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KAKIMIZU COMPLEXES OF SURFACES AND 3-MANIFOLDS

JENNIFER SCHULTENS

Abstract. The Kakimizu complex is usually defined in the context of knots, where it is known to be quasi-Euclidean. We here generalize the definition of the Kakimizu complex to surfaces and 3-manifolds (with or without boundary). Interestingly, in the setting of surfaces, the complexes and the techniques turn out to replicate those used to study the Torelli group, i.e., the "nonlinear" subgroup of the mapping class group. Our main results are that the Kakimizu complexes of a surface are contractible and that they need not be quasi-Euclidean. It follows that there exist (product) 3-manifolds whose Kakimizu complexes are not quasi-Euclidean.

The existence of Seifert's algorithm, discovered by Herbert Seifert, proves, among other things, that every knot admits a Seifert surface; i.e., for every knot K, there is a compact orientable surface whose boundary is K. It is worth noting that the existence of a Seifert surface for a knot K also follows from the existence of submanifolds representing homology classes of manifolds or pairs of submanifolds, in this case the pair $(K, \mathbb{S}^3)$. This point of view proves useful in generalizing our understanding of Seifert surfaces to other classes of surfaces in 3-manifolds.

Adding a trivial handle to a Seifert surface produces an isotopically distinct surface. Adding additional handles produces infinitely many isotopically distinct surfaces. These are not the multitudes of surfaces of primary interest here. The multitudes of surfaces of primary interest here are, for example, the infinite collection of Seifert surfaces produced by Julian R. Eisner [4]. Eisner realized that "spinning" a Seifert surface around the decomposing annulus of a connected sum of two non-fibered

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knots produces homeomorphic but non-isotopic Seifert surfaces. This abundance of Seifert surfaces led Osamu Kakimizu [16] to define a complex, now named after him, whose vertices are isotopy classes of Seifert surfaces of a given knot and whose n-simplices are $(n+1)$-tuples of vertices that admit pairwise disjoint representatives.

Our understanding of the topology and geometry of the Kakimizu complex continues to evolve. Both Kakimizu's work [16] and, independently, a result of Martin Scharlemann and Abigail Thompson [24] imply that the Kakimizu complex is connected. Makoto Sakuma and Kenneth J. Shackleton [23] exhibit diameter bounds in terms of the genus of a knot. Piotr Przytycki and the author [21] establish that the Kakimizu complex is contractible. Finally, Jesse Johnson, Roberto Pelayo, and Robin Wilson [15] prove that the Kakimizu complex of a knot is quasi-Euclidean.

This paper grew out of a desire to study concrete examples of Kakimizu complexes of 3-manifolds other than knot complements. A natural case to consider is product manifolds, where relevant information is captured by the surface factor. The challenge lies in adapting the idea of the Kakimizu complex to a more general setting: codimension 1 submanifolds of n-manifolds.

As it turns out, in the case of 1-dimensional submanifolds of a surface, the Kakimizu complexes are related to the homology curve complexes investigated by Allen Hatcher [9]; Ingrid Irmer [12]; Mladen Bestvina, Kai-Uwe Bux, and Dan Margalit [2]; and Hatcher and Margalit [10] discussed in §2. These complexes are of interest in the study of the Torelli group, which is the kernel of the action of the mapping class group of a manifold on the homology of the manifold. The Torelli group of a surface, in turn, acts on the homology curve complexes. This group action has been used to study the topology of the Torelli group of a surface, for instance by Bestvina, Bux, and Margalit in their investigation of the dimension of the Torelli group [2], by Irmer [13], by Hatcher and Margalit [10], and by Andrew Putman in [22]

Hatcher [9] proved that the homology curve complex is contractible and computed its dimension. Irmer studied geodesics of the homology curve complex and exhibited quasi-flats. These insights guide our investigation of the Kakimizu complex of a surface. Specifically, we prove similar, and in some cases analogous, results in the setting of the Kakimizu complex of a surface. Our main results are that the Kakimizu complexes of a surface are contractible and that they need not be quasi-Euclidean.

One example stands out: the Kakimizu complex of a genus 2 surface. In [2], Bestvina, Bux, and Margalit reprove a theorem of Geoffrey Mess that the Torelli group of a genus 2 surface is an infinitely generated free group. They do so by showing that it acts on a tree with infinitely many edges emerging from each vertex. As it turns out, the Kakimizu complex of the genus 2 surface is also a tree with infinitely many edges emerging from each vertex. In particular, the Kakimizu complex of the genus 2 surface is Gromov hyperbolic. A product manifold with the genus 2 surface as a factor will thus also have some Gromov hyperbolic Kakimizu complexes. This is interesting as it shows that in addition to examples of 3-manifolds with quasi-Euclidean Kakimizu complexes, as proved by Johnson, Pelayo, and Wilson [15], there are 3-manifolds with Gromov hyperbolic Kakimizu complexes. Kakimizu complexes exhibit more than one geometry!

1. The Kakimizu Complex of a Surface

The work here follows in the footsteps of [21]. Whereas the setting for [21] is surfaces in 3-manifolds, the setting here is 1-manifolds in 2-manifolds. It is worth pointing out that although we discuss only 1-manifolds in 2-manifolds and 2-manifolds in 3-manifolds, the definitions and arguments carry over verbatim to the setting of codimension 1 submanifolds in manifolds of any dimension.

Recall that an element of a finitely generated free abelian group G is *primitive* if it is an element of a basis for G. In the following we will always assume (1) S is a compact (possibly closed) connected oriented 2-manifold and (2) α is a primitive element of $H_1(S, \partial S, \mathbf{Z})$.

Definition 1.1. A *Seifert curve* (see Figure 1) for (S, α) is a pair (w, c), where c is a union, $c_1 \sqcup \cdots \sqcup c_n$, of pairwise disjoint oriented simple closed curves and arcs in S, and w is an n-tuple of natural numbers $(w^1, \ldots, w^n)$ such that the homology class $w^1[[c_1]] + \cdots + w^n[[c_n]]$ equals α. Moreover, we require that $S \backslash c$ is connected. We call c the underlying curve of (w, c). We will denote $w^1[[c_1]] + \cdots + w^n[[c_n]]$ by $w \circ c$.

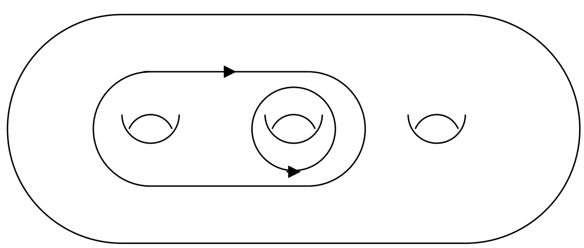

Figure 1. A Seifert curve (weights are 1)

Our definition of a Seifert curve disallows null homologous subsets. Indeed, a null homologous subset would bound a component of $S \backslash c$ and would hence be separating. In fact, c contains no bounding subsets. Conversely, if $w \circ d = \alpha$ and d contains no bounding subsets, then $S \backslash d$ is connected.

Lemma 1.2. *If (w, c) represents α, then w is determined by the underlying curve c.*

Proof. Suppose that (w, c) and (w', c) represent α where $w = (w^1, \cdots, w^n)$ and $w' = ((w')^1, \cdots, (w')^n)$. Then

$$w^1[[c_1]] + \cdots + w^n[[c_n]] = \alpha = (w')^1[[c_1]] + \cdots + (w')^n[[c_n]];$$

hence,

$$(w^1 - (w')^1)[[c_1]] + \cdots (w^n - (w')^n)[[c_n]] = 0.$$

Since c has no null homologous subsets, this ensures that

$$w^1 - (w')^1 = 0, \ldots, w^n - (w')^n = 0.$$

Thus,

$$w^1 = (w')^1, \cdots, w^n = (w')^n. \qquad \square$$

Since the underlying curve c of a Seifert curve (w, c) determines w, we will occasionally speak of a Seifert curve c, when w does not feature in our discussion.

Definition 1.3. Given a Seifert curve (w, c), we denote the curve obtained by replacing, for all i, the curve c_i with w_i parallel components of c_i, with $h(w, c)$. This defines a function from Seifert curves to unweighted curves.

Conversely, let $d = d_1 \sqcup \cdots \sqcup d_m$ be a disjoint union of (unweighted) pairwise disjoint simple closed curves and arcs such that parallel components are oriented to be parallel oriented curves and arcs. We denote the weighted curve obtained by replacing parallel components with one weighted component whose weight is equivalent to the number of these parallel components with $h^{-1}(d)$.

