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by

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L-NORMALITY

LUTFI KALANTAN AND MAHA MOHAMMED SAEED

Abstract. A topological space X is called *L-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|A} : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$. We will investigate this property and produce some examples to illustrate the relation between L-normality and other weaker kinds of normality.

A. V. Arhangel'skiĭ introduced in 2012, when he was visiting the Department of Mathematics at King Abdulaziz University, a new, weaker version of normality, called C-normality [8]. A topological space X is called *C-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|C} : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. We use the idea of this definition to introduce another new, weaker version of normality and call it *L-normality.* The purpose of this paper is to investigate this property. We prove that normality implies L-normality but the converse is not true in general. We present some examples to show relationships between L-normality and other weaker versions of normality such as C-normality, almost normality, mild normality, quasi-normality, and π-normality. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$, and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space, a Tychonoff space is a T_1 completely regular space, and a T_3 space is a T_1 regular space. We do not assume T_2 in the definition of compactness and we do not assume regularity in the definition of Lindelöfness. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of

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all ordinals α such that $\alpha < \gamma$. The first infinite ordinal is ω_0, the first uncountable ordinal is ω_1, and the successor cardinal of ω_1 is ω_2.

1. L-normality

Definition 1.1. A topological space X is called *C-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$ [8]. A topological space X is called *L-normal* if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$.

A function $f : X \longrightarrow Y$ witnessing the C-normality of X need not be continuous. But it will be if it has the property that for each convergent sequence $x_n \longrightarrow x$ in X we have $f(x_n) \longrightarrow f(x)$. This happens if X is a Hausdorff sequential space or a k-space. Similarly, a function $f : X \longrightarrow Y$ witnessing the L-normality of X need not be continuous. But it will be if X is of countable tightness.

Theorem 1.2. *If X is L-normal and of countable tightness and $f : X \longrightarrow Y$ is a witness of the L-normality of X, then f is continuous.*

Proof. Let A be any non-empty subset of X. Let $y \in f(\overline{A})$ be arbitrary. Let $x \in X$ be the unique element such that $f(x) = y$. Then $x \in \overline{A}$. Pick a countable subset $A_0 \subseteq A$ such that $x \in \overline{A_0}$. Let $B = \{x\} \cup A_0$; then B is a Lindelöf subspace of X, and hence $f_{|_B} : B \longrightarrow f(B)$ is a homeomorphism. Now, let $V \subseteq Y$ be any open neighborhood of y; then $V \cap f(B)$ is open in the subspace $f(B)$ containing y. Thus, $f^{-1}(V) \cap B$ is open in the subspace B containing x. Thus, $(f^{-1}(V) \cap B) \cap A_0 \neq \emptyset$. So $(f^{-1}(V) \cap B) \cap A \neq \emptyset$. Hence, $\emptyset \neq f((f^{-1}(V) \cap B) \cap A) \subseteq f(f^{-1}(V) \cap A) = V \cap f(A)$. Thus, $y \in \overline{f(A)}$. Therefore, f is continuous. $\square$

Since any compact space is Lindelöf, then any L-normal space is C-normal. The converse is not true in general. Obviously, no Lindelöf non-normal space is L-normal. So, no countable complement topology on an uncountable set X is L-normal, but it is C-normal because the only compact subspaces are the finite subspaces, and the countable complement topology is T_1, so compact subspaces are discrete. Hence, the discrete topology on X and the identity function will witness C-normality. We will give more Tychonoff C-normal spaces which are not L-normal.

By definition, it is clear that a Lindelöf L-normal space must be normal. If X is countable, then X may not be L-normal, for example, $(\mathbb{Q}, \mathcal{T}_0)$, where $\mathcal{T}_0$ is the particular point topology [14]. It is also obvious that any normal space is L-normal, just by taking $X = Y$ and f to be the identity function. The converse is not true in general. Here is an example of a Tychonoff L-normal space which is neither normal nor locally Lindelöf.

