http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 50, 2017

Pages 97–100

http://topology.nipissingu.ca/tp/

Atomic Maps and $\mathcal{T}$-closed Sets

by

Sergio Macías

Electronically published on August 26, 2016

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 50 (2017)
Pages 97-100

E-Published on August 26, 2016

ATOMIC MAPS AND $\mathcal{T}$-CLOSED SETS

SERGIO MACÍAS

In Memoriam Ms. Isabel Caravaca

Abstract. We give a partial answer to a question by David P. Bellamy, Leobardo Fernández, and Sergio Macías by showing that if $f\colon X \twoheadrightarrow Y$ is an atomic map between continua, then the cardinality of the $\mathcal{T}$-closed sets of X is equal to the cardinality of the $\mathcal{T}$-closed sets of Y. We present an example showing that the converse implication is not true.

1. Introduction

$\mathcal{T}$-closed sets have been considered by several authors (see for example [2] and [8]), the first study of the properties of this type of sets is in [1]. We present a partial answer to [1, Question 3.18] by showing that if $f\colon X \twoheadrightarrow Y$ is an atomic map between continua, then the cardinality of the $\mathcal{T}$-closed sets of X is equal to the cardinality of the $\mathcal{T}$-closed sets of Y (Theorem 3.6). We present an example showing that the converse implication is not true (Example 3.7). We also extend [1, Theorem 4.12] to atomic maps between continua (Corollary 3.8).

2. Definitions

If Z is a metric space, then given a subset A of Z, the interior of A is denoted by $Int_Z(A)$ and the boundary of A is denoted by $Bd_Z(A)$. A *map* is a continuous function.

A *continuum* is a compact, connected, metric space. A continuum is *decomposable* if it is the union of two of its proper subcontinua. A

2010 *Mathematics Subject Classification.* Primary 54B20; 54C60.

Key words and phrases. atomic map, continuum, monotone map, set function $\mathcal{T}$, $\mathcal{T}$-closed set.

©2016 Topology Proceedings.

continuum is *indecomposable* if it is not decomposable. A subcontinuum K of a continuum X is *terminal* if, for each subcontinuum L of X such that $L \cap K \neq \emptyset$, we have that either $L \subset K$ or $K \subset L$. A continuum X is *aposyndetic* provided that, for each pair of points x_1 and x_2 of X, there exists a subcontinuum W of X such that $x_1 \in Int_X(W) \subset W \subset X \setminus \{x_2\}$.

Given a continuum X, we define the set function $\mathcal{T}$ as follows: if A is a subset of X, then

$$\mathcal{T}(A) = X \setminus \{x \in X \mid \text{there exists a subcontinuum } W \text{ of } X \text{ such that } x \in Int_X(W) \subset W \subset X \setminus A\}.$$

We write $\mathcal{T}_X$ if there is a possibility of confusion. Let us observe that, for any subset A of X, $\mathcal{T}(A)$ is a closed subset of X and $A \subset \mathcal{T}(A)$.

Given a continuum X, a subset A of X is a *$\mathcal{T}$-closed set* provided that $\mathcal{T}(A) = A$. We denote the family of $\mathcal{T}$-closed sets of a continuum X by $\mathfrak{T}(X)$. Note that $X \in \mathfrak{T}(X)$. For properties of $\mathcal{T}$-closed sets see [1].

A surjective map $f\colon Z \twoheadrightarrow Y$ between continua is *monotone* provided that $f^{-1}(C)$ is connected for every connected subset C of Y. The surjective map f is *atomic* if for each subcontinuum K of Z such that $f(K)$ is nondegenerate, then $K = f^{-1}(f(C))$.

3. Atomic Maps

We present a partial answer to [1, Question 3.18] by showing that if $f\colon X \twoheadrightarrow Y$ is an atomic map between continua, then $|\mathfrak{T}(X)| = |\mathfrak{T}(Y)|$ (Theorem 3.6). We present an example of a monotone map between decomposable continua X and Y, that is not atomic, such that $|\mathfrak{T}(X)| = |\mathfrak{T}(Y)|$ (Example 3.7). We also extend [1, Theorem 4.12] to atomic maps between continua (Corollary 3.8).

Lemma 3.1. *Let X be a continuum. If W is a proper terminal subcontinuum of X, then $Int_X(W) = \emptyset$.*

Proof. Suppose W is a proper terminal subcontinuum of X and $Int_X(W) \neq \emptyset$. Note that $Bd_X(Int_X(W)) \subset W$. Let $x \in X \setminus W$ and let C be the component of $X \setminus Int_X(W)$ containing x. By [7, Theorem 5.4], $C \cap Bd_X(Int_X(W)) \neq \emptyset$. Hence, $C \cap W \neq \emptyset$ and $C \setminus W \neq \emptyset$. Since W is a terminal subcontinuum of X, $W \subset C$, a contradiction to the fact that $C \cap Int_X(W) = \emptyset$. Therefore, $Int_X(W) = \emptyset$. □

Lemma 3.2. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be an atomic map. Then for every $x \in X$, $f^{-1}(f(x)) \subset \mathcal{T}_X(\{x\})$.*

Proof. Let $x \in X$ and let $z \in X \setminus \mathcal{T}_X(\{x\})$. Then there exists a subcontinuum W of X such that $z \in Int_X(W) \subset W \subset X \setminus \{x\}$. Since $f^{-1}(f(x))$ is a terminal subcontinuum of X [6, (1.2)], by Lemma 3.1, we

have that $f^{-1}(f(x))\cap W = \emptyset$. In particular, $z \in X\setminus f^{-1}(f(x))$. Therefore, $f^{-1}(f(x)) \subset \mathcal{T}_X(\{x\})$. □

Since atomic maps are monotone [5, (3.7)], we have the following as a consequence of Lemma 3.2 and [4, Lemma 3.2].