Definition 1.4. For each pair (S, α), the isomorphism between $H_1(S, \partial S)$ and $H^1(S)$ identifies an element a^* of $H^1(S)$ corresponding to α that lifts to a homomorphism $h_a : \pi_1(S) \to \mathbf{Z}$. We denote the covering space corresponding to $N_\alpha = kernel(h_a)$ by $(p_\alpha, \hat{S}_\alpha, S)$, or simply $(p, \hat{S}, S)$, and call it the *infinite cyclic covering space associated with α.*

We now describe the Kakimizu complex of (S, α). As *vertices* we take Seifert curves (w, c) of (S, α), considered up to isotopy of underlying curves. We write $[(w, c)]$. Consider a pair of vertices v and v' and representatives (w, c) and (w', c'). Here $S \backslash c$ and $S \backslash c'$ are connected, hence path-connected. It follows that lifts of $S \backslash c$ and $S \backslash c'$ to the covering space associated with α are simply path components of $p^{-1}(S\backslash c)$ and $p^{-1}(S\backslash c')$. We obtain a graph $\Gamma(S,\alpha)$ by spanning an *edge* $e = (v, v')$ on the vertices v and v' if and only if the representatives (w, c) and (w', c') of v and v' can be chosen so that a lift of $S\backslash c$ to the covering space associated with α intersects exactly two lifts of $S\backslash c'$. (Note that in this case, c and c' are necessarily disjoint.) See Figure 2.

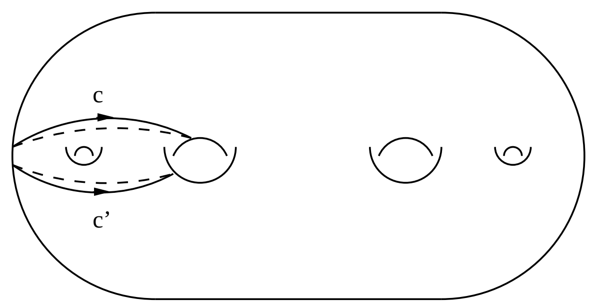

FIGURE 2. Two Seifert curves corresponding to vertices of distance 1 (weights are 1)

Definition 1.5. Let X be a simplicial complex. If, whenever the 1-skeleton of a simplex σ is in X, the simplex σ is also in X, then X is said to be *flag*.

Definition 1.6. The *Kakimizu complex of* (S, α), denoted by $Kak(S, \alpha)$, is the flag complex with $\Gamma(S, \alpha)$ as its 1-skeleton.

Remark 1.7. The Kakimizu complex is defined for a pair (S, α). For simplicity we use the expression "the Kakimizu complex of a surface" in general discussions, rather than the more cumbersome "the Kakimizu complex of a pair (S, α), where S is a surface and α is a primitive element of $H_1(S, \partial S, \mathbf{Z})$." Note that the Kakimizu complex of a surface is thus unique only in conjunction with a specified α.

Figure 3 provides an example of a pair (w, c) and (w', c') of disjoint (disconnected) Seifert curves that do not span an edge. The arc from one side of c to the other side of c intersects c' twice with the same orientation and a lift of $S\backslash c$ will hence meet at least three distinct lifts of $S\backslash c'$. For a 3-dimensional analogue of Figure 3, see [1].

Example 1.8. The Kakimizu complexes of the disk and sphere are empty. The annulus has a non-empty but trivial Kakimizu complex $Kak(A, \alpha)$ consisting of a single vertex. Specifically, let $A = annulus$ and let α be a generator of $H_1(A, \partial A, \mathbb{Z}) = \mathbb{Z}$. Then α is represented by a spanning

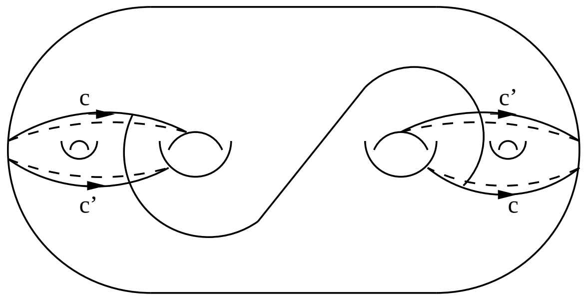

FIGURE 3. Two Seifert curves corresponding to vertices of distance strictly greater than 1 (weights are 1)

arc with weight 1. The spanning arc is, up to isotopy, the only possible underlying curve for a representative of α. Thus, $Kak(A, \alpha)$ consists of a single vertex.

Similarly, the torus has non-empty but trivial Kakimizu complexes, each consisting of a single vertex. Specifically, let $T = torus$ and let β be a primitive element of $H_1(T, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$. Again, there is, up to isotopy, only one underlying curve for representatives of β. There are infinitely many choices for β, but in each case, $Kak(T, \beta)$ consists of a single vertex.

Having understood the above examples, we restrict our attention to the case where S is a compact orientable hyperbolic surface with geodesic boundary for the remainder of this paper.

Definition 1.9. Let (w, c) and (w', c') be Seifert curves. We say that (w, c) and (w', c') (or simply c and c') are *almost disjoint* if, for all i and j, the component c_i of c and the component c'_j of c' are either disjoint or coincide.

Remark 1.10. Let σ be a simplex in $Kak(S, \alpha)$ of dimension n. Denote the vertices of σ by $v_0, \ldots, v_n$ and let $c_0, \ldots, c_n$ be geodesic representatives of the underlying curves of Seifert curves for $v_0, \ldots, v_n$ such that arc components of $c_0, \ldots, c_n$ are perpendicular to ∂S. It is a well-known fact that closed geodesics that can be isotoped to be disjoint must be disjoint or coincide. The same is true for the geodesic arcs considered here and for combinations of closed geodesics and geodesic arcs, because their doubles are closed geodesics in the double of S. Hence, for all pairs i and j, the component c_i of c and the component c'_j of c' are either disjoint or coincide.

Definition 1.11. Consider $Kak(S, \alpha)$. Let $(p, \hat{S}, S)$ be the infinite cyclic cover of S associated with α. Let τ be the generator of the group of covering transformations of $(p, \hat{S}, S)$ (which is $\mathbb{Z}$) corresponding to 1. Note that τ is canonical up to sign.

Let (w, c) and (w', c') be Seifert curves in (S, α). Let S_0 denote a lift of $S \backslash c$ to $\hat{S}$, i.e., a path component of $p^{-1}(S \backslash c)$. Set $S_i = \tau^i(S_0)$ and $c_i = closure(S_i) \cap closure(S_{i+1})$. Let S_0' be a lift of $S \backslash c'$ to $\hat{S}$. Set $d_K(c, c) = 0$ and for $c \neq c'$, set $d_K(c, c')$ equal to one less than the number of translates of S_0 met by S_0'. Let v and v' be vertices in $Kak(S, \alpha)$. Set $d_k(v, v) = 0$ and for $v \neq v'$, set $d_K(v, v')$ equal to the minimum of $d_K(c, c')$ for (w, c) and (w', c') representatives of v and v', respectively.

Definition 1.12. Let C and D be disjoint separating subsets of $\hat{S}$. We say that D lies *above* C if D lies in the component of $\hat{S} \backslash C$ containing $\tau(C)$. We say that D lies *below* C if D lies in the component of $\hat{S} \backslash C$ containing $\tau^{-1}(C)$.

Remark 1.13. Here $d_K(c, c')$ is finite: Indeed, $w \circ c = w' \circ c' = \alpha$ and so $[(w, c)]$ and $[(w', c')]$ are in the kernel, N_α, of the cohomology class dual to α. Specifically, the cohomology class dual to α is represented by the weighted intersection pairing with (w, c) and also the weighted intersection pairing with (w', c'). Thus, let c^j be a component of c; then the value of the cohomology class dual to α evaluated at $[[c^j]]$ is given by the weighted intersection pairing of (w, c) with c^j which is 0, likewise for other components of c and c'. Thus, each component of c and c' lies in the kernel of this homomorphism and hence in N_α. Thus, lifts of c and c' are homeomorphic to c and c', respectively, and in particular, they are compact 1-manifolds. It follows that $d_K(c, c')$ is finite, whence for all vertices v and v' of $Kak(S, \alpha)$, $d_K(v, v')$ is also finite.

It is not hard to verify, but important to note, the following proposition (see [16, Proposition 1.4]).

