1 \leq i \leq n$, let $X_i \cong \mathcal{K}$ where the only $1 \leq j \leq n$ with $X_i \cap X_j \neq \emptyset$ is precisely when $j = i$.

Given a finite set $T := \{a_1, a_2, \dots, a_n\}$ with $n > 1$, let $\mathcal{T}$ be any connected T_0 topology for T. Since T is finite and $\mathcal{T}$ a topology, the intersection

$$B_i := \bigcap_{a_i \in U \in \mathcal{T}} U$$

is open. Moreover, the collection $\{B_i\}_{i=1}^n$ forms a connected basis for the connected T_0 topology $\mathcal{T}$. Without loss of generality, assume that the points $\{a_i\}_{i=1}^n$ are indexed so that this connected basis $\{B_i\}_{i=1}^n$ satisfies $|B_j| \leq |B_k|$ whenever $1 \leq j \leq k \leq n$.

Since $\mathcal{T}$ is a T_0 topology, it separates points, and thus $\{a_1\} = B_1$ is open. However, since $\mathcal{T}$ is a connected topology, it cannot be discrete and there is at least one index i such that B_i is not a singleton. Let $2 \leq m \leq n$ be the first such index. Begin the construction below sequentially starting with $k = m$ and proceeding through $k = n$.

Now since $\mathcal{T}$ is a T_0 topology and from the way the basic open sets were determined, this basic open set B_k can be uniquely decomposed as the disjoint union of one or more of the previous basic open sets and the singleton $\{a_k\}$. Let $\{j_m\} \subset [1, k-1] \cap \mathbb{N}$ be the indices of the basic open sets comprising that specific decomposition of B_k. For each corresponding B_{j_m}, there is an $a_i \in B_{j_m}$ such that for all $j > i$, $a_j \notin B_{j_m}$. Join the continua X_i and X_k by a copy of the basic building block $\bar{C}$, used in the prior two sections, with $\mathcal{K}_1$ identified with X_i and $\mathcal{K}_\omega$ identified with X_k.

Do this for each B_{j_m} in the decomposition of B_k and then continue in this fashion through the remaining basic open sets to the basic open set B_n. The resulting continuum comprised of $\{X_i\}_{i=1}^n$ and all of the homeomorphic copies of $\bar{C}$ added to them in this process form the desired continuum X.

Claim 1. *The continuum X constructed above is such that $\hat{X}$, the T-closed decomposition of X, is topologically equal to the set T endowed with the topology $\mathcal{T}$.*

Proof. The claim will be proven in two steps. The first step is to show that the sets $\hat{X} = T$, which itself is done in two small parts.

For each $1 \le i \le n$, define the subset $A_i \subset X$ to be the set comprised of X_i and all (if there are any) of the homeomorphic copies of C (note not $\bar{C}$ but rather just C) that were attached to it. Thus, X is the pairwise disjoint union of the subsets in the collection $\{A_i\}_{i=1}^n$.

Suppose $V \subset X$ is a T-closed set such that $A_j \cap V \neq \emptyset$ for some $1 \le j \le n$. Let $p \in A_j \cap V$, then p is a point in at least one copy of $\mathcal{K}$ (denote it by $\mathcal{K}^*$) that lies within A_j, and thus $\mathcal{K}^* \subset T(p) \subset V$. Let $\mathcal{K}^\dagger$ be an arbitrary copy of $\mathcal{K}$ in A_j. By the construction of A_j, there is a finite chain of copies of $\mathcal{K}$ linking $\mathcal{K}^*$ to $\mathcal{K}^\dagger$ (possibly from one attached copy of C to another going through X_j). Since V is T-closed, if n is the number of links in that chain, then $\mathcal{K}^\dagger \subset T^n(p) \subset V$. Thus, it follows that $A_j \cap V \neq \emptyset$ implies $A_j \subset V$ which means that T-closed sets do not distinguish between pairs of points lying within a given A_j.

To see that T-closed sets distinguish between each of the A_i's, first observe that in constructing the space X that whenever a chain $\bar{C}$ was added joining say X_i to X_j, it was the order of the indices that determined how the attachment was made. Without loss of generality, suppose $i < j$, then, in adding the chain $\bar{C}$ between them, the initial link $\mathcal{K}_1$ was identified with X_i while the limiting link $\mathcal{K}_\omega := \bar{C} \setminus C$ was identified with X_j. Thus, for a given $A_k \subset X$, it holds that $\bar{A}_k \cap A_l = \emptyset$ whenever $l < k$.

