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MICHAŁ STUKOW

Abstract. Let a and b be two simple closed curves on an orientable surface S such that their geometric intersection number is greater than 1. The group generated by corresponding Dehn twists t_a and t_b is known to be isomorphic to the free group of rank 2. In this paper we extend this result to the case of a nonorientable surface.

1. Introduction

Let N be a smooth, nonorientable, compact surface. We will mainly focus on the local properties of N; hence, we allow N to have some boundary components and/or punctures. Let $\mathcal{H}(N)$ be the group of all diffeomorphisms $h\colon N \to N$ such that h is the identity on each boundary component and h fixes the set of punctures (setwise). By $\mathcal{M}(N)$ we denote the quotient group of $\mathcal{H}(N)$ by the subgroup that comprises the maps isotopic to the identity with an isotopy which fixes the boundary pointwise. $\mathcal{M}(N)$ is known as the *mapping class group* of N. The mapping class group $\mathcal{M}(S)$ of an orientable surface S is defined analogously, but we consider only orientation preserving maps. Usually, we will use the same letter to denote a map and its isotopy class.

Important elements of the mapping class group $\mathcal{M}(S)$ are Dehn twists. Dehn twists generate $\mathcal{M}(S)$; thus, obtaining a good understanding of

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possible relations between them is important. One of the basic results in this direction is the following theorem.

Theorem 1.1 (Ishida [3]). *If a and b are simple closed curves on an orientable surface S such that the geometric intersection number of a and b is greater than 1, then the group generated by Dehn twists t_a and t_b is free of rank 2.*

The main goal of this paper is to extend the above result to the case of a nonorientable surface: see Theorem 13.2. Let us mention that Dan Margalit observed that if we lift the statement of Theorem 13.2 to the oriented double cover S, then we obtain some special cases of the well-known conjecture [4] that two elements of the Torelli subgroup of S either commute or generate a free group. For more details about this correspondence see [6].

The paper is organized as follows. In §2, we establish some basic notation. Section 3 contains some examples that show how the nonorientable case differs from the orientable one. In §4 we recall some language introduced in [5], namely the notion of adjacent and joinable segments. Sections 5, 6, and 8 are devoted to the study of properties of curves in the neighborhood of $a \cup b$. The main theorem of the paper (Theorem 13.2) is proved in §13. This proof is based on five propositions (7.1, 9.12, 10.4, 11.3, and 12.3) that are proved in §7 and sections 9 to 12.

2. Preliminaries

By a *circle* on N we mean an oriented simple closed curve that is disjoint from the boundary of N. Usually, we identify a circle with its image. If two circles a and b intersect, then we always assume that they intersect transversely. According to whether a regular neighborhood of a circle is an annulus or a Möbius strip, we call the circle *two-sided* or *one-sided*, respectively.

We say that a circle is *generic* if it bounds neither a disk with fewer than two punctures nor a Möbius strip without punctures. It is known ([5, Corollary 4.5]) that if N is not a closed Klein bottle, then the circle a is generic if and only if t_a has infinite order in $\mathcal{M}(N)$.

For any two circles a and b we define their *geometric intersection number* as follows:

$$I(a,b) = \inf\{|a' \cap b| \,:\, a' \text{ is isotopic to } a\}.$$

We say that circles a and b *form a bigon* if a disk exists whose boundary is the union of an arc of a and an arc of b. The following proposition provides a useful tool for checking if two circles are in a minimal position (with respect to $|a \cap b|$).

Proposition 2.1 (Epstein [1]). *Let a and b be generic circles on N. Then $|a \cap b| = I(a,b)$ if and only if a and b do not form a bigon.*

3. Disappointing Examples

Let a and b be two circles in an oriented surface S such that $I(a,b) \geq 2$. The key observation that leads to the conclusion that Dehn twists t_a and t_b generate a free group is the following lemma.

Lemma 3.1 ([3, Lemma 2.3]). *Assume that circles $a, b, c \subset S$ satisfy $I(a,b) \geq 2$. Then for any nonzero integer k*

$$I(c,a) > I(c,b) \quad \Longrightarrow \quad I(t_a^k(c), a) < I(t_a^k(c), b).$$

The above lemma allows us to apply the so-called "ping-pong lemma" (Lemma 13.1) and easily conclude that $\langle t_a, t_b \rangle$ is a free group.

Relations between Dehn twists and geometric intersection numbers are known to become more complicated if we allow the surface to be nonorientable. Some results in this direction were obtained in [5], but they were too weak to prove a nonorientable version of the above lemma. The main goal of this section is to show that there is a reason for this condition, namely, Lemma 3.1 is not true on nonorientable surfaces. Moreover, finding general families of counterexamples is possible. Hence, no easy fix seems to exist for this situation (for a nontrivial fix, see propositions 7.1, 9.12, 10.4, 11.3, and 12.3).

Example 3.2. Let a, b, and c be two-sided circles indicated in Figure 1. (Shaded disks are crosscaps; that is, the interiors are to be removed and the boundary points are to be identified by the antipodal map.) In

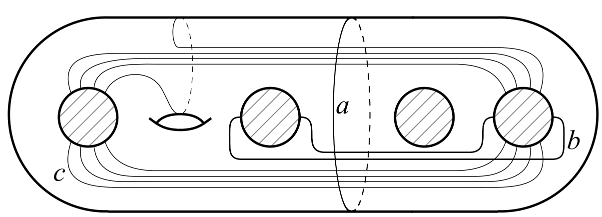

Figure 1. Circles a, b, and c – Example 3.2.

particular, $I(a,b) = 2$ and $I(c,a) = 8 > I(c,b) = 4$. However, checking the following is straightforward:

$$I(t_a(c), a) = 8 > I(t_a(c), b) = 4.$$

The above example can be generalized in the obvious way (by changing c) to the example where $I(a,b) = 2$, $I(c,a) = 2n > I(c,b) = n$, and $I(t_a(c), a) = 2n > I(t_a(c), b) = n$, where $n \geq 1$.

Example 3.3. Let a, b, and c be two-sided circles indicated in Figure 2.

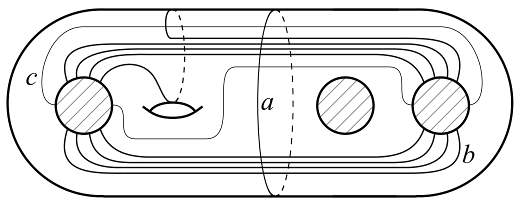

FIGURE 2. Circles a, b, c – Example 3.3.

In particular, $I(a, b) = 8$ and $I(c, a) = 2 > I(c, b) = 1$. However, checking the following is straightforward:

$$I(t_a(c), a) = 2 > I(t_a(c), b) = 1.$$

The above example can be generalized in the obvious way (by changing b) to the example where $I(a, b) = 2n$, $I(c, a) = 2 > I(c, b) = 1$, and $I(t_a(c), a) = 2 > I(t_a(c), b) = 1$, where $n \geq 1$.

The above examples are disappointing, because they show that the geometric intersection number is too weak to notice the action of a twist. Moreover, this situation can happen for arbitrary large complexity [that is, for arbitrary large values of $I(a, b)$ and $I(c, a)$].

Example 3.4. Let a and c be two-sided circles as indicated in Figure 3.

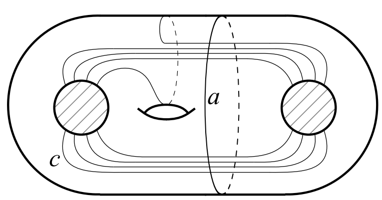

FIGURE 3. Circles a and c – Example 3.4.