Proposition 1.14. *The function d_K is a metric on the vertex set of $Kak(S, \alpha)$.*

2. Relation to Homology Curve Complexes

In [9], Hatcher introduces the *cycle complex* of a surface:

> By a *cycle* in a closed oriented surface S we mean a nonempty collection of finitely many disjoint oriented smooth simple closed curves. A cycle c is *reduced* if no subcycle of c is the oriented boundary of one of the complementary regions of c in S (using either orientation of the region). In particular, a reduced cycle contains no curves that bound disks in S, and no pairs of circles that are parallel but oppositely oriented.

Define the *cycle complex* $C(S)$ to be the simplicial complex having as its vertices the isotopy classes of reduced cycles in S, where a set of $k+1$ distinct vertices spans a k-simplex if these vertices are represented by disjoint cycles $c_0, \ldots, c_k$ that cut S into $k+1$ cobordisms $C_0, \ldots, C_k$ such that the oriented boundary of C_i is $c_{i+1} - c_i$, subscripts being taken modulo $k+1$, where the orientation of C_i is induced from the given orientation of S and $-c_i$ denotes c_i with the opposite orientation. The cobordisms C_i need not be connected. The faces of a k-simplex are obtained by deleting a cycle and combining the two adjacent cobordisms into a single cobordism. One can think of a k-simplex of $C(S)$ as a cycle of cycles. The ordering of the cycles $c_0, \ldots, c_k$ in a k-simplex is determined up to cyclic permutation.

Cycles that span a simplex represent the same element of $H_1(S)$ since they are cobordant. Thus we have a well-defined map $\pi_0 : C(S) \to H_1(S)$. This has image the nonzero elements of $H_1(S)$ since on the one hand, every cycle representing a nonzero homology class contains a reduced subcycle representing the same class (subcycles of the type excluded by the definition of reduced can be discarded one by one until a reduced subcycle remains), and on the other hand, it is an elementary fact, left as an exercise, that a cycle that represents zero in $H_1(S)$ is not reduced.

For a nonzero class $x \in H_1(S)$ let $C_x(S)$ be the subcomplex of $C(S)$ spanned by vertices representing x, so $C_x(S)$ is a union of components of $C(S)$. [9, p. 1].

Lemma 2.1. *When both are defined, i.e., when S is closed, connected, of genus at least 2, and α is primitive, $Vert(Kak(S, \alpha))$ is isomorphic to a proper subset of $Vert(C_\alpha(S))$.*

Proof. Let v be a vertex of $Kak(S, \alpha)$. If we choose a representative (w, c), then $h(w, c)$ is a disjoint collection of (unweighted) curves and arcs. The requirement on the Seifert curve (w, c) that $S \backslash c$ be connected implies that the multi-curve $h(w, c)$ is reduced and thus represents a vertex of $C_\alpha(S)$. Abusing notation slightly, we denote the map from $Vert(Kak(S, \alpha))$ to $Vert(C_\alpha)$ thus obtained by h. There is an inverse, h^{-1}, defined on the image of h; hence, h is injective.

It is not hard to identify reduced multi-curves that contain bounding subsets that are not the oriented boundary of a subsurface. Hence, $Vert(Kak(S,\alpha))$ is a proper subset of $Vert(C_\alpha(S))$. □

Lemma 2.2. *Suppose that S is hyperbolic and let σ be an n-simplex in $Kak(S,\alpha)$. Denote the vertices of σ by $v_0,\dots,v_n$. Then there are representatives of $v_0,\dots,v_n$ with underlying curves $c_0,\dots,c_n$ such that the following hold:*

1. *$c_i \cap c_j = \emptyset$ for all $i \neq j$;*
2. *$S\backslash(c_0 \cup \cdots \cup c_n)$ is partitioned into subsurfaces $P_0,\dots,P_n$ such that $\partial P_i = c_i - c_{i-1}$.*

Proof. Let $(p,\hat{S},S)$ be the covering space associated with α and let σ be a simplex in $Kak(S,\alpha)$. Let $c_0,\dots,c_n$ be geodesic representatives of the underlying curves of $v_0,\dots,v_n$ such that arc components of $c_0,\dots,c_n$ are perpendicular to ∂S. By Remark 1.10, c_i and c_j are almost disjoint for all $i \neq j$. Consider a lift S_0 of $S\backslash c_0$ to $\hat{S}$. For each $j \neq 0$, c_j lifts to a separating collection $\hat{c}_j$ of simple closed curves and simple arcs. Moreover, since S_0 is homeomorphic to $S\backslash c_0$, the lifts $\hat{c}_i$ and $\hat{c}_j$ are almost disjoint as long as $i \neq j$. By reindexing $c_0,\dots,c_n$ if necessary and performing small isotopies that pull apart equal components, we can thus ensure that $\hat{c}_i$ lies above $\hat{c}_j$ for $i > j$.

Note that the lift of $S\backslash c_0$ is homeomorphic to $S\backslash c_0$. In particular, $c_i \cap c_j = \emptyset$ for all $i \neq j$. Moreover, the surface with interior below $\hat{c}_i$ and above $\hat{c}_{i-1}$ projects to a subsurface P_i of S for $i = 1,\dots,n$. The subsurfaces $P_1,\dots,P_n$ exhibit the required properties. □

Remark 2.3. When $w_0,\dots,w_n = 1$, Lemma 2.2 ensures that $c_0 = h(w_0,c_0) = h(1,c_0)$, $\dots$, $c_n = h(w_n,c_n) = h(1,c_n)$ form a cycle of cycles. In this case h extends over the simplex σ to produce a simplex $h(\sigma)$ in C_α. However, h does not extend over simplices in which weights are not all 1. See Figure 4.

Hatcher [9] proves that for each $x \in H_1(S)$, $C_x(S)$ is contractible. (In particular, it is therefore connected and hence constitutes just one component of $C(S)$.) In §4 we prove an analogous result for $Kak(S,\alpha)$, using a technique from the study of the Kakimizu complex of 3-manifolds.

The cyclic cycle complex and the Kakimizu complex are simplicial complexes. The complex defined by Bestvina, Bux, and Margalit (see [2]) is not simplicial, but can be subdivided to obtain a simplicial complex. See the final comments in [10, §2]. There is a subcomplex of the cyclic cycle complex that equals this subdivision of the complex defined by Bestvina, Bux, and Margalit. This is the complex of interest in the context the Torelli group. Bestvina, Bux, and Margalit exploited the action of the

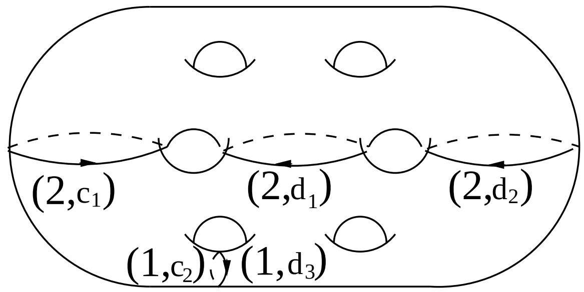

FIGURE 4. An edge of $Kak(S,\alpha)$ that does not map into C_α

Torelli group on this complex to compute the dimension of the Torelli group. Hatcher and Margalit [10] used it to identify generating sets for the Torelli group.

In [12], Irmer defines the homology curve complex of a surface:

> Suppose S is a closed oriented surface. S is not required to be connected but every component is assumed to have genus $g \geq 2$.
>
> Let α be a nontrivial element of $H_1(S,\mathbb{Z})$. The *homology curve complex*, $\mathcal{H}C(S,\alpha)$, is a simplicial complex whose vertex set is the set of all homotopy classes of oriented multi-curves in S in the homology class α. A set of vertices $m_1,\ldots,m_k$ spans a simplex if there is a set of pairwise disjoint representatives of the homotopy classes.
>
> The *distance*, $d_{\mathcal{H}}(v_1,v_2)$, between two vertices v_1 and v_2 is defined to be the distance in the path metric of the one-skeleton, where all edges have length one. [12, p. 1]

It is not hard to see the following (cf. Remark 1.10 and Figure 3).

Lemma 2.4. *When both are defined, i.e., when S is closed, connected, of genus at least 2 and α is primitive, $Kak(S,\alpha)$ is a subcomplex of $\mathcal{H}C(S,\alpha)$. Moreover, for vertices v and v' of $Kak(S,\alpha)$,*

$$d_K(v,v') \geq d_{\mathcal{H}}(v,v').$$

Irmer [12] shows that distance between vertices of $\mathcal{H}C(S,\alpha)$ is bounded above by a linear function on the intersection number of representatives. The same is true for vertices of the Kakimizu complex. Irmer also constructs quasi-flats in $\mathcal{H}C(S,\alpha)$. Her construction carries over to the setting of the Kakimizu complex. See §6.