For any $A_i \neq A_j$, there is a T-closed set containing one of them that does not intersect (and hence contain) the other. Without loss of generality, assume that $i < j$. Define the set

$$Y_j := \bigcup_{k \ge j} A_k,$$

and thus $X \setminus Y_j = \bigcup_{l<j} A_l$. For each $k \ge j$, the closure $\bar{A}_k$ is a subset of Y_j, which makes Y_j a closed set. Now since each A_l is connected, the set $X \setminus Y_j$ has, perforce, at most a finite number of components, all of which are open. For any $l < j$ and any point $p \in A_l \subset X \setminus Y_j$, $T(p)$ is exactly the copies of $\mathcal{K}$ in A_l containing p. If $T(p) \cap X_l = \emptyset$, then define a subcontinuum $W \subset X \setminus Y_j$ by $W := T(T(p))$, which is $T(p)$ together with the copies of $\mathcal{K}$ in A_l intersecting $T(p)$. If, on the other hand, $T(p) \cap X_l \neq \emptyset$, then first let Z be the finite (possibly empty) union of all the copies of the chain $\bar{C}$ that were attached between X_l and any X_m with $m < l$. In this case, define the subcontinuum $W \subset X \setminus Y_j$ as the finite union of subcontinua, $W := T(T(p)) \cup Z$, joined together at X_l.

In either event, W is a subcontinuum of $X \setminus Y$ containing the point p. Pick an open set Q such that $T(p) \subseteq Q \subseteq W \subset X \setminus Y_j$. Thus, by the definition of T, we have $p \notin T(Y_j)$, and since p was an arbitrary element of $X \setminus Y_j$, Y_j is T-closed. Hence, Y_j is a T-closed set containing A_j but not containing A_i.

To recap, every point of a given A_i belongs to the same T-closed subsets as A_i itself does, and that for any two distinct A_i and A_j that there is a T-closed subset containing one but not the other. For each $1 \leq i \leq n$, the set A_i decomposes (in the sense of T-closed sets) exactly to the point $a_i \in T$. Thus, the sets $\hat{X} = T$.

All that remains, as the second step of proving this claim, is to show that $\hat{X}$, the T-closed decomposition of X, has the given topology $\mathcal{T}$.

Define the set valued map $f : T \to \{A_i\}_{i=1}^n$ by $f(a_i) = A_i$ and, for each $1 \leq i \leq n$, let $\mathcal{B}_i$ be the union of the subsets of X in the f-image of the previously chosen connected basic open set B_i. By finite induction, each of the sets $\mathcal{B}_i$ will be seen to be open. Since $B_1 = \{a_1\}$, the subset $\mathcal{B}_1 = A_1$. Let $Y_1 := X \setminus A_1 = \bigcup_{i=2}^n A_i$. As observed in the first step of this proof, the set Y_1 has all its limits points and thus is closed. Thus, its compliment A_1 is an open set. For any $1 \leq k \leq n$, the basis element B_{k+1} can be written as the disjoint union of the singleton set $\{a_{k+1}\}$ and basic open sets B_{j_m} for some finite subsequence $\{j_m\}$ where each $j_m \leq k$. Let U be an open neighborhood of A_{k+1} that is sufficiently "small" in the sense that for any $1 \leq l \leq n$ if $A_l \cap U \neq \emptyset$, then $\bar{A}_l \cap A_{k+1} \neq \emptyset$. Recall that $Y_{k+1} := \bigcup_{i=k+2}^n A_i$ has all of its limit points; hence, it is closed. Then the set $\mathcal{B}_{k+1}$ is open, since it can be written as

$$\big(U \cap (X \setminus Y_{k+1})\big) \cup \bigcup_{i \in \{j_m\}} \mathcal{B}_i$$

which is the finite union of open sets. Thus, by finite induction, for $1 \leq i \leq n$ the set $\mathcal{B}_i$ is open. Since $\{B_i\}_{i=1}^n$ was a basis for the topology $\mathcal{T}$ on the set T, the T-closed decomposition of X is endowed with an equivalent topology. $\square$

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