The action of t_a on c is trivial because a bounds a Möbius strip; this is the case even though $I(a, c) = 8$ (or in general $I(a, c) = 2n$, $n \geq 1$). Such a situation cannot happen on an oriented surface S; if $I(a, c) > 0$ for some curve c on S, then t_a is automatically nontrivial.

4. Joinable Segments of a and b

For the rest of the paper assume that a and b are two generic two-sided circles in a nonorientable surface N such that $|a \cap b| = I(a, b) \geq 2$.

Following [5], by a *segment* of b (with respect to a), we mean any unoriented arc p of b that satisfies $a \cap p = \partial p$. Similarly, we define an *oriented segment* of b. If p is an oriented segment, then by $-p$, we denote the segment equal to p as an oriented segment but with the opposite orientation.

We call a segment p of b *one-sided* (*two-sided*) if the union of p and an arc of a connecting ∂p is a one-sided (two-sided) circle. An oriented segment is one-sided (two-sided) if the underlying unoriented segment is one-sided (two-sided).

If $P, Q \in a \cap b$ are two intersection points of $a \cap b$ consecutive on b, then by PQ we denote an oriented segment of b with endpoints P and Q. Oriented segments PP' and QQ' of b are called *adjacent* if both are one-sided and an open disk Δ exists on N with the following properties

(1) $\partial\overline{\Delta}$ consists of the segments PP' and QQ' of b and the arcs PQ and $P'Q'$ of a;
(2) Δ is disjoint from $a \cup b$ (Figure 4).

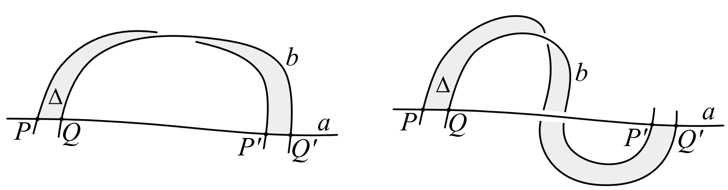

Figure 4. Adjacent segments of b.

Oriented segments $p \neq q$ are called *joinable* if oriented segments $p_1, \ldots, p_k$ exist such that $p_1 = p$, $p_k = q$, and p_i is adjacent to p_{i+1} for $i = 1, \ldots, k-1$ (Figure 5).

Unoriented segments are called adjacent (joinable) if they are adjacent (joinable) as oriented segments for some choice of orientations.

In exactly the same way, we define segments of a (with respect to b) and their properties.

Remark 4.1. The main reason for the importance of adjacent/joinable segments of b is that they provide natural reductions of the intersection points of $t_a(b)$ and b (Figure 5).

In fact, as observed in [5], these segments are the only nontrivial source of such reductions.

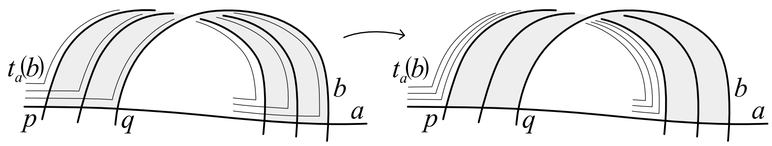

FIGURE 5. Joinable segments of b and reduction of intersection points of b and $t_a(b)$.

Let us recall some basic properties of joinable segments.

Proposition 4.2 ([5, lemmas 3.4, 3.7, and 3.8])**.**

(1) *Initial (terminal) points of oriented joinable segments of b are on the same side of a.*
(2) *Let p and q be oriented segments such that q begins at the terminal point of p (this includes the case $q = -p$). Then p and q are not joinable.*

Still following [5], by a *double segment* of b, we mean an unordered pair of two different oriented segments of b that have the same initial point. Exactly $I(a, b)$ double segments exist, which correspond to intersection points of a and b.

Two double segments are called *joinable* if an oriented segment p exists in the first double segment and q exists in the other such that p and q are joinable.

Studying the action of a twist t_a on a circle b is important in order to obtain some obstructions for possible reductions of intersection points between $t_a(b)$ and b. The basic result in this direction is the following proposition.

Proposition 4.3 ([5, Lemma 3.9])**.** *Suppose $I(a, b) \geq 2$. Then, for each double segment P, a double segment $Q \neq P$ exists, which is not joinable to P.*

5. Curves in the Neighborhood of $a \cup b$

A regular neighborhood $N_{a\cup b}$ of $a \cup b$ is fixed. Topologically, $N_{a\cup b}$ is the union of regular neighborhoods N_a and N_b of a and b, respectively. By changing N_a, N_b, and $N_{a\cup b}$ into their closures, we can assume that all these sets are closed. If we define

$$N_{a\setminus b} = \overline{N_a \setminus N_b},\ N_{b\setminus a} = \overline{N_b \setminus N_a},\ N_{a\cap b} = N_a \cap N_b,$$

then

$$N_{a\cup b} = N_a \cup N_b = N_{a\setminus b} \cup N_{b\setminus a} \cup N_{a\cap b},$$

where each three sets on the right-hand side consist of $I(a,b)$ disks with disjoint interiors. These disks correspond to the intersection points of a and b (Figure 6).

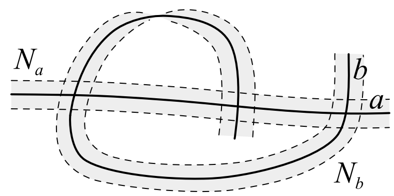

FIGURE 6. Neighborhood of $a \cup b$ as the union of $3 \cdot I(a,b)$ rectangles.

We consider these disks as rectangles with two opposite sides parallel to a and the other two parallel to b. The rectangles in $N_{a\setminus b}$ and $N_{b\setminus a}$ have a one-to-one correspondence with the segments of a and b, respectively.

If r is one of the rectangles that constitutes $N_{a\cup b}$ and c is a circle in $N_{a\cup b}$ that intersects $a \cup b$ transversally, then by an *arc of* $r \cap c$, we mean a connected component of $r \cap c$.

Let $\mathcal{C}$ be the family of generic circles on N that satisfy the following properties:

(1) Each circle in $\mathcal{C}$ is contained in $N_{a\cup b}$ and intersects $a \cup b$ transversally.
(2) Each intersection point of c and $a \cup b$ is contained in $N_{a\cap b}$.
(3) If $c \in \mathcal{C}$ and r is one of the rectangles in $N_{a\cup b}$, then each arc of $c \cap r$ has endpoints on two different sides of r.

The third condition simply means that c does not turn back when crossing a rectangle. Each generic circle contained in $N_{a\cup b}$ is obviously isotopic to a circle in $\mathcal{C}$.

Let $c \in \mathcal{C}$ and let r be one of the rectangles in $N_{a\cup b}$. If an arc of c contained in r crosses both sides of r parallel to a, then we say that r contains an arc of c *parallel* to b.

If every rectangle in N_b contains an arc of c parallel to b, then we say that c *winds around* b. Clearly, the sufficient condition for a circle $c \in \mathcal{C}$ to wind around b is that each rectangle in $N_{a\cap b}$ contains an arc of c parallel to b.

If r is a rectangle in $N_{b\setminus a}$ and r contains an arc q of c parallel to b, then we say that q is a *segment* of c. Moreover, if p is a segment of b that corresponds to r, then we say that q is parallel to p. Similarly, we define segments of c parallel to segments of a.