3. The Projection Map, Distances and Geodesics

In [16], Kakimizu defined a map on the vertices of the Kakimizu complex of a knot. He used this map to prove several things, for instance that the metric, d_K, on the vertices of the Kakimizu complex equals graph distance (quoted and re-proved here as Theorem 3.6). In [21], Kakimizu's map was rebranded as a projection map.

We wish to define

$$\pi_{Vert(Kak(S,\alpha))} : Vert(Kak(S,\alpha)) \to Vert(Kak(S,\alpha))$$

on the vertex set of $Kak(S,\alpha)$. Let $(p, \hat{S}, S)$ be the infinite cyclic covering space associated with α. Let v and v' be vertices in $Kak(S,\alpha)$ such that $v \neq v'$. Here $v = [(w,c)]$ for some compact oriented 1-manifold c and $v' = [(w',c')]$ for some compact oriented 1-manifold c'. We may assume, in accordance with Definition 1.11 and Remark 1.13, that (w,c) and (w',c') are chosen so that $d_K(c,c') = d_K(v,v')$. Define τ, S_i, S'_i, c_i and, by analogy, c'_i, as in Definition 1.11.

Instead of working only with c'_0, we will now also work with $h(w',c'_0)$. Take $m = max\{i \mid S_{i+1} \cap S'_0 \neq \emptyset\}$. Consider a connected component C of $S_{m+1} \cap S'_0$. Its frontier consists of a subset of c'_0 and a subset of c_m. The subset of c'_0 lies above the subset of c_m. In particular, C lies above c_m and below c'_0; hence, the orientations of the subset of c'_0 are opposite those of the subset of c_m. See Figure 5. Because the subset of c'_0 and the subset of c_m cobound C, they are homologous. It follows that the lowest components of the corresponding subset of $h(w',c'_0)$ are also homologous to the subset of c_m.

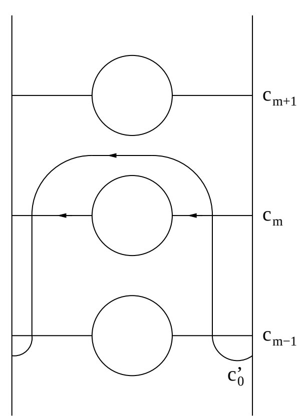

Figure 5. The setup with c_m, c'_0 (weights are 1)

Replacing the lowest of the corresponding subsets of $h(w', c'_0)$ with the subset of c_m and isotoping this portion of c_m to lie below c_m yields a multi-curve d_1 with the following properties:

- d_1 is homologous to $h(w', c'_0)$ via a homology that descends to a homology in S (because C is homeomorphic to a subset of S);
- d_1 has lower geometric intersection number with c_m than $h(w', c'_0)$;
- d_1 lies above $h(w', c'_{-1})$ and can be isotoped to lie below and thus be disjoint from $h(w', c'_0)$; moreover, its projection can be isotoped to be disjoint from $h(w', c')$.
- For $(x_1, e_1) = h^{-1}(d_1)$, we have $x_1 \circ e_1$ homologous to $w' \circ c'_0$ via a homology that descends to a homology in S.

See figures 5 and 6.

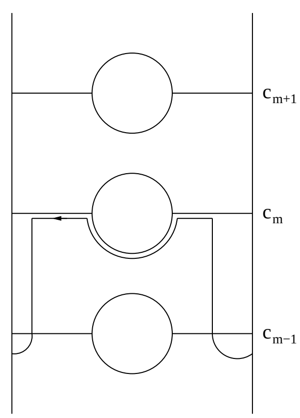

FIGURE 6. d_1

Working with $h^{-1}(d_1), d_1$ instead of $c'_0, h(w', c'_0)$, we perform such replacements in succession to obtain a sequence of multi-curves $d_1, \ldots, d_k$ such that the following hold:

- d_j is homologous to d_{j-1} via a homology that descends to a homology in S;
- d_j has lower geometric intersection number with c_m than d_{j-1};
- d_j can be isotoped to lie below $h(w', c'_0), d_1, \ldots, d_{j-1}$; moreover, its projection can be isotoped to be disjoint from $h(w', c')$.
- For $(x_j, e_j) = h^{-1}(d_j)$, we have $x_j \circ e_j$ homologous to $w' \circ c'_0$ via a homology that descends to S.
- d_k lies above $h(w', c'_{-1})$ and below c_m.

See figures 7, 8, and 9.

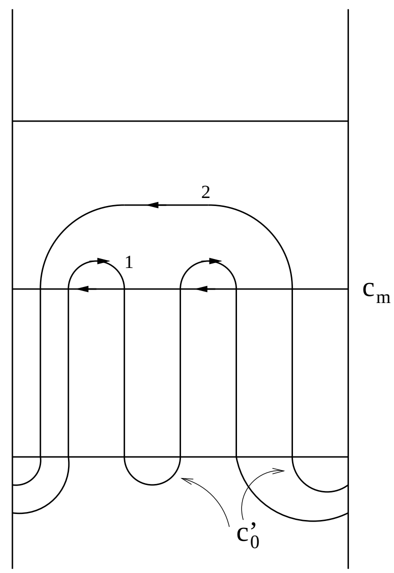

FIGURE 7. A different pair of weighted multi-curves

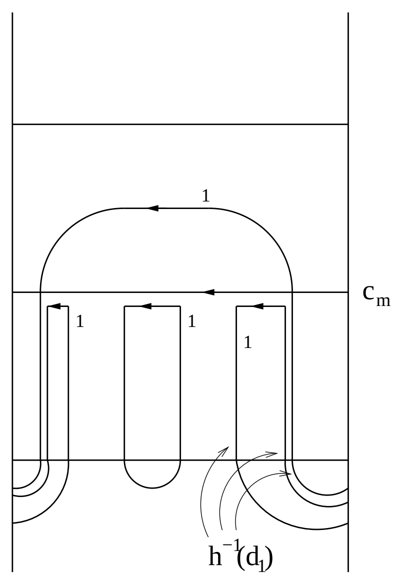

FIGURE 8. $h^{-1}(d_1)$

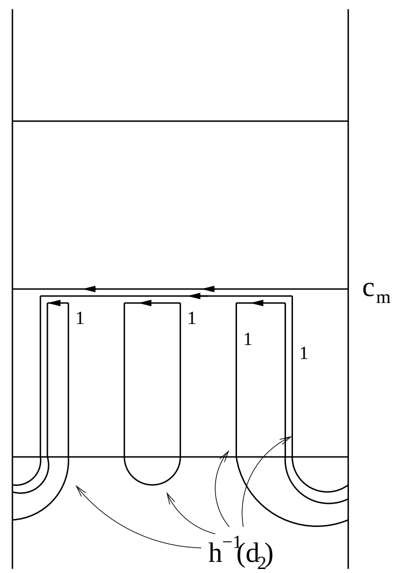

FIGURE 9. $h^{-1}(d_2)$

This proves the following.

Lemma 3.1. *The homology class* $[[p(d_k)]] = p_\#(x_k \circ e_k) = \alpha$.

We make two observations: (1) A result of Ulrich Oertel [19] shows that the isotopy class of $p(e_k)$ does not depend on the choices made. (2) It is important to realize that $(x_k, p(e_k))$ may not be a Seifert curve, because $S \backslash p(e_k)$ is not necessarily connected.

If $S \backslash p(h^{-1}(e_k))$ is connected, set $p_c(c') = (x_k, p(e_k))$. Otherwise, choose a component D of $S \backslash p(e_k)$. If the frontier of D is null homologous, then remove the frontier of D from $p(e_k)$. See figures 10, 11, 12.

If the frontier of D is not null homologous (because the orientations do not match up), choose an arc a in its frontier with smallest weight. Denote the weight of a by w^a. We eliminate the component a of $p(e_k)$ by adding $\pm w^a$ to the weights of the other components of $p(e_k)$ in the frontier of D in such a way that the resulting weighted multi-curve still has homology α.

After a finite number of such eliminations, we obtain a weighted multi-curve that is a subset of $p(e_k)$, has homology α, and whose complement in S is connected. After reversing orientation on components with negative weights, we obtain a Seifert curve $p_c(c')$.