Corollary 3.3. *Let X and Y be continua, where Y is aposyndetic, and let $f\colon X \twoheadrightarrow Y$ be an atomic map. Then, for every $x \in X$, $\mathcal{T}_X(\{x\}) = f^{-1}(f(x))$.*

Lemma 3.4. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be an atomic map. If $A \in \mathfrak{T}(X)$, then $A = f^{-1}(f(A))$.*

Proof. Let $A \in \mathfrak{T}(X)$. We know that $A \subset f^{-1}(f(A))$. Let $x \in f^{-1}(f(A))$. There exists $a \in A$ such that $f(a) = f(x)$. Thus, by Lemma 3.2, $x \in f^{-1}(f(a)) \subset \mathcal{T}_X(\{a\}) \subset \mathcal{T}(A) = A$. Therefore, $A = f^{-1}(f(A))$. □

Theorem 3.5. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be an atomic map. If $A \in \mathfrak{T}(X)$, then $f(A) \in \mathfrak{T}(Y)$.*

Proof. Let $A \in \mathfrak{T}(X)$. By [3, Theorem 3.1.64(a)], we have that $\mathcal{T}_Y(f(A)) \subset f\mathcal{T}_X f^{-1}(f(A))$. By Lemma 3.4, $A = f^{-1}(f(A))$. Thus, we obtain that $\mathcal{T}_Y(f(A)) \subset f\mathcal{T}_X(A) = f(A)$. Hence, $\mathcal{T}_Y(f(A)) = f(A)$. Therefore, $f(A) \in \mathfrak{T}(Y)$. □

The following theorem gives a partial answer to [1, Question 3.18].

Theorem 3.6. *Let X and Y be continua and let $f\colon X \twoheadrightarrow Y$ be an atomic map. Then $|\mathfrak{T}(X)| = |\mathfrak{T}(Y)|$.*

Proof. Note that atomic maps are monotone [5, (3.7)]. Also observe that by Theorem 3.5, we have that $|\mathfrak{T}(X)| \leq |\mathfrak{T}(Y)|$. By [1, Corollary 3.17], we know that $|\mathfrak{T}(X)| \geq |\mathfrak{T}(Y)|$. Therefore, $|\mathfrak{T}(X)| = |\mathfrak{T}(Y)|$. □

The following example shows that there exist monotone maps, that are not atomic, between decomposable continua with the same cardinality of $\mathcal{T}$-closed sets.

Example 3.7. Let Z be the Knaster indecomposable continuum [3, Example 2.4.7], let X be the union of three copies of Z glued by their endpoints, and let Y be two copies of Z also glued by their endpoints. Let v be the common point of the two copies of Z in Y. Let $f\colon X \twoheadrightarrow Y$ be the map that sends two of the copies of Z in X homeomorphically to the two copies of Z in Y and sends the third copy of Z in X to $\{v\}$. Then f is a monotone map, f is not an atomic map, and $|\mathfrak{T}(X)| = |\mathfrak{T}(Y)| = 1$.

The next corollary extends [1, Theorem 4.12] to atomic maps between continua.

Corollary 3.8. *Let X and Y be continua, let $f\colon X \twoheadrightarrow Y$ be an atomic map, and let A be a nonempty closed subset of X. Then $A \in \mathfrak{T}(X)$ if and only if there exists a nonempty closed subset B of Y such that $B \in \mathfrak{T}(Y)$ and $A = f^{-1}(B)$.*

Proof. Assume that $A \in \mathfrak{T}(X)$. By Lemma 3.4, $A = f^{-1}(f(A))$, and by Theorem 3.5, $f(A) \in \mathfrak{T}(Y)$. Hence, if $B = f(A)$, we are done.

Suppose that there exists $B \in \mathfrak{T}(Y)$ and $A = f^{-1}(B)$. Since atomic maps are monotone [5, (3.7)], by [1, Theorem 3.16], $A = f^{-1}(B) \in \mathfrak{T}(X)$. □

References

[1] David P. Bellamy, Leobardo Fernández, and Sergio Macías, *On $\mathcal{T}$-closed sets*, Topology Appl. **195** (2015), 209–225.

[2] R. W. FitzGerald and P. M. Swingle, *Core decomposition of continua*, Fund. Math. **61** (1967), 33–50.

[3] Sergio Macías, *Topics on Continua*. Pure and Applied Mathematics Series, Vol. 275. Boca Raton, FL: Chapman & Hall/CRC, 2005.

[4] ———, *On continuously type A' θ-continua*, JP J. Geom. Topol. **18** (2015), no. 1, 1–14.

[5] T. Maćkowiak, *Continuous mappings on continua*, Dissertationes Math. (Rozprawy Mat.) **158**, (1979).

[6] ———, *Singular arc-like continua*, Dissertationes Math. (Rozprawy Mat.) **257**, (1986).

[7] Sam B. Nadler, Jr., *Continuum Theory. An Introduction*. Monographs and Textbooks in Pure and Applied Mathematics, Vol. 158. New York: Marcel Dekker, Inc., 1992.

[8] Eldon J. Vought, *Monotone decompositions into trees of Hausdorff continua irreducible about a finite subset*, Pacific J. Math. **54** (1974), no. 2, 253–261.

Instituto de Matemáticas; Universidad Nacional Autónoma de México; Circuito Exterior, Ciudad Universitaria; México, D.F., C. P. 04510, México

E-mail address: sergiom@matem.unam.mx