Lemma 5.1. *Let $c \in \mathcal{C}$ such that c winds around b, and let a' be one of the components of ∂N_a. If $|c \cap a| = I(c,a)$ and Δ is a bigon formed by c and a', then $\Delta \subset N_a$.*

Proof. Suppose that a bigon Δ, which is not contained in N_a, with sides $p \subset a'$ and $q \subset c$ exists. If Δ and N_a are on the same side of p (Figure 7), then we can find a smaller bigon $\Delta' \subset \Delta$ with sides $p' \subset \partial N_a$ and $q' \subset c$

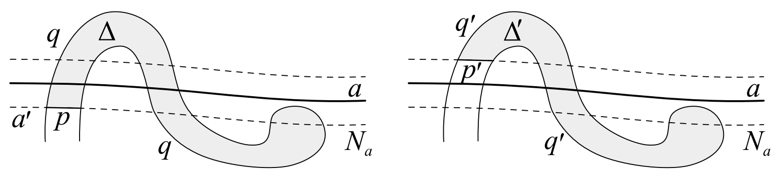

Figure 7. Case of Δ and N_a being on the same side of a' – Lemma 5.1.

such that Δ' and N_a are on different sides of p' (because $\Delta \setminus N_a \neq \emptyset$). Hence, we can assume that Δ and N_a are on different sides of p.

If c intersects the interior of p, then we can pass to a smaller bigon that is still not contained in N_a. Hence, we can assume that $p \cap c$ consists of two points A and B (note that q may still intersect a').

Let r_A and r_B be rectangles of $N_{a\cap b}$ that correspond to A and B, respectively. If $r_A = r_B$, then the segments of q that start at A and B terminate at the same rectangle of $N_{a\cap b}$ (Figure 8). Hence, we can pass

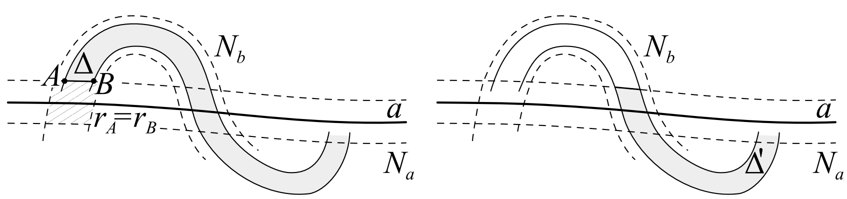

Figure 8. The case of $r_A = r_B$ – Lemma 5.1.

to a smaller bigon $\Delta' \subset \Delta$ by removing these segments of q. The obtained bigon Δ' is still not contained in N_a because this would imply that c is not in $\mathcal{C}$ (c would need to turn back in one of the rectangles of N_a). Hence, we can assume that $r_A \neq r_B$.

Let $c_A, c_B \subset N_a$ be arcs of c that start at A and B, respectively.

Recall that we assumed that c winds around b. Hence, r_A and r_B contain arcs of c that are parallel to b. Therefore, c_A either crosses a in

r_A or c_A turns in r_A in the direction of p, and after running parallel to p, c must turn and cross a in r_B. In fact, c cannot turn towards p and it cannot cross r_B because r_B contains an arc of c parallel to b (Figure 9(i)). Similar analysis applied to c_B shows that the arc of $c \cap N_a$ that

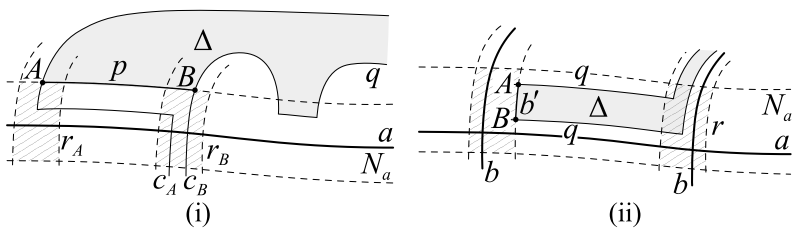

FIGURE 9. Configurations of arcs – lemmas 5.1 and 5.2.

contains c_B must intersect a; it can do it in r_B, or in r_A after running parallel to p. However, this implies that c and a form a bigon, which is a contradiction. $\square$

Lemma 5.2. *Let A and B be the endpoints of an arc b' contained in one of the components of $\partial N_b \cap N_a$. Let $c \in \mathcal{C}$ be such that c winds around a, and let $q \subset c$ be an arc with endpoints A and B which starts and ends on the same side of the component of ∂N_b containing b'. Then b' and q do not form a bigon with interior disjoint from $b \cup c$.*

Proof. Suppose to the contrary that b' and q form a bigon Δ with interior disjoint from $b \cup c$ and consider the arcs of q that start at A and B (Figure 9(ii)). If these arcs enter some rectangle r of $N_{a\cup b}$, then they must be parallel in r; that is, they are disjoint and intersect the same sides of r. Clearly, this condition is true for rectangles in $N_{a\cup b} \setminus N_{a\cap b}$ (since $c \in \mathcal{C}$) and for rectangles in $N_{a\cap b}$; this follows from our assumptions that the interior of Δ is disjoint from $b \cup c$ and that c winds around a.

However, this implies that the arcs of q which start A and B will never meet. $\square$

Let $c \in \mathcal{C}$ and p be one of the arcs of $c \cap N_a$. Four different possible configurations of p exist (Figure 10) and are referred to as types A–D.

Remark 5.3. If c winds around b, then arcs of $c \cap N_a$ of types B–D can pass through only one rectangle in $N_{a\setminus b}$. Otherwise, c would intersect itself.

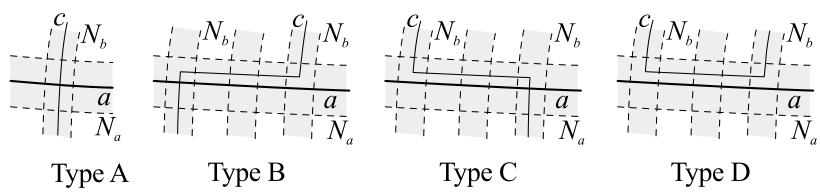

FIGURE 10. Possible configurations of arcs of $c \cap N_a$.

6. Rigidity of Circles in $\mathcal{C}$

Remark 6.1. As we mentioned in Remark 4.1, adjacency between segments of b is the only nontrivial source of reductions of the intersection points between b and $t_a(b)$. However, if we consider the intersection points between b and $t_a(c)$ for $c \in \mathcal{C}$, then other kinds of reductions exist. For example, if Δ is a component of $N \setminus N_{a\cup b}$, which is a disk, then $t_a(c)$ and b may reduce along Δ (Figure 11). As we will see later, this type

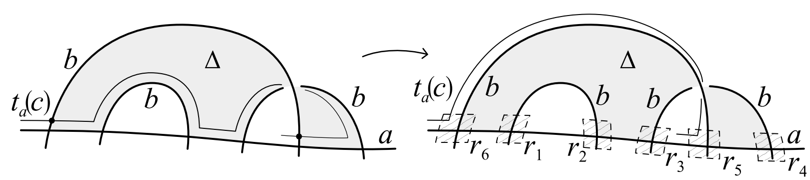

FIGURE 11. Exterior hexagon Δ and the reduction of intersection points between b and $t_a(c)$.

of reduction is rather exceptional, but we need additional definitions to control it.

Suppose that Δ is a component of $N \setminus N_{a\cup b}$, which is a disk. If the boundary of $\overline{\Delta}$ intersects exactly n boundaries of rectangles $r_1, r_2, \ldots, r_n$ in $N_{a\cap b}$, then we say that Δ is an *exterior n-gon* with *vertices* $r_1, r_2, \ldots, r_n$ (Figure 11). Note that $r_1, \ldots, r_n$ does not need to be pairwise distinct ($\overline{\Delta}$ may intersect r_i in each of its four corners).