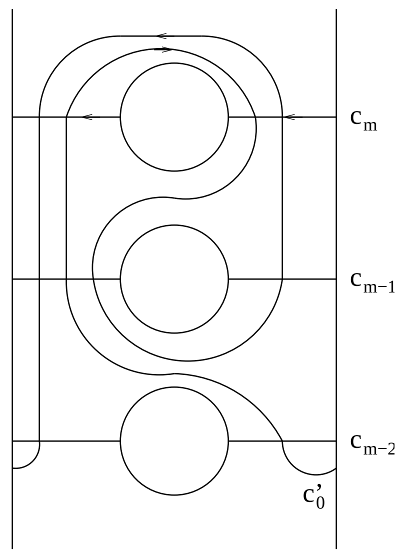

FIGURE 10. *The setup with c_m, c_0' (weights are 1)*

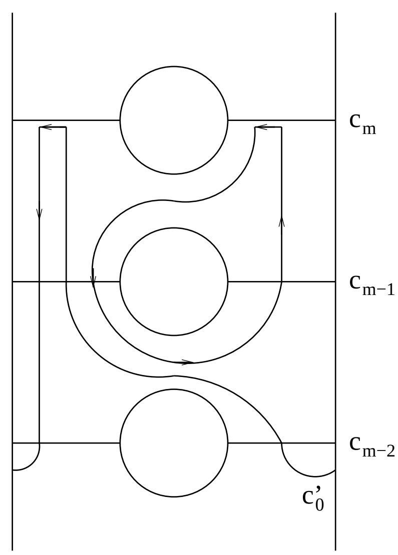

FIGURE 11. d_k

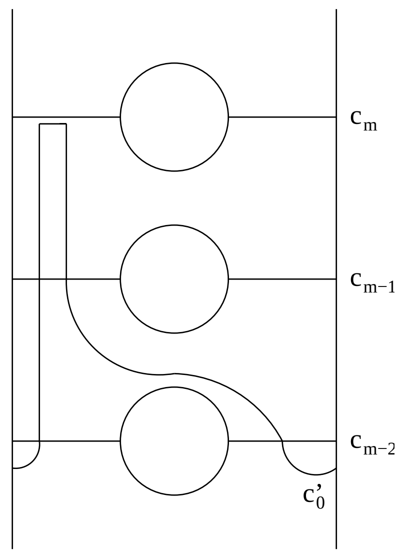

FIGURE 12. *A subset of* d_k

Lemma 3.2. *The homology class* $[[p_c(c')]] = \alpha$.

Proof. This follows from Lemma 3.1 and the observations above. □

Definition 3.3. We denote the isotopy class $[p_c(c')]$ by $\pi_v(v')$.

Lemma 3.4. *For* $v \neq v'$, *the following hold:*

$$d_K(\pi_v(v'), v') = 1$$

and

$$d_K(\pi_v(v'), v) \leq d_K(v', v) - 1.$$

It will follow from Theorem 3.6 below that the inequality is in fact an equality.

Proof. By construction, e_k lies strictly between c'_0 and c'_{-1}. So $\tau(e_k)$ lies strictly between c'_1 and c'_0. Thus, the lift of $S \backslash p_c(c')$ with frontier in $e_k \cup \tau(e_k)$ meets S'_0 and S'_1 and is disjoint from S'_i for $i \neq 0, 1$. It follows that the lift of $S \backslash p_c(c')$ with frontier contained in $e_k \cup \tau(e_k)$ also meets S'_0 and S'_1 and is disjoint from S'_i for $i \neq 0, 1$. Hence,

$$d_K(\pi_v(v'), v') = 1.$$

In addition, suppose that $c'_0 \cap S_i \neq \emptyset$ if and only if $i \in \{n, \ldots, m+1\}$. Then $c'_1 \cap S_i \neq \emptyset$ if and only if $i \in \{n+1, \ldots, m+2\}$. Hence, the lift of $S \backslash c'$ that lies strictly between c'_0 and c'_1 meets exactly $S_n, \ldots, S_{m+2}$.

By construction, $e_k \cap S_i$ can be nonempty only if $i \in \{n, \ldots, m\}$ and thus $\tau(e_k) \cap S_i$ can be nonempty only if $i \in \{n+1, \ldots, m+1\}$. Hence, the lift of $S \backslash p_c(c')$ with frontier in $e_k \cup \tau(e_k)$ can meet S_i only if $i \in \{n, \ldots, m+1\}$. It follows that the lift of $S \backslash p_c(c')$ with frontier contained in $e_k \cup \tau(e_k)$ can meet S_i only if $i \in \{n, \ldots, m+1\}$, whence

$$d_K(\pi_v(v'), v) \leq m + 1 - n - 1 = d_K(v', v) - 1. \qquad \square$$

Definition 3.5. The *graph distance* on a complex $\mathcal{C}$ is a function that assigns to each pair of vertices v and v' the least possible number of edges in an edge path in $\mathcal{C}$ from v to v'.

Theorem 3.6 (Kakimizu)**.** *The function d_K equals graph distance.*

Proof. Denote the graph distance between v' and v by $d(v', v)$. If $d_K(v', v) = 1$, then $d(v', v) = 1$ and vice versa by definition. So suppose $d_K(v', v) = m > 1$ and consider the path with vertices

$$v', \pi_v(v'), \pi_v^2(v'), \ldots, \pi_v^{m-1}(v'), \pi_v^m(v') = v.$$

By Lemma 3.4, $d_K(\pi_v(v'), v') = 1$ and $d_K(\pi_v^i(v'), \pi_v^{i-1}(v')) = 1$. Thus, the existence of this path guarantees that $d(v', v) \leq m$. Hence, $d(v', v) \leq d_K(v', v)$. Let $v' = v_0, v_1, \ldots, v_n = v$ be the vertices of a path realizing $d(v', v)$. By the triangle inequality and the fact that $d(v_{i-1}, v_i) = 1 = d_K(v_{i-1}, v_i)$,

$$\begin{aligned} d_K(v', v) \leq d_K(v_0, v_1) + \cdots + d_K(v_{n-1}, v_n) = 1 + \cdots + 1 = \\ d(v_0, v_1) + \cdots + d(v_{n-1}, v_n) = d(v', v) \end{aligned} \qquad \square$$

The following theorem is a reinterpretation of a theorem of Scharlemann and Thompson [24] that was proved using different methods.

Theorem 3.7. *The Kakimizu complex is connected.*

Proof. Let v and v' be vertices in $Kak(S, \alpha)$. By Remark 1.13, $d_K(v, v')$ is finite. By Theorem 3.6, $d(v, v')$ is finite. In particular, there is a path between v and v'. $\square$

Definition 3.8. A *geodesic* between vertices v and v' in a Kakimizu complex is an edge-path that realizes $d(v, v')$.

Theorem 3.9. *The path with vertices $v', \pi_v(v'), \pi_v^2(v'), \ldots, \pi_v(v')^{m-1}$, $\pi_v^m(v') = v$ is a geodesic.*

Proof. This follows from Theorem 3.6 because the path

$$v', \pi_v(v'), \pi_v^2(v'), \ldots, \pi_v(v')^{m-1}, \pi_v^m(v') = v$$

realizes $d(v', v)$. $\square$

Remark 3.10. Theorem 3.9 tells us that geodesics in the Kakimizu complex joining two given vertices are, at least theoretically, constructible.

Note that, typically, $\pi_v(v') \neq \pi_{v'}(v)$. See Figure 13 for a step in the construction of $\pi_{v'}(v)$.

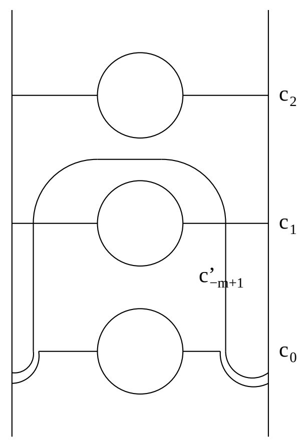

FIGURE 13. $u(c'_{-m+1}, c_0)$

4. Contractibility

The proof of contractibility presented here is a streamlined version of the proof given in the 3-dimensional case in [21]. Those familiar with Hatcher's work in [9] will note certain similarities with his first proof of contractibility of $C_\alpha(S)$ in the case that α is primitive.