Example 1.3. We modify the Dieudonné plank [14] to define a new topological space. Let

$$X = ((\omega_2 + 1) \times (\omega_0 + 1)) \setminus \{\langle \omega_2, \omega_0 \rangle\}.$$

Write $X = A \cup B \cup N$, where $A = \{\langle \omega_2, n\rangle : n < \omega_0\}$, $B = \{\langle \alpha, \omega_0\rangle : \alpha < \omega_2\}$, and $N = \{\langle \alpha, n\rangle : \alpha < \omega_2 \text{ and } n < \omega_0\}$. The topology τ on X is generated by the following neighborhood system: For each $\langle \alpha, n\rangle \in N$, let $\mathcal{B}(\langle \alpha, n\rangle) = \{\{\langle \alpha, n\rangle\}\}$. For each $\langle \omega_2, n\rangle \in A$, let $\mathcal{B}(\langle \omega_2, n\rangle) = \{V_\alpha(n) = (\alpha, \omega_2] \times \{n\} : \alpha < \omega_2\}$. For each $\langle \alpha, \omega_0\rangle \in B$, let $\mathcal{B}(\langle \alpha, \omega_0\rangle) = \{V_n(\alpha) = \{\alpha\} \times (n, \omega_0] : n < \omega_0\}$. Then X is a Tychonoff non-normal space which is not locally compact nor locally Lindelöf as any basic open neighborhood of any element in A is not Lindelöf. Now, a subset $C \subseteq X$ is Lindelöf if and only if C is countable because if C is uncountable, then either $C \cap B$ or $C \cap N$ is uncountable, then a basic open set for each element in C would give an open cover for C which has no countable subcover.

Now, define $Y = X = A \cup B \cup N$. Generate a topology τ' on Y by the following neighborhood system: Elements of $B \cup N$ have the same local base as in X. For each $\langle \omega_2, n\rangle \in A$, let $\mathcal{B}(\langle \omega_2, n\rangle) = \{\{\langle \omega_2, n\rangle\}\}$. Then Y is a T_4 space because it is paracompact. Consider the identity function $id : X \longrightarrow Y$. Let $C \subset X$ be any Lindelöf subspace. Then $id_{|_C} : C \longrightarrow id(C) = C$ is a bijective. Let $\langle a, b\rangle$ be any element in C. If $\langle a, b\rangle \in N$, then $\{\langle a, b\rangle\}$, which is open in C as a subspace of X and Y, will give that $id_{|_C}$ is continuous. If $\langle a, b\rangle \in B$ and W is any basic open set of $\langle a, b\rangle$ in C as a subspace of Y, then W is also a basic open set of $\langle a, b\rangle$ in C as a subspace of X; hence, $id_{|_C}$ is continuous. If $\langle a, b\rangle \in A$, then the smallest open neighborhood of $\langle a, b\rangle$ in C as a subspace of Y is $\{\langle a, b\rangle\}$. Since C is Lindelöf in X, then it is countable. Write $(\omega_2 \times \{b\}) \cap C = \{\langle \alpha_k, b\rangle : k \in K \text{ where } K \text{ is countable } \}$. Pick $\beta < \omega_2$ such that $\alpha_k < \beta$ for each $k \in K$. Then $V_\beta(b)$ is a basic open set of $\langle a, b\rangle$ in X; hence, $V_\beta(b) \cap C = \{\langle a, b\rangle\}$ is an open neighborhood of $\langle a, b\rangle$ in C as a subspace of X. Thus, $id_{|_C}$ is continuous. From the three cases, we conclude that $id_{|_C}$ is continuous. Since the topology on X is coarser than the topology on Y, then the inverse function of $id_{|_C}$ is also continuous. We conclude that $id_{|_C}$ is a homeomorphism. Therefore, the modified Dieudonné plank X is L-normal.