Let p be an arc of $c \in \mathcal{C}$, which is parallel to b in a rectangle r of $N_{a\cap b}$. Fix some orientation of p and follow p to the rectangle r_1 of $N_{a\cap b}$ following r. We say that p is *rigid* in r with respect to b if p is parallel to b in r_1. Equivalently (from the perspective of p intersecting N_a), p is of type A in r and r_1.

We say that $c \in \mathcal{C}$ *winds strongly* around b if for every rectangle r in $N_{a\cap b}$ an oriented arc p that is parallel to b in r exists such that both p and $-p$ are rigid in r. Equivalently, for each of the three rectangles r_1,

r_2, and r_3 of $N_{a\cap b}$, which are consecutive along b, an arc of $c \cap N_b$ exists, which is parallel to b in r_1, r_2, and r_3. In particular, if a circle $c \in \mathcal{C}$ winds strongly around b, then c winds around b.

Let R be a double segment of b and let $p \neq q$ be oriented segments of b that are not contained in R and do not start at the same point of $a \cap b$. Assume also that p and q start on different sides of a. If we fix an orientation of N_a, then two possible mutual positions of p, q, and R exist (Figure 12). We say that the triple $\{p, q, R\}$ is *positively oriented* if the configuration is as in Figure 12(i).

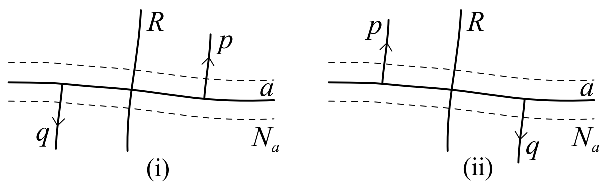

FIGURE 12. Two possible orientations of $\{p, q, R\}$.

As we will see later, some special configurations of generic two-sided circles a and b exist in N, which will require additional analysis (see sections 9–12). These special configurations are defined by the following properties:

(S1) $I(a, b) \geq 4$ and oriented segments p and q of b exist, which start on different sides of a such that each double segment of b contains an oriented segment joinable to p or q (see Figure 38 in §9).

(S2) $I(a, b) \geq 4$ and oriented segments p and q of b, and a double segment R of b exist, such that a and b are not in the special position (S1), p and q start on different sides of a, p starts and terminates on different sides of a, each double segment of b different from R contains an oriented segment joinable to p or q, and $\{p, q, R\}$ is positively oriented (see Figure 38 in §9).

(S3) $I(a, b) \geq 4$ and there are oriented segments p and q of b and a double segment R of b such that p starts and terminates on one side of a, q starts and terminates on the other side of a, each double segment of b different from R contains an oriented segment joinable to p or q, and $\{p, q, R\}$ is positively oriented (see Figure 42 in §10).

If one of the ordered pairs of circles (a, b) or (b, a) satisfies one of the above conditions, then we say that the unordered pair $\{a, b\}$ is *special.*

Let X_a be the set of isotopy classes of circles c in N which satisfy the following conditions:

(1) $c \in \mathcal{C}$,
(2) $I(c,a) = |c \cap a|$ and $I(c,b) = |c \cap b|$,
(3) $I(c,a) < I(c,b)$,
(4) c winds strongly around a.

Similarly, we define X_b by requiring (1)–(2) above and additionally

(3′) $I(c,b) < I(c,a)$,
(4′) c winds strongly around b.

7. The Case of $I(a,b) \geq 4$ and $\{a,b\}$ Is Not Special

The main goal of this section is to prove the following proposition.

Proposition 7.1. *Let a and b be two generic two-sided circles in N such that $I(a,b) \geq 4$ and $\{a,b\}$ is not special. Then for any integer $k \neq 0$ we have*

$$t_a^k(X_b) \subseteq X_a \quad \textit{and} \quad t_b^k(X_a) \subseteq X_b.$$

Proof. Of course proving that $t_a^k(X_b) \subseteq X_a$ is sufficient. No canonical choice exists for the orientation of the neighborhood N_a. However, in our figures, we will assume that t_a^k twists to the right.

Construction of $t_a^k(c)$. Fix a circle $c \in X_b$. It is enough to prove that $t_a^k(c) \in X_a$.

We begin by constructing the circle $d = t_a^k(c)$. Outside N_a and on each arc of $d \cap N_a$ of type D, d is equal to c. For each arc of $c \cap N_a$ of types A–C, d circles $|k|$ times around a. In particular, d winds around a and

$$I(d,a) = |d \cap a| = |c \cap a| = I(c,a)$$
$$|d \cap b| = I(c,a) \cdot I(a,b) \cdot |k|.$$

Now the problem is that d may not be an element of $\mathcal{C}$ and d does not need to be in a minimal position with respect to b.

Before we start to reduce d, observe that if an arc of d enters N_a and turns to the left, then after passing through one rectangle in $N_{a \setminus b}$, it must turn back or leave N_a through the same side of N_a as it entered (Figure 13). In fact, arcs of d turning to the left in N_a came from arcs of $c \cap N_a$ of

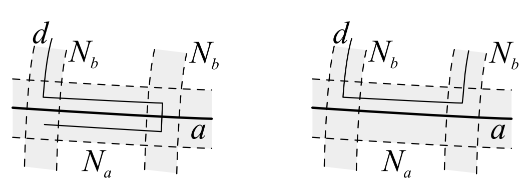

Figure 13. Arcs of d turning to the left in N_a.

types C and D. As we observed in Remark 5.3 such arcs can pass through only one rectangle in $N_{a\setminus b}$.

Reduction of type I. Suppose that one of the rectangles r in $N_{a\cup b}$ contains an arc p of d such that the endpoints of p are on the same side of r (d turns back in r). Clearly, this situation cannot happen for r being one of the rectangles in $N_{b\setminus a}$ (because in such a rectangle, d coincides with c) or r being a rectangle in $N_{a\setminus b}$ (by construction, d runs parallel to a in each such rectangle). Hence, r must be a rectangle in $N_{a\cap b}$ and p must intersect the b-side of r (otherwise, c would not be an element of $\mathcal{C}$). Hence, we have the situation illustrated in Figure 14, and we can replace

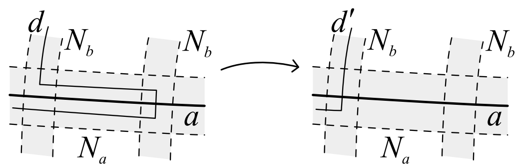

FIGURE 14. Reduction of type I.

d with the circle d' shown in the same figure. In such a case, we say that we reduced d by a *reduction of type I.*

Reductions of type I correspond to arcs of $c\cap N_a$ of type C. Hence, on each arc of $d\cap N_a$, at most one reduction of type I exists.

Let d_1 be the circle obtained from d by performing all possible reductions of type I. In particular, $d_1 \in \mathcal{C}$.

Remark 7.2. The only arcs of $d_1 \cap N_a$ that turn to the left after entering N_a are arcs that correspond to (in fact, are equal to) arcs of $c\cap N_a$ of type D.