Lemma 4.1. *Suppose that v, v^1, and v^2 are vertices in $Kak(S,\alpha)$. Then there are representatives c, c^1, and c^2 with $v = [(w,c)]$, $v^1 = [(w^1,c^1)]$, and $v^2 = [(w^2,c^2)]$ that realize $d_K(v,v^1), d_K(v,v^2)$, and $d_K(v^1,v^2)$.*

Proof. Let c, c^1, and c^2 be geodesic representatives of the underlying curves of representatives of v, v^1, and v^2 such that arc components of c, c^1, and c^2 are perpendicular to ∂S. Lifts of c, c^1, and c^2 to $(p,\hat{S},S)$, the infinite cyclic covering of S associated with α, are also geodesics. Points of intersection lift to points of intersection. Geodesics that intersect cannot be isotoped to be disjoint. Hence, c, c^1, and c^2, with appropriate weights, realize $d_K(v,v^1)$, $d_K(v,v^2)$, and $d_K(v^1,v^2)$. $\square$

Lemma 4.2. *Suppose that v, v^1, and v^2 are vertices in $Kak(S,\alpha)$ such that $d_K(v,v^i) > 1$ and $d_K(v^1,v^2) = 1$. Then $d_K(\pi_v(v^1),\pi_v(v^2)) \leq 1$.*

Proof. In the case that, say, $v^1 = v$, note that $d_K(v^1, v^2) = 1$ means that $d_K(v, v^2) = 1$. Thus, $\pi_v(v^1) = \pi_v(v) = v$ and $\pi_v(v^2) = v$. Thus, $d_K(\pi_v(v^1), \pi_v(v^2)) = 0$. In the case that, say, $d_K(v^1, v) = 1$ and $v^2 \neq v$, note that $\pi_v(v^1) = v$ and

$$d_K(v, v^2) \leq d_K(v, v^1) + d_K(v^1, v^2) = 1 + 1;$$

thus,

$$d_K(v, \pi_v(v^2) \leq 1,$$

by Lemma 3.4, and $d_K(\pi_v(v^1), \pi_v(v^2)) \leq 1$. Hence, we will assume for the rest of this proof that $d_K(v, v^i) > 1$.

By Lemma 4.1, there are representatives (w, c), (w^1, c^1), and (w^2, c^2) of v, v^1, and v^2 that realize $d_K(v, v^1)$, $d_K(v, v^2)$, and $d_K(v^1, v^2)$. Let $(p, \hat{S}, S)$ be the infinite cyclic cover of S associated with α. Define τ, S_i, S_i^1, S_i^2, c_i, c_i^1, c_i^2 as in Definition 1.11 but with a caveat: Label S_i^1 and S_i^2 so that S_0^1 and S_0^2 meet S_1 and meet S_j only if $j \leq 1$.

Since $d_K(c^1, c^2) = d_K(v^1, v^2)$, c^1 and c^2 must be disjoint. Since c_0^1 is separating, c_0^2 lies either above or below c_0^1. Without loss of generality, we will assume that c_0^2 lies above c_0^1 (and below $\tau(c_0^1)$). See figures 14 and 15. Note that $h(w^2, c_0^2)$ also lies above $h(w^1, c_0^1)$. Proceeding as in the discussion preceding Lemma 3.2, construct e_k^1 whose projection contains $p_c(c^1)$ and then e_l^2 whose projection contains $p_c(c^2)$, noting that this construction can be undertaken so that e_l^2 lies above e_k^1 (and below $\tau(e_k^1)$).

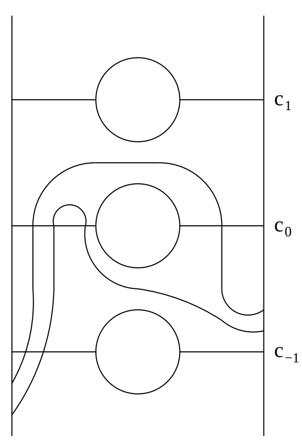

FIGURE 14. c_0^1 *and* c_0^2

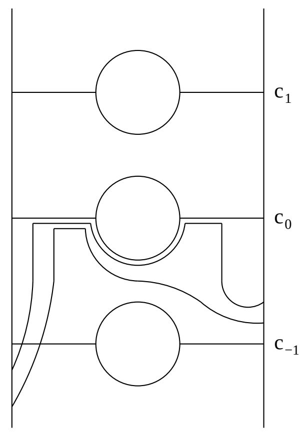

FIGURE 15. d_k^1 *and* d_l^2

Consider the lift of $S\backslash p_c(c^2)$ with frontier in $e_l^2 \cup \tau(e_l^2)$. This lift of $S\backslash p_c(c^2)$ meets at most the two lifts of $S\backslash p_c(c^1)$ whose frontiers lie in $e_k^1 \cup \tau(e_k^1)$ and $\tau(e_k^1) \cup \tau^2(e_k^1)$, whence

$$d_K(\pi_v(v^1), \pi_v(v^2)) \leq 1. \qquad \square$$

Lemma 4.3. *If* $d_K(v^1, v^2) = m$*, then* $d_K(\pi_v(v^1), \pi_v(v^2)) \leq m$*.*

Proof. Let $v^1 = v_0, v_1, \ldots, v_{m-1}, v_m = v^2$ be the vertices of a path from v^1 to v^2 that realizes $d_K(v^1, v^2)$. By Lemma 4.2, $d_K(\pi_v(v_i), \pi_v(v_{i+1})) \leq d_K(v_i, v_{i+1}) = 1$ for $i = 0, \ldots, m-1$. Hence,

$$d_K(\pi_v(v^1), \pi_v(v^2)) \leq d_K(\pi_v(v_0), \pi_v(v_1)) + \cdots + d_K(\pi_v(v_{m-1}), \pi_v(v_m)) \leq$$
$$d_K(v_0, v_1) + \cdots + d_K(v_{m-1}, v_m) \leq m. \qquad \square$$

Theorem 4.4. *The Kakimizu complex of a surface is contractible.*

Proof. Let $Kak(S, \alpha)$ be a Kakimizu complex of a surface. It is well known (see [8, p. 358, Exercise 11]) that it suffices to show that every finite subcomplex of $Kak(S, \alpha)$ is contained in a contractible subcomplex of $Kak(S, \alpha)$. Let $\mathcal{C}$ be a finite subcomplex of $Kak(S, \alpha)$. Choose a vertex v in $\mathcal{C}$ and denote by $\mathcal{C}'$ the smallest flag complex containing every geodesic of the form given in Theorem 3.9 for v' a vertex in $\mathcal{C}$. Since $\mathcal{C}$ is finite, it follows that $\mathcal{C}'$ is finite.

Define $c : Vert(\mathcal{C}') \to Vert(\mathcal{C}')$ on vertices by $c(v') = \pi_v(v')$. By Lemma 4.2, this map extends to edges. Since $\mathcal{C}'$ is flag, the map extends to simplices and thus to all of $\mathcal{C}'$. By Lemma 4.3 this map is continuous. It is not hard to see that c is homotopic to the identity map. In particular,

c is a contraction map. (Specifically, c^d, where d is the diameter of $\mathcal{C}'$, has the set $\{v\}$ as its image.) $\square$

5. Dimension

In [9], Hatcher proves that the dimension of $C_\alpha(S)$ is $2g(S)-3$, where $g(S)$ is the genus of the closed oriented surface S. An analogous argument derives the same result in the context of $Kak(S,\alpha)$.

Lemma 5.1. *Let S be a closed connected orientable surface with $genus(S)$ greater than or equal to 2 and let α be a primitive class in $H_1(S,\partial S)$. The dimension of $Kak(S,\alpha)$ is $-\chi(S)-1 = 2genus(S)-3$.*

Proof. It is not hard to build a simplex of $Kak(S,\alpha)$ of dimension $2genus(S)-3$. See, for example, Figure 16, where 0, 1, 2, and 3 are multicurves (each of weight 1) representing the vertices of a simplex. Thus, the dimension of $Kak(S,\alpha)$ is greater than or equal to $2genus(S)-3$.

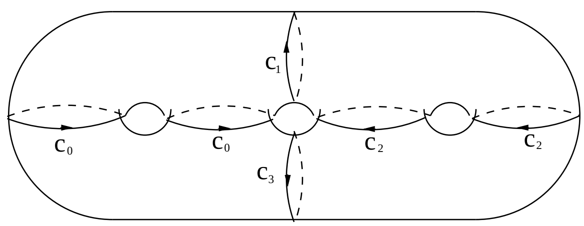

FIGURE 16. A simplex in a genus 3 surface

Conversely, let σ be a simplex of maximal dimension in $Kak(S,\alpha)$. Label the vertices of σ by $v_0,\dots,v_n$ and let $c_0,\dots,c_n$ be geodesic representatives of the underlying curves of representatives of $v_0,\dots,v_n$. By Lemma 2.2, $S\backslash(c_0\cup\cdots\cup c_n)$ consists of subsurfaces $P_0,\dots,P_n$ with frontiers $c_0-c_n, c_1-c_0,\dots,c_n-c_{n-1}$. Since c_i and c_{i-1} are not isotopic, no P_i can consist of annuli. In addition, no P_i can be a sphere; hence, each must have negative Euler characteristic. Thus, the number of P_i's is at most $-\chi(S)$. That is,

$$n \leq -\chi(S) = 2genus(S)-2.$$

In other words, the dimension of σ and hence the dimension of $Kak(S,\alpha)$ are less than or equal to $2genus(S)-3$. $\square$

We can extend this argument to compact surfaces by introducing the following notion of complexity.