Theorem 1.4. *L-normality is a topological property.*

Proof. Let X be an L-normal space and $X \cong Z$. Let Y be a normal space and $f : X \longrightarrow Y$ be a bijective such that $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each Lindelöf subspace C of X. Let $g : Z \longrightarrow X$ be a homeomorphism. Then $f \circ g : Z \longrightarrow Y$ satisfies all requirements. □

Theorem 1.5. *L-normality is an additive property.*

Proof. Let X_α be an L-normal space for each $\alpha \in \Lambda$. We show that their sum $\oplus_{\alpha\in\Lambda} X_\alpha$ is L-normal. For each $\alpha \in \Lambda$, pick a normal space Y_α and a bijective function $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ such that $f_{\alpha_{|C_\alpha}} : C_\alpha \longrightarrow f_\alpha(C_\alpha)$ is a homeomorphism for each Lindelöf subspace C_α of X_α. Since Y_α is normal for each $\alpha \in \Lambda$, then the sum $\oplus_{\alpha\in\Lambda} Y_\alpha$ is normal ([3, 2.2.7]). Consider the function sum (see [3, 2.2.E]), $\oplus_{\alpha\in\Lambda} f_\alpha : \oplus_{\alpha\in\Lambda} X_\alpha \longrightarrow \oplus_{\alpha\in\Lambda} Y_\alpha$ defined by $\oplus_{\alpha\in\Lambda} f_\alpha(x) = f_\beta(x)$ if $x \in X_\beta, \beta \in \Lambda$. Now, a subspace $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is Lindelöf if and only if the set $\Lambda_0 = \{\alpha \in \Lambda : C \cap X_\alpha \neq \emptyset\}$ is countable and $C \cap X_\alpha$ is Lindelöf in X_α for each $\alpha \in \Lambda_0$. If $C \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is Lindelöf, then $(\oplus_{\alpha\in\Lambda} f_\alpha)_{|C}$ is a homeomorphism because $f_{\alpha_{|C\cap X_\alpha}}$ is a homeomorphism for each $\alpha \in \Lambda_0$. □

Theorem 1.6. *If X is T_3 separable L-normal and of countable tightness, then X is normal (T_4).*

Proof. Let Y be a normal space and $f : X \longrightarrow Y$ be a bijective witness to L-normality of X. Then f is continuous because X is of countable tightness. Let D be a countable dense subset of X. We show that f is closed. Let H be any non-empty closed proper subset of X. Suppose that $f(p) = q \in Y \setminus f(H)$; then $p \notin H$. Using regularity, let U and V be disjoint open subsets of X containing p and H, respectively. Then $U \cap (D \cup \{p\})$ is open in the Lindelöf subspace $D \cup \{p\}$ containing p, so $f(U \cap (D \cup \{p\}))$ is open in the subspace $f(D \cup \{p\})$ of Y containing q. Thus, $f(U \cap (D \cup \{p\})) = f(U) \cap f(D \cup \{p\}) = W \cap f(D \cup \{p\})$ for some open subset W in Y with $q \in W$.

We claim that $W \cap f(H) = \emptyset$. Suppose otherwise, and take $y \in W \cap f(H)$. Let $x \in H$ such that $f(x) = y$. Note that $x \in V$. Since D is dense in X, D is also dense in the open set V. Thus, $x \in \overline{V \cap D}$. Now since W is open in Y and f is continuous, $f^{-1}(W)$ is an open set in X; it also contains x. Thus, we can choose $d \in f^{-1}(W) \cap V \cap D$. Then $f(d) \in W \cap f(V \cap D) \subseteq W \cap f(D \cup \{p\}) = f(U \cap (D \cup \{p\}))$. So $f(d) \in f(U) \cap f(V)$, a contradiction.

Thus, $W \cap f(H) = \emptyset$. Note that $q \in W$. As $q \in Y \setminus f(H)$ was arbitrary, $f(H)$ is closed. So f is a homeomorphism and X is normal. □

We conclude from the above theorem that the Niemytzki plane [14] and Mrówka space $\Psi(\mathcal{A})$, where $\mathcal{A} \subset [\omega_0]^{\omega_0}$ is mad [2], are examples of Tychonoff spaces which are not L-normal. L-normality is not multiplicative because, for example, the Sorgenfrey line is T_4, but its square is

Tychonoff separable first countable space which is not L-normal because it is not normal. Also, L-normality is not hereditary: take any compactification of the Sorgenfrey line square. We still do not know if L-normality is hereditary with respect to closed subspaces.