We now argue that d_1 winds around a. In fact, if we fix a rectangle r in $N_{a\cap b}$ and r' is another rectangle in $N_{a\cap b}$, then (because c winds around b) r' contains an arc q of c parallel to b (Figure 15). Now $t_a^k(q)$

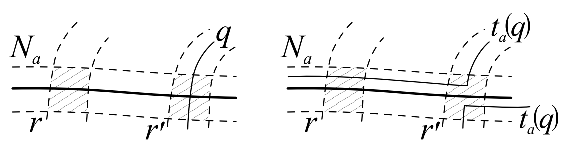

FIGURE 15. Action of t_a on an arc q in a rectangle r'.

is strictly monotone in N_a with respect to a. Hence, this arc does not admit a reduction of type I. In particular, $t_a^k(q)$ gives an arc of d_1, which is parallel to a in r.

For further reference, note the following observation.

Lemma 7.3. *Let $q \subset d_1$ be an arc with endpoints $A, B \in \partial N_a$ such that $q \cap b = \emptyset$. Assume also that q intersects both $N_{a\setminus b}$ and $N_{b\setminus a}$. Then q is an arc of c.*

Proof. By construction, d_1 coincides with c in every rectangle of $N_{b\setminus a}$ and if q enters N_a in a rectangle r (Figure 16), then q must leave r through

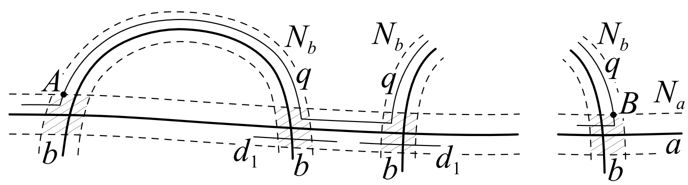

FIGURE 16. Configuration of arcs – Lemma 7.3.

the b-side of r given that d_1 winds around a. Afterwards, q goes through one rectangle of $N_{a\setminus b}$, and then it must turn and leave N_a (because q cannot intersect b). Hence, each arc of $q \cap N_a$ is an arc of type D, which proves that q is an arc of c. □

Reduction of type II. Suppose that there exist arcs p and q of b and d_1, respectively, such that

- p and q form a bigon with interior disjoint from $b \cup d_1$,
- $p \setminus N_{a\cap b}$ is a subarc of a two-sided segment of b.

In such a case, we can remove the bigon formed by p and q (Figure 17), and we say that we reduced d_1 to d_1' by a *reduction of type II.*

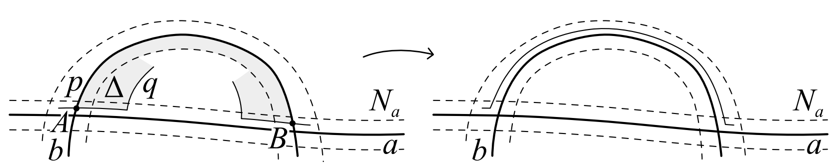

FIGURE 17. Reduction of II.

Let us describe the possible reductions of type II in more detail. Let A and B be vertices of the bigon Δ formed by p and q, and let r_A and r_B be the rectangles of $N_{a\cap b}$ that contain A and B, respectively. By Lemma 5.2, the arc $q_A \subset q$, which connects A with the boundary of N_a, is either entirely contained in r_A or it passes through one rectangle of $N_{a\setminus b}$ and then leaves N_a (it can pass at most one rectangle of $N_{a\setminus b}$ because otherwise it would intersect b). In other words, the situation illustrated in Figure 18 is not possible. The same is true for the arc $q_B \subset q$, which

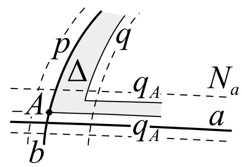

FIGURE 18. Types II and III reductions – impossible configuration of arcs.

connects B with the boundary of N_a. For the same reason, either both arcs q_A and q_B are entirely contained in r_A and r_B, or each of them passes through one rectangle of $N_{a\setminus b}$.

Given that we assume that p corresponds to a two-sided segment of b, at one endpoint of q, say B, q turns to the left as it enters N_a. We claim that q consists of q_A, q_B, and a single segment of c parallel to b. In order to prove this statement, showing that q_B is entirely contained in r_B is sufficient. Suppose to the contrary that q_B passes through a rectangle of $N_{a\setminus b}$ (Figure 19). Given that q_B turns to the left after entering N_a, by

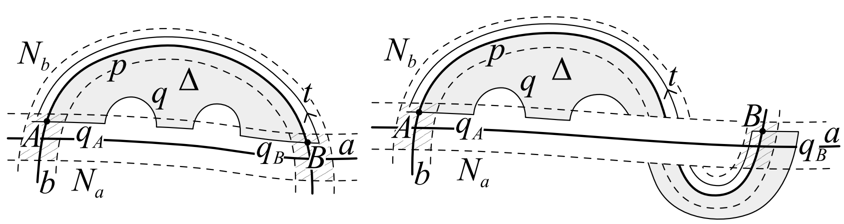

FIGURE 19. Reduction of type II – impossible configurations of arcs.

Remark 7.2, this arc must be an arc of type D. Hence, after crossing p in B it must follow an arc t of d_1, which turns left in r_B and leaves N_a (it must leave N_a in r_B by Remark 5.3). After leaving N_a, t runs parallel to p in a rectangle of $N_{b\setminus a}$. In particular, q_B and t are arcs of c (given that $(q_B \cup t) \cap N_a$ is an arc of type D). Moreover, by Lemma 7.3, $q \setminus (q_A \cup q_B)$ is also an arc of c. Hence, $(q \setminus q_A) \cup t \subset c$ and an arc of ∂N_a form a bigon Δ' not contained in N_a (Figure 20). This finding contradicts Lemma 5.1.

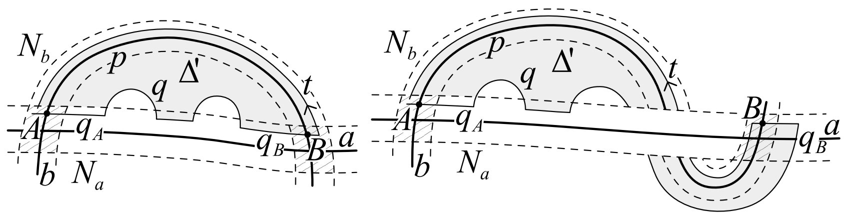

FIGURE 20. Reduction of type II – extending Δ to Δ'.

Therefore, we proved that if d_1' and d_1 differ by a reduction of type II that corresponds to the bigon formed by arcs p and q of b and d_1, respectively, then p and q are in the same rectangle of $N_{b\setminus a}$, and q is parallel to p. This condition means that d_1 and d_1' intersect rectangles in $N_{a\cup b}$ in exactly the same way. Hence, $d_1' \in \mathcal{C}$ and d_1' winds around a.

Let d_2 be the circle obtained from d_1 by performing all possible reductions of type II. As we observed, $d_2 \in \mathcal{C}$ and d_2 winds around a.

Remark 7.4. In the following, we use Lemma 7.3 with d_2 instead of d_1. This approach is somewhat problematic, because some arcs of d_1 that satisfy the assumptions of that lemma may admit reductions of type II. To solve this problem, we mimic reductions of type II on the level of c. To be more precise, if q is an arc of $c \cap N_{b\setminus a}$ which as an arc of d_1 admits a reduction of type II, then we push q so that it coincides with the corresponding arc of d_2 (Figure 21). Then we extend this push to the

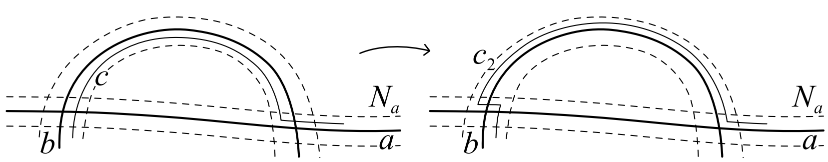

FIGURE 21. Applying reductions of type II to c.

isotopy of c. If c_2 is a circle obtained from c by all possible reductions of type II (in the sense described above), then c_2 still winds around b.