Definition 5.2. Let S be a compact surface and let P be an open subset of S whose boundary consists of open subarcs of ∂S and, possibly, components of ∂S. Define

$$c(P, S) = -2\chi(P) + \textit{number of open subarcs in } \partial P.$$

The following lemma is immediate.

Lemma 5.3. *Let $\mathcal{C}$ be a union of simple closed curves and simple arcs in S. Then*

$$c(S, S) = c(S \backslash \mathcal{C}, S).$$

Theorem 5.4. *Let S be a compact connected orientable surface with $\chi(S) \leq -1$ and let α be a primitive class in $H_1(S, \partial S)$. The dimension of $Kak(S, \alpha)$ is $-2\chi(S) - 1 = 4genus(S) + 2b - 5$, where b is the number of boundary components of S.*

Proof. To build a simplex of $Kak(S, \alpha)$ of dimension $4genus(S) + 2b - 5$, we can, for instance, employ the vertices pictured in Figure 17, where 0, 1, 2, 3, 4 and 5 are multi-curves (each of weight 1) representing the vertices of a simplex. Thus, the dimension of $Kak(S, \alpha)$ is greater than or equal to $4g(S) + 2b - 5$.

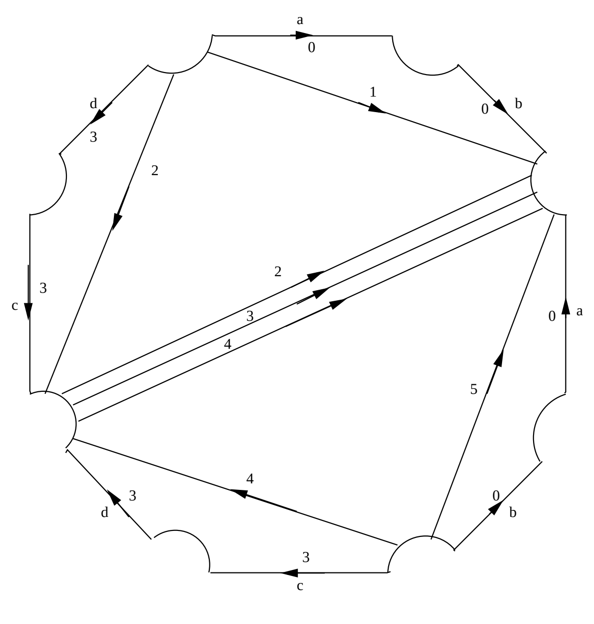

FIGURE 17. A simplex in a punctured genus 2 surface

Conversely, let σ be a simplex of maximal dimension in $Kak(S,\alpha)$. Label the vertices of σ by $v_0,\ldots,v_n$ and let $c_0,\ldots,c_n$ be geodesic representatives of the underlying curves of representatives of $v_0,\ldots,v_n$ such that arc components of $c_0,\ldots,c_n$ are perpendicular to ∂S. By Lemma 2.2, $S\backslash(c_0\cup\cdots\cup c_n)$ consists of subsurfaces $P_0,\ldots,P_n$ with frontiers containing $c_0-c_n, c_1-c_0,\ldots,c_n-c_{n-1}$. Since c_i and c_{i-1} are not isotopic, P_i cannot consist of annuli or disks with exactly two open subarcs of ∂S in their boundary. In addition, no P_i can be a sphere or a disk with exactly one open subarc of ∂S in its boundary; hence, each must have positive complexity. Thus, the number of P_i's is at most $c(S,S\backslash(c_0\cup\cdots\cup c_n))$. That is,

$$n \leq c(S,S\backslash(c_0\cup\cdots\cup c_n)) = c(S,S) = -2\chi(S).$$

In other words, the dimension of σ and hence the dimension of $Kak(S,\alpha)$ is less than or equal to $-2\chi(S)-1 = 4genus(S)+2b-5$. □

6. Quasi-flats

In this section we explore an idea of Irmer (see [12, §7]).

Consider Figure 18. Denote the surface depicted by S and the homology class of c by α. The curves t_1 and t_2 are homologous, as are t_3 and t_4. Denote by v the vertex $(1,c)$ of $Kak(S,\alpha)$, by v_1 the vertex corresponding to the result d_1 obtained from c by Dehn twisting n times around t_1 and $-n$ times around t_2, and by v_2 the vertex corresponding to the result d_2 obtained from c by Dehn twisting n times around t_3 and $-n$ times around t_4. Then d_1 and d_2 are homologous to c, so we obtain three vertices v, v_1, and v_2 in $Kak(S,\alpha)$ (all weights are 1). Note the following:

$$d(v,v_i) = d_K(v,v_i) = n$$
$$d(v_1,v_2) = d_K(v_1,v_2) = n.$$

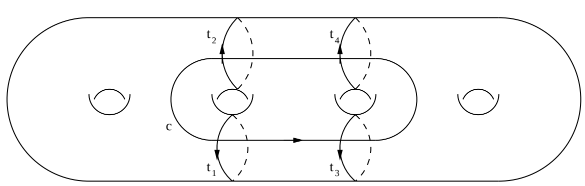

Figure 18. Building a quasi-flat by Dehn twists

For $i=1,2$, we consider the geodesics g_i with vertices $v_i,\pi_v(v_i),\ldots,\pi_v^n(v_i)=v$. In addition, consider the geodesic g_3 with vertices $v_2,\pi_{v_1}(v_2),\ldots,\pi_{v_1}^n(v_2)=v_1$ and note that $\pi_{v_1}^i(v_2)$ is represented by a curve obtained from c by Dehn twisting i times around t_1, $-i$ times around t_2, $n-i$ times around t_3, and $-(n-i)$ times around t_4.

Definition 6.1. Let (X, d) be a metric space. A *triangle* is a 6-tuple (v^1, v^2, v^3, g^1, g^2, g^3), where v^1, v^2, and v^3 are vertices and the edges g^1, g^2, and g^3 satisfy the following: g^1 is a distance minimizing path between v^1 and v^2; g^2 is a distance minimizing path between v^2 and v^3; g^3 is a distance minimizing path between v^3 and v^1.

A triangle (v^1, v^2, v^3, g^1, g^2, g^3) is *δ-thin* if each g^i lies in a δ-neighborhood of the other two edges. A metric space (X, d) is *δ-hyperbolic* if every triangle in (X, d) is δ-thin. It is *hyperbolic* if there is a $\delta > 0$ such that (X, d) is δ-hyperbolic.

For n even, the midpoint m_1 of the geodesic g_1 is the vertex corresponding to the result d_1' obtained from c by Dehn twisting $\frac{n}{2}$ times around t_1 and $-\frac{n}{2}$ times around t_2. Likewise, the midpoint m_2 of the geodesic g_2 is the vertex corresponding to the result d_2' obtained from c by Dehn twisting $\frac{n}{2}$ times around t_3 and $-\frac{n}{2}$ times around t_4. The midpoint m_3 of g_3 is represented by a curve obtained from c by Dehn twisting $\frac{n}{2}$ times around t_1 and around t_3 and $-\frac{n}{2}$ times around t_2 and t_4.

Lemma 6.2. *Let S be the closed oriented surface of genus 4. Then $Kak(S, \alpha)$ is not hyperbolic.*

Proof. For S the closed genus 4 surface, the triangle $(v, v_1, v_2, g_1, g_2, g_3)$ described depends on n, so we will denote it by T_n. In T_n we have the following:

$$d(v, m_3) = d_K(v, m_3) = n$$
$$d(v_1, m_2) = d_K(v_1, m_2) = n$$
$$d(v_2, m_1) = d_K(v_2, m_1) = n.$$

In particular, g_3 is contained in a δ-neighborhood of the two geodesics g_1 and g_2 only if n is less than δ. Thus, the triangle T_n in $Kak(S, \alpha)$ is not δ-thin for $n \geq \delta$. It follows that $Kak(S, \alpha)$ is not hyperbolic. □

Definition 6.3. Let (X, d) be a metric space. A *quasi-flat* in (X, d) is a quasi-isometry from $\mathbb{R}^n$ to (X, d), for $n \geq 2$.