Recall that a *Dowker space* is a T_4 space whose product with I, $I = [0,1]$ with its usual metric, is not normal. Mary Ellen Rudin [10] used the existence of a Suslin line to obtain a Dowker space which is hereditarily separable and first countable. Using CH, I. Juhász, K. Kunen, and Rudin constructed a first countable hereditarily separable real compact Dowker space [4]. W. Weiss [15] constructed a first countable separable locally compact Dowker space whose existence is consistent with $\mathsf{MA} + \neg\ \mathsf{CH}$ [15]. By Theorem 1.6, such spaces are consistent examples of Dowker spaces whose product with I are not L-normal.

Since any second countable space is Lindelöf and any T_3 second countable space is metrizable [3, 4.2.9], we conclude with the following theorem.

Theorem 1.7. *Every T_1 second countable L-normal space is metrizable.*

2. L-normality and Other Properties

Now, we study some relationships between L-normality and some other weaker versions of normality. First, we recall some definitions.

Definition 2.1. A subset A of a space X is a *closed domain* [3], also called *regularly closed* or *κ-closed*, if $A = \overline{\text{int}A}$. A space X is *mildly normal* [13], also called *κ-normal* [11], if, for any two disjoint closed domains A and B of X, there exist two disjoint open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$; see also [9] and [5]. A space X is called *almost normal* [12], [7] if, for any two disjoint closed subsets A and B of X one of which is closed domain, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. A subset A of a space X is called *π-closed* [6] if A is a finite intersection of closed domains. A space X is called *π-normal* [6] if, for any two disjoint closed subsets A and B of X one of which is π-closed, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. A space X is called *quasi-normal* [16] if, for any two disjoint π-closed subsets A and B of X, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$; see also [6].

It is clear from the definitions that

normal $\Longrightarrow$ π-normal $\Longrightarrow$ almost normal $\Longrightarrow$ mildly normal.

normal $\Longrightarrow$ π-normal $\Longrightarrow$ quasi-normal $\Longrightarrow$ mildly normal.

Now, $(\mathbb{R}, \mathcal{CC})$, where $\mathcal{CC}$ is the countable complement topology, is not L-normal. But, since the only closed domains are $\emptyset$ and $\mathbb{R}$, then it is π-normal, hence quasi-normal, almost normal, and mildly normal. Here is an example of an L-normal space which is not mildly normal, hence neither quasi-normal, almost normal, nor π-normal.

Example 2.2. *The modified Dieudonné plank X is L-normal but not mildly normal.*

Proof. X is not normal because A and B are closed disjoint subsets which cannot be separated by two disjoint open sets. Let $E = \{n < \omega_0 : n \text{ is even}\}$ and $O = \{n < \omega_0 : n \text{ is odd}\}$. Let K and L be subsets of ω_2 such that $K \cap L = \emptyset$, $K \cup L = \omega_2$, and the cofinality of K and L are ω_2; for instance, let K be the set of limit ordinals in ω_2 and L be the set of successor ordinals in ω_2. Then $K \times E$ and $L \times O$ are both open being subsets of N. Define $C = \overline{K \times E}$ and $D = \overline{L \times O}$; then C and D are closed domains in X, being closures of open sets, and they are disjoint. Note that $C = \overline{K \times E} = (K \times E) \cup (K \times \{\omega_0\}) \cup (\{\omega_2\} \times E)$ and $D = \overline{L \times O} = (L \times O) \cup (L \times \{\omega_0\}) \cup (\{\omega_2\} \times O)$. Let $U \subseteq X$ be any open set such that $C \subseteq U$. For each $n \in E$ there exists an $\alpha_n < \omega_2$ such that $V_{\alpha_n}(n) \subseteq U$. Let $\beta = \sup\{\alpha_n : n \in E\}$; then $\beta < \omega_2$. Since L is cofinal in ω_2, then there exists $\gamma \in L$ such that $\beta < \gamma$ and then any basic open set of $\langle \gamma, \omega_0 \rangle \in D$ will meet U. Thus, C and D cannot be separated. Therefore, the modified Dieudonné plank X is L-normal but is not mildly normal. $\square$