Hence, it satisfies the assumptions of Lemma 5.1. Moreover, we have the following replacement for Lemma 7.3.

Lemma 7.5. *Let $q \subset d_2$ be an arc with endpoints $A, B \in \partial N_a$ such that $q \cap b = \emptyset$. Assume also that q intersects both $N_{a\setminus b}$ and $N_{b\setminus a}$. Then q is an arc of c_2.*

Reduction of type III. Suppose that there exist arcs p and q of b and d_2, respectively, such that

- p and q form a bigon with interior disjoint from $b \cup d_2$,
- $p \setminus N_{a\cap b}$ is a subarc of a one-sided segment of b.

In such a case we can remove the bigon formed by p and q; see Figure 22. We say that we reduced d_2 to d_2' by a *reduction of type III.*

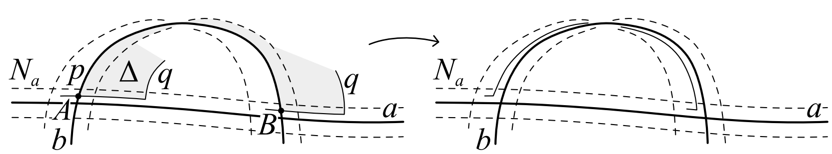

FIGURE 22. Reduction of type III.

Let us attempt to understand reductions of type III in further detail.

Let A and B be the vertices of the bigon Δ formed by p and q, and let r_A and r_B be the rectangles of $N_{a\cap b}$ that contain A and B, respectively. By Lemma 5.2, the arc $q_A \subset q$ that connects A with the boundary of N_a is either entirely contained in r_A or it passes through one rectangle of $N_{a\setminus b}$ and then leaves N_a (it can pass through at most one rectangle of $N_{a\setminus b}$ because otherwise it would intersect b). In other words, the situation illustrated in Figure 18 is not possible. The same is true for the arc $q_B \subset q$, which connects B with the boundary of N_a. For the same reason, either both arcs q_A and q_B are entirely contained in r_A and r_B, or each arc passes through one rectangle of $N_{a\setminus b}$.

If $q \setminus (q_A \cup q_B)$ intersects N_a, then by Lemma 7.5, this is an arc of c_2 (see Remark 7.4). In particular, all components of $(q \setminus (q_A \cup q_B)) \cap N_a$ are arcs of type D (on arcs of other types, d_2 does not coincide with c_2).

Observe also that q turns to the right when it enters N_a (because q is one-sided, it enters N_a in the same way on both ends). In fact, q_A and q_B would be arcs of type D otherwise (Remark 7.2), which are not involved in reductions of type II. Hence, q_A and q_B would be arcs of c (Figure 23). Therefore, the whole q would be an arc of c, and that would imply that c and b form a bigon.

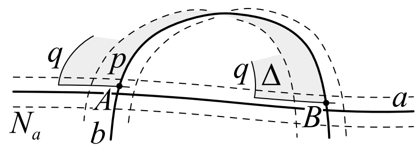

FIGURE 23. Reduction of type III – impossible configuration of arcs.

Remark 7.6. Observe that $d_2' \in \mathcal{C}$. In fact, if d_2' turns back in one of the rectangles r of $N_{a\cup b}$, then r must be a rectangle that contains one of the vertices of the bigon, which leads to the reduction. (In all other rectangles, we either did not change anything, or d_2' runs parallel to b in them.) Since d_2' enters r through the a-side, after turning back, it must leave r also through the a-side. Hence, we have the situation shown in the right-hand side of Figure 24. The left-hand side of the same figure shows how the situation looked before the reduction.

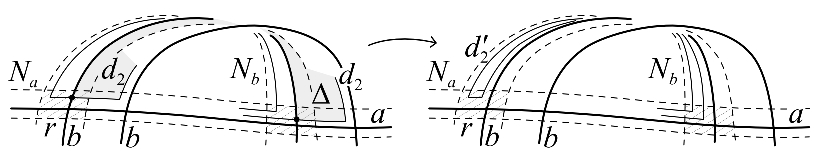

FIGURE 24. Reduction of type III – impossible configuration of arcs.

Let t be an arc of d_2 following q past the intersection point $p \cap q \cap r$. Arc t turns right in r and leaves r as an arc parallel to p; see Figure 25. The same figure shows how the reduction disk Δ can be deformed to the

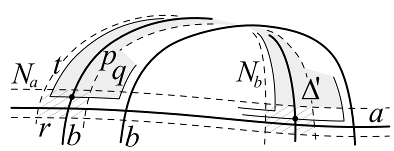

FIGURE 25. Reduction of type III – impossible configuration of arcs.

closed disk Δ' with boundary composed of q, t, and an arc of ∂N_a. Note that $q \cap \Delta'$ is an arc of c_2 by Lemma 7.5 and t is an arc of c_2 because d_2

intersects r as an arc of type D. Therefore, the existence of Δ' contradicts Lemma 5.1. Hence, we proved that $d_2' \in \mathcal{C}$ (in particular, d_2' does not admit reductions of type I).

Remark 7.7. Reduction of type III does not create any new arcs of $d_2' \cap N_a$ which turn to the left after entering N_a. This reduction does not affect the segments of d_2' which run parallel to two-sided segments of b. Hence, d_2' does not admit reductions of type II.

Before we go further, we divide possible reductions of type III into three subtypes. Let p and q be arcs of b and d_2, respectively, which define a reduction of type III.

- If p and q enter N_a in the same rectangles of $N_{a\cap b}$, then we say that p and q define a *reduction of type IIIa*.
- If p and q enter N_a in different rectangles of $N_{a\cap b}$ and q meets a single rectangle of $N_{b\setminus a}$, then we say that p and q define a *reduction of type IIIc*.
- Otherwise, we say that p and q define a *reduction of type IIIb*.

Reduction of type IIIa. Let d_3 be the circle obtained from d_2 by performing all possible reductions of type IIIa. As we already observed, $d_3 \in \mathcal{C}$ and d_3 does not admit reductions of types I, II, and IIIa (remarks 7.6 and 7.7). Moreover, d_3 intersects the rectangles of $N_{a\cup b}$ in exactly the same way as d_2; hence, d_3 winds around a.

Remark 7.8. We now follow Remark 7.4 to obtain a version of Lemma 7.3 for d_3. As in the construction of c_2, we mimic the reductions of type IIIa on c_2 (Figure 26). However, we do not need to mimic all reductions of

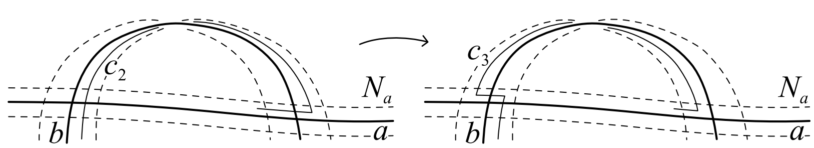

FIGURE 26. Applying reductions of type IIIa to c_2.

type IIIa. To be more precise, we apply to c_2 only these reductions of type IIIa, which are determined by an arc q of c_2, which enters N_a on one side as an arc of type D (q cannot enter N_a on both sides as an arc of type D; this condition would imply that c and b bound a bigon). If q is an arc of c that enters N_a on both sides as an arc of types A–C, then the corresponding arc of d_3 does not satisfy the assumptions of Lemma

7.3 (on both ends, it intersects b). Hence, we do not consider these arcs as they do not mimic the setup of Lemma 7.3.