Note the following:

$$d(m_1, m_2) = d_K(m_1, m_2) = \frac{n}{2}$$
$$d(m_1, m_3) = d_K(m_1, m_3) = \frac{n}{2}$$
$$d(m_2, m_3) = d_K(m_2, m_3) = \frac{n}{2}.$$

Thus, the triangle T_n scales like a Euclidean triangle. It is not too hard to see that a triangle with this property can be used to construct a quasi-isometry between $\mathbb{R}^2$ and an infinite union of such triangles lying in $Kak(S,\alpha)$. Thus, $Kak(S,\alpha)$ contains quasi-flats. It is also not hard to adapt this construction to show that, for S an oriented surface, $Kak(S,\alpha)$ is not hyperbolic and contains quasi-flats if the genus of S is greater than or equal to 4, or the genus of S is greater than or equal to 2 and $\chi(S) \leq -6$.

7. Genus 2

We consider the example of a closed orientable surface S of genus 2. A non-trivial primitive homology class α can always be represented by a non-separating simple closed curve with weight 1. Moreover, a Seifert curve in a closed orientable surface of genus 2, since its underlying curve is non-separating, can have at most two components. Figure 19 depicts multi-curves c and $d_1 \cup d_2$ such that $[[c]] = [[d_1]] + [[d_2]]$.

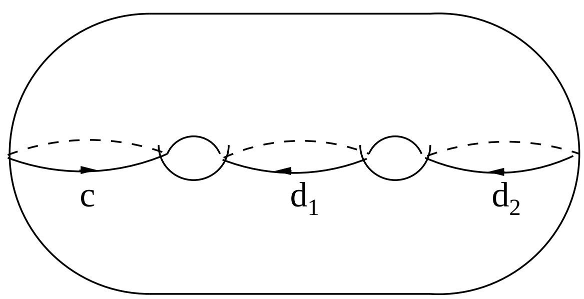

FIGURE 19. Underlying curves c and $d_1 \cup d_2$ in a genus 2 surface (all weights are 1)

We refer to a Seifert curve with one component as *type* 1 and a Seifert curve with two components as *type* 2. Since α is primitive, a Seifert curve of type 1 must have weight 1. It follows that distinct Seifert curves of type 1 must intersect. A Seifert curve of type 1 can be disjoint from a Seifert curve of type 2 (see Figure 19) and distinct Seifert curves of type 2 can be disjoint (see figures 20 and 21).

Let c be the underlying curve of a Seifert curve of type 1 and $d = d_1 \cup d_2$ the underlying curve of a Seifert curve of type 2 that are disjoint. Then the three disjoint simple closed curves $c \cup d$ cut S into pairs of pants. Any Seifert curve that is disjoint from $c \cup d$ must have its underlying curve parallel to either c or d. Note that, since the weight of c is 1, the weights for d_1 and d_2 must also be 1.

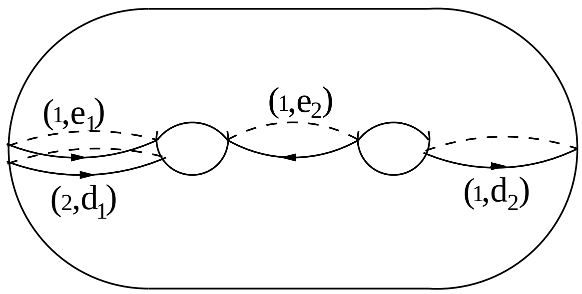

FIGURE 20. Seifert curves $(2, 1, d)$ and $(1, 1, e)$

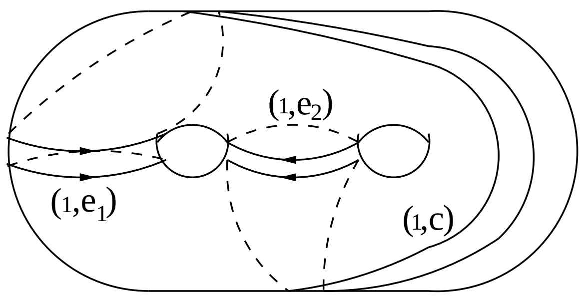

FIGURE 21. Seifert curves $(1, 1, e)$ and $(1, c)$

Consider the link of $[(1, c)]$ in $Kak(S, \alpha)$. It consists of equivalence classes of Seifert curves of type 2. The Seifert curves of type 2 have underlying curves that are pairs of curves lying in $S \backslash c$, are not parallel to c, and are separating in $S \backslash c$ but not in S. There are infinitely many such pairs of curves. More specifically, $S \backslash c$ is a twice punctured torus, so the curves are parallel curves that separate the two punctures and can be parameterized by $\mathbf{Q}$. Distinct such curves cannot be isotoped to be disjoint and hence correspond to distance two vertices of $Kak(S, \alpha)$. This confirms that $Kak(S, \alpha)$ has dimension $1 = (2)(2) - 3$ near $[(1, c)]$, as prescribed by Theorem 5.1.

For $d = d_1 \cup d_2$ as in Figure 19 or 20, we consider $S \backslash (d_1 \cup d_2)$, a sphere with four punctures. The link of $[(w_1, w_2, d_1 \cup d_2)]$ contains isotopy classes of Seifert curves of type 1. These are essential curves that are separating in $S \backslash (d_1 \cup d_2)$ but not in S and that partition the punctures of $S \backslash (d_1 \cup d_2)$ appropriately. There are infinitely many such curves. They too can be parameterized by $\mathbf{Q}$. Note that distinct Seifert curves of type 1 cannot be isotoped to be disjoint and hence correspond to vertices of $Kak(S,\alpha)$ of distance two or more.

In addition, the link of $[(w_1, w_2, d_1 \cup d_2)]$ contains vertices $[(u_1, u_2, e_1 \cup e_2)]$ such that one component of $e_1 \cup e_2$, say e_1, is parallel to a component of $d_1 \cup d_2$, say d_1, and $S \backslash (d_1 \cup d_2 \cup e_1 \cup e_2)$ consists of two pairs of pants and one annulus. Seifert curves of this type can also be parameterized by $\mathbf{Q}$, since e_2 is a curve in a twice punctured torus that partitions the punctures appropriately and e_1 is parallel to d_1. Note that, since the weights of d_1 and d_2 are w_1 and w_2, we must have

$$\begin{array}{ccccc} w_1 & = & u_1 & \pm & u_2 \\ w_2 & = & u_2\,. & & \end{array}$$

In summary, $Kak(S,\alpha)$ is a tree, each of whose vertices has a countably infinite discrete (i.e., 0-dimensional) link.

Recall that Johnson, Pelayo, and Wilson showed that the Kakimizu complex of a knot in the 3-sphere is quasi-Euclidean. The Kakimizu complex of the genus 2 surface is an infinite graph, thus Gromov hyperbolic. In particular, it is not quasi-Euclidean.

8. 3-Manifolds

The definitions given for Seifert curve, infinite cyclic cover, Kakimizu complex, and so forth carry over to codimension 1 submanifolds in manifolds of any dimension. In particular, they carry over to Seifert surfaces and Kakimizu complexes in the context of compact (possibly closed) 3-manifolds. One need merely replace 1's by 2's and 2's by 3's. Instead of Seifert curves, one considers Seifert surfaces. Seifert surfaces are weighted essential surfaces that represent a given relative second homology class and have connected complements. This ties into and generalizes some of the work in [21].

Let S be a compact oriented surface. Take $M = S \times I$. Incompressible surfaces in a product manifold are either horizontal or vertical. Vertical surfaces have the form $c \times I$, where c is a multi-curve in S. It follows that $Kak(M, [[c \times I]]) = Kak(S, [[c]])$, where $[[\cdot]]$ denotes the homology class of $\cdot$.

Theorem 8.1. *There exist* 3*-manifolds with Gromov hyperbolic Kakimizu complex.*

Proof. Let S be the closed oriented surface of genus 2, α a primitive homology class in $H_1(S)$, and c a compact 1-manifold representing α. Then $Kak(S,\alpha)$ is the graph discussed in §7. In particular, $Kak(S,\alpha)$ is quasi-hyperbolic. Take $M = S \times I$. Then $Kak(M, [[c \times I]]) = Kak(S,\alpha)$ is also quasi-hyperbolic. $\square$

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