Theorem 2.3. *If X is a C-normal space such that each Lindelöf subspace is contained in a compact subspace, then X is L-normal.*

Proof. Let X be any C-normal space such that if A is any Lindelöf subspace of X, then there exists a compact subspace B such that $A \subseteq B$. Let Y be a normal space and $f : X \longrightarrow Y$ be a bijective function such that $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace C of X. Now, let A be any Lindelöf subspace of X. Pick a compact subspace B of X such that $A \subseteq B$; then $f_{|_B} : B \longrightarrow f(B)$ is a homeomorphism; hence, $f_{|_A} : A \longrightarrow f(A)$ is a homeomorphism as $(f_{|_B})_{|_A} = f_{|_A}$. $\square$

The next example is an application of the above theorem.

Example 2.4. Consider the product space $\omega_1 \times (\omega_1 + 1)$. It is not almost normal because the diagonal $\triangle = \{\langle \alpha, \alpha \rangle : \alpha < \omega_1\}$ is a closed domain which is disjoint from the closed set $K = \omega_1 \times \{\omega_1\}$ and they cannot be separated by two disjoint open sets (see [7]). But $\omega_1 \times (\omega_1 + 1)$ is C-normal being locally compact and local compactness implies C-normality (see [8]).

Now we characterize all Lindelöf subspaces of $\omega_1 \times (\omega_1 + 1)$.

Claim 2.5. *A subspace A of $\omega_1 \times (\omega_1 + 1)$ is Lindelöf if and only if A has the following properties:*

(1) *There is an* $\alpha \in \omega_1$ *such that* $A \subseteq \alpha \times (\omega_1 + 1)$.

(2) *If* $\beta \in \omega_1$ *and* $A \cap (\{\beta\} \times (\omega_1 + 1))$ *is uncountable, then* $\langle \beta, \omega_1 \rangle \in A$.

Proof. Let A be any Lindelöf subspace of $\omega_1 \times (\omega_1 + 1)$. If condition (1) does not hold, then $\{[0, \alpha] \times (\omega_1 + 1) : \alpha < \omega_1\}$ would be an open cover for A which has no countable subcover, a contradiction. Now, assume that A is Lindelöf and satisfies condition (1) but not (2); i.e., there exists a $\beta \in \omega_1$ such that $A \cap (\{\beta\} \times (\omega_1 + 1))$ is uncountable, but $\langle \beta, \omega_1 \rangle \notin A$. Pick $\alpha \in \omega_1$ such that $A \subseteq \alpha \times (\omega_1 + 1)$. It is clear that $\beta \leq \alpha$, but we may assume, without loss of generality, that $\beta < \alpha$. The family $\{(\beta, \alpha] \times (\omega_1 + 1)\} \cup \{[0, \beta] \times [0, \gamma] : \langle \beta, \gamma \rangle \in A \cap (\{\beta\} \times (\omega_1 + 1))\} \cup \{U_\zeta : \langle \zeta, \omega_1 \rangle \in A, \zeta < \beta$ and U_ζ is a basic open neighborhood of $\langle \zeta, \omega_1 \rangle\}$ is an open cover for A which has no countable subcover, a contradiction. Now let A be any subset of $\omega_1 \times (\omega_1 + 1)$ that satisfies both conditions. Since for any $\alpha \in \omega_1$ we have that α is countable and for each basic open set G of $\langle \beta, \omega_1 \rangle$ we have that $(A \cap (\{\beta\} \times (\omega_1 + 1))) \setminus G$ is countable, then it is clear that A will be Lindelöf as a countable union of countable sets is countable. So the claim is proved. $\square$

We conclude from the above claim that each Lindelöf subspace A of $\omega_1 \times (\omega_1 + 1)$ is contained in a compact subspace B of $\omega_1 \times (\omega_1 + 1)$ of the form $B = (\alpha + 1) \times (\omega_1 + 1)$, where α satisfies condition (1) above. Thus, by Theorem 2.3, $\omega_1 \times (\omega_1 + 1)$ is L-normal.