Let c_3 be the arc obtained from c_2 by reductions of type IIIa described above. Since c_3 still winds around b, it satisfies the assumptions of Lemma 5.1. As a replacement for Lemma 7.3, we have the following lemma.

Lemma 7.9. *Let $q \subset d_3$ be an arc with endpoints $A, B \in \partial N_a$ such that $q \cap b = \emptyset$. Assume also that q intersects both $N_{a\setminus b}$ and $N_{b\setminus a}$. Then, q is an arc of c_3.*

Reduction of type IIIb. Let d_3' be the circle obtained from d_3 by a single reduction of type IIIb. As in the general definition of the reduction of type III, by A and B we denote vertices of the bigon formed by p and q; r_A and r_B are rectangles of $N_{a\cap b}$ that contain A and B, respectively; and q_A and q_B are arcs of q that connect A and B with the boundary of N_a.

Remark 7.10. By definition, $q \setminus (q_A \cup q_B)$ is not a single segment of d_3 parallel to b. Hence, it intersects N_a at least once. By Lemma 7.9, $q \setminus (q_A \cup q_B)$ is an arc of c_3, so it intersects N_a only in arcs of type D (on arcs of other types, d_3 does not coincide with c_3).

The existence of a bigon with vertices A and B between d_3 and b implies that q and p bound an exterior n-gon Δ (see §6 and the right-hand side of Figure 27). Moreover, if the reduction is not of type IIIc, then $n \geq 6$.

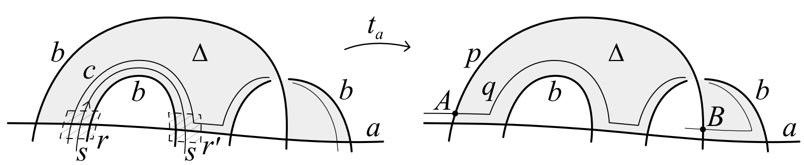

FIGURE 27. Exterior hexagon Δ and the reduction of intersection points between b and $t_a(c)$.

In such a case, an oriented arc s of c which is of type A in a rectangle r of $N_{a\cap b}$ (see the left-hand side of Figure 27) and which is rigid in r with respect to b cannot yield an arc of d_3 which allows a reduction of type IIIb (because if we follow s to the next rectangle r' of $N_{a\cap b}$, then s is of type A in r' and not of type D).

Remark 7.11. Our assumption that c winds strongly around b (see §6) implies that for each rectangle r of $N_{a\cap b}$, an arc s of c exists which is of type A in r and such that s yields an arc of d_3, which does not allow reductions of type IIIb on either side of r; for such s, an arc of c that is rigid on both sides of r should be selected.

Lemma 7.12. *If s' is an arc of c of type A in a rectangle r of $N_{a\cap b}$, then s' can yield a reduction of type IIIb only on one side of r.*

Proof. Suppose to the contrary that s' yields an arc s of d_3 which allows reductions of types IIIb on both sides of r and assume that r_1, r, and r_2 are rectangles of $N_{a\cap b}$ that are consecutive along b (Figure 28). By

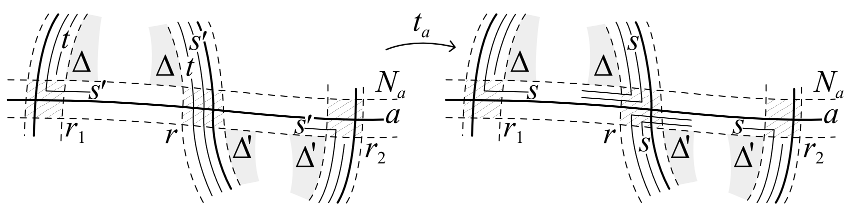

FIGURE 28. Configuration of arcs – Lemma 7.12.

Remark 7.10, s coincides with s' in r_1 and r_2 and enters each of these rectangles as an arc of type D. However, this condition implies that each arc t of c that is of type A in r must enter either r_1 or r_2 as an arc of type D (because it must follow s' along the boundary of one of the exterior n-gons leading to reductions of s). This contradicts our assumption that c winds strongly around b. □

Lemma 7.13. *The arcs of $c \cap N_a$ that correspond to q_A and q_B are arcs of types A or B.*

Proof. If, for example, q_A came from an arc of c of type D, then by Lemma 7.9, $q \setminus q_B$ is an arc of c_3 and if we follow this arc past the point A, we obtain an arc of c_3 which, together with an arc of ∂N_a, bounds a disk Δ that is not contained in N_a (Figure 29(i)); this contradicts Lemma 5.1.

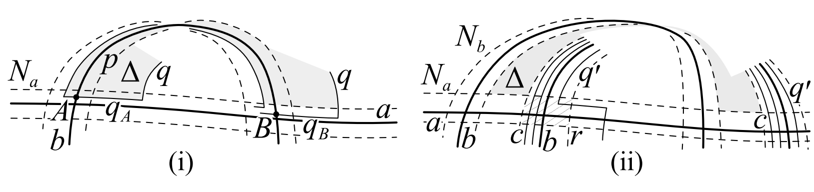

FIGURE 29. Reduction of type IIIb – impossible configurations of arcs.

As for arcs of type C, if, for example, $q \setminus q_B$ was formed from the arc q' of c of type C and r is the rectangle of $N_{a\cap b}$ in which q_A meets ∂N_a, then all the arcs of c that are parallel to b in r are on one side of q' in r and they are bounded along the boundary of Δ by q' (Figure 29(ii)). Hence, all these arcs must lead to arcs of d_3, which allow a reduction of type IIIb across Δ. In particular, if we follow these arcs along the boundary of Δ, then they enter N_a as arcs of type D. However, this contradicts our assumption that arcs that are rigid with respect to b exist on both sides of r (because c winds strongly along b).

Hence, we proved that q_A and q_B were formed from arcs of c which are of types A or B in r_A and r_B. □

Note that the distinction between types B and C of arcs of $c \cap N_a$ is a consequence of our assumption that t_a twists to the right in N_a.

Remark 7.14. As a consequence of Lemma 7.13, if q' is an arc of d_3' obtained from q by a reduction of type IIIb, then the arcs of $d_3' \cap N_a$ that follow q' (on each end) are not arcs of type D (because they intersect a). Hence, q' is not involved in any further reductions of d_3' of type IIIb (Remark 7.10). In other words, no cascade reductions of type IIIb exist; each arc of $d_3 \cap N_{b\setminus a}$ is involved in at most one such reduction. However, q' may be further reduced by reductions of type IIIc. We postpone this problem to the analysis of reductions of type IIIc.

Remark 7.15. If q' is as in the previous remark (that is, q' is an arc of d_3' obtained from q by a reduction of type IIIb), then q' does not admit a reduction of type IIIa. This conclusion follows because the arcs of $d_3' \cap N_a$ that follow q' (on each end) do not intersect b in rectangles of $N_{a\cap b}$ in which they enter N_a (Figure 22). The only arc of $d_3' \cap N_{b\setminus a}$ in which d_3' differs from d_3 is q'. Thus, no new reductions of type IIIa exist on other arcs of $d_3' \cap N_{b\setminus a}$.