We discovered that the Alexandroff duplicate space of an L-normal space is L-normal. Recall that the *Alexandroff duplicate space* $A(X)$ *of a space* X is defined as follows: Let X be any topological space. Let $X' = X \times \{1\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, 1 \rangle$ in X' by x', and for a subset $B \subseteq X$, let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U$ is open in X with $x \in U\}$. Then $\mathcal{B} = \{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ will generate a unique topology on $A(X)$ such that $\mathcal{B}$ is its neighborhood system. $A(X)$ with this topology is called the Alexandroff duplicate of X [1].

Theorem 2.6. *If* X *is* L*-normal, then its Alexandroff duplicate* $A(X)$ *is also* L*-normal.*

Proof. Let X be any L-normal space. Pick a normal space Y and a bijective function $f : X \longrightarrow Y$ such that $f_{|_C} : C \longrightarrow f(C)$ is a homeomorphism for each Lindelöf subspace $C \subseteq X$. Consider the Alexandroff duplicate spaces $A(X)$ and $A(Y)$ of X and Y, respectively. It is well known that the Alexandroff duplicate of a normal space is normal; hence, $A(Y)$ is also normal. Define $g : A(X) \longrightarrow A(Y)$ by $g(a) = f(a)$ if $a \in X$, and if $a \in X'$, let

b be the unique element in X such that $b' = a$, then define $g(a) = (f(b))'$. Then g is a bijective function. Now, a subspace $C \subseteq A(X)$ is Lindelöf if and only if $C \cap X$ is Lindelöf in X, and for each open set U in X with $C \cap X \subseteq U$, we have that $(C \cap X') \setminus U'$ is countable. Let $C \subseteq A(X)$ be any Lindelöf subspace. We show $g_{|C} : C \longrightarrow g(C)$ is a homeomorphism. Let $a \in C$ be arbitrary. If $a \in C \cap X'$, let $b \in X$ be the unique element such that $b' = a$. For the smallest basic open neighborhood $\{(f(b))'\}$ of the point $g(a)$, we have that $\{a\}$ is open in C and $g(\{a\}) \subseteq \{(f(b))'\}$. If $a \in C \cap X$, let W be any open set in Y such that $g(a) = f(a) \in W$. Consider $H = (W \cup (W' \setminus \{(f(a))'\})) \cap g(C)$ which is a basic open neighborhood of $f(a)$ in $g(C)$. Since $f_{|C \cap X} : C \cap X \longrightarrow f(C \cap X)$ is a homeomorphism, then there exists an open set U in X with $a \in U$ and $f_{|C \cap X}(U \cap C) \subseteq W \cap f(C \cap X)$. Now, $(U \cup (U' \setminus \{a'\})) \cap C = G$ is open in C such that $a \in G$ and $g_{|C}(G) \subseteq H$. Therefore, $g_{|C}$ is continuous. Now, we show that $g_{|C}$ is open. Let $K \cup (K' \setminus \{k'\})$, where $k \in K$ and K is open in X, be any basic open set in $A(X)$, then $(K \cap C) \cup ((K' \cap C) \setminus \{k'\})$ is a basic open set in C. Since $X \cap C$ is Lindelöf in X, then $g_{|C}(K \cap (X \cap C)) = f_{|X \cap C}(K \cap (X \cap C))$ is open in $Y \cap f(C \cap X)$ since $f_{|X \cap C}$ is a homeomorphism. Thus, $K \cap C$ is open in $Y \cap f(X \cap C)$. Also, $g((K' \cap C) \setminus \{k'\})$ is open in $Y' \cap g(C)$ being a set of isolated points. Thus, $g_{|C}$ is an open function. Therefore, $g_{|C}$ is a homeomorphism. $\square$

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