Let d_4 be the circle obtained from d_3 by performing all possible reductions of type IIIb. As we observed in the general analysis of reductions of type III, $d_4 \in \mathcal{C}$ (Remark 7.6) and d_4 does not admit any reductions of types I and II (remarks 7.6 and 7.7). We also proved that d_4 does not admit reductions of type IIIa (Remark 7.15).

Remark 7.16. A segment q of d_4 obtained by a reduction of type IIIb cannot enter N_a as an arc of type D (Lemma 7.13). We will show later (Lemma 7.20) that in such a case, arcs that follow/precede q in N_a must intersect b. Hence, Lemma 7.9 remains valid with d_3 replaced by d_4 (arcs modified by a reduction of type IIIb cannot satisfy assumptions of that lemma).

Reduction of type IIIc. Let d_4' be the circle obtained from d_4 by a single reduction of type IIIc. As in the general definition of the reduction of type III, by A and B we denote vertices of the bigon formed by p and q; r_A and r_B are rectangles of $N_{a\cap b}$ that contain A and B, respectively; and q_A and q_B are arcs of q that connect A and B with the boundary of N_a.

Remark 7.17. Reduction of type IIIc may be considered a special case of a reduction of type IIIb, where the exterior disk is a rectangle. For this reason, these two types of reductions have some common properties:

- the arcs of c that correspond to q_A and q_B cannot be of type D (Lemma 7.13);
- if an arc q' is an arc of d_4' obtained by a reduction of type IIIc, then q' is not involved in reductions of type IIIa or IIIb (remarks 7.14 and 7.15);
- d_4' does not admit reductions of type IIIa or IIIb.

Because the proofs of the above properties can be copied verbatim from the analysis of the reduction of type IIIb, we skip the proofs.

Let d_5 be the circle obtained from d_4 by performing all possible reductions of type IIIc. As we observed in the general analysis of reductions of type III, $d_5 \in \mathcal{C}$ (Remark 7.6) and d_5 does not admit any reductions of types I and II (Remark 7.7). As we noted above (Remark 7.17), d_5 also does not admit reductions of types IIIa and IIIb.

Remark 7.18. As in Remark 7.16, a segment q of d_5 obtained by a reduction of type IIIc cannot enter N_a as an arc of type D. We will show later (Lemma 7.20) that in such a case, arcs that follow/precede q in N_a must intersect b. Hence, Lemma 7.9 remains valid with d_3 replaced by d_5. (Arcs modified by a reduction of type IIIc cannot satisfy assumptions of that lemma).

Remark 7.19. An arc s of $d_4 \cap N_a$ can be involved in multiple reductions of type IIIc (Figure 30); that is, the arcs that start at the end-points of s can be involved in several reductions of type IIIc. However, all these reductions can change the initial and the terminal rectangles r_1 and r_2 of s only to the rectangles joinable to r_1 and r_2, respectively (see §4).

Lemma 7.20. *Let s' be an arc of $c \cap N_a$ of type A, B, or C, and let s be the arc of $d_5 \cap N_a$ which corresponds to s'. Then s is parallel to a in at least one rectangle of $N_{a\cap b}$.*

Proof. Let s'' be the arc of d_3 that corresponds to s' (that is, s'' is obtained from $t_a^k(s')$ by reductions of types I, II, and IIIa), and let p and q be

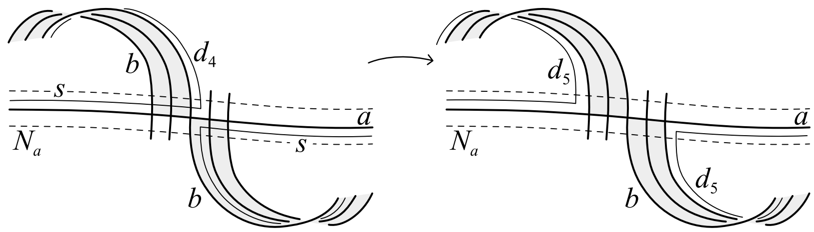

FIGURE 30. Cascade reductions of type IIIc.

segments of b that correspond to the segments of d_5 that start at the endpoints of s.

Assume first that s' is an arc of type C. Arcs of this type do not allow reductions of type IIIb (Lemma 7.13). Hence, the only type of reductions that could decrease the number of rectangles in which s'' is parallel to a is the reduction of type IIIc. If we assume that no rectangles of $N_{a\cap b}$ exist in which s is parallel to a, then each double segment of b must contain a segment joinable either to p or q (Remark 7.19). However, this implies that a and b are in the special position (S1), which contradicts our assumption that $\{a, b\}$ is not special.

If s' is an arc of type A or B, then the situation is completely analogous. If s' is of type A, then s'' can admit on one side one reduction of type IIIb (Lemma 7.12), and if s' is of type B, then s'' can admit reductions of type IIIb on both sides (one on each side – Remark 7.14). Hence, the assumption that no rectangle of $N_{a\cap b}$ exists in which s is parallel to a implies that a and b are in the special position (S1). □

Strong winding of d_5. We need to show that for each three rectangles r_1, r, and r_2 of $N_{a\cap b}$, which are consecutive along a, an arc of d_5 exists, which is parallel to a in r_1, r, and r_2. Without loss of generality, we can assume that the configuration of rectangles is as in Figure 31; that is, r_2

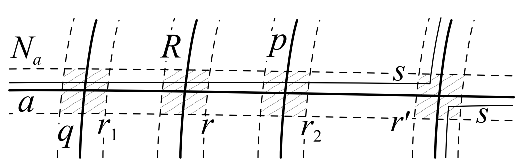

FIGURE 31. Intersection of $t_a^k(s')$ with rectangles of $N_{a\cap b}$.

is on the right of r. Let p and q be segments of b such that p goes up from r_2 and q goes down from r_1, and let R be the double segment of b that corresponds to r.

Let r' be any rectangle of $N_{a\cap b}$ that is different from r_1, r, and r_2 (such rectangles exist given that $I(a,b) \geq 4$), and let s' be an arc of $c \cap N_{a\cap b}$ that is rigid on both sides of r' (it exists because c winds strongly around b). If s is an arc of d_3 that corresponds to s', then s does not allow reductions of type IIIb on either side of r'. Hence, it can only be reduced by reductions of type IIIc. By Remark 7.19, if we assume that s can be reduced so that the corresponding arc of d_5 is not parallel to a in r_1 or r_2, then one of the segments of b that starts in r' must be joinable to either p or q.

From the above analysis, if we assume that each arc s' of $c \cap N_{a\cap b}$ leads to an arc s of $d_5 \cap N_a$ that is not parallel to a in either r_1 or r_2, then each double segment of b different from R is joinable either to p or q. Moreover, the triple $\{p, q, R\}$ is positively oriented (see §6). Hence, we are in the special position (S2) or (S3), which is a contradiction.

Bigons formed by d_5 and b. Let us prove that d_5 and b are in the minimal position so they do not form any bigon. Suppose on the contrary that Δ is a bigon with vertices A and B bounded by arcs p and q of d_5 and b, respectively. By taking the innermost bigon, we can assume that the interior of Δ is disjoint from $d_5 \cup b$.

Since d_5 winds around a, each rectangle of $N_{a\cap b}$ contains an intersection point of d_5 and b. Hence, q is either an arc of b in a single rectangle $r_{A,B}$ in $N_{a\cap b}$, or q is a segment of b that connects two different rectangles r_A and r_B of $N_{a\cap b}$. In the second case, d_5 would admit a reduction of type II or III (depending on whether q is two-sided or not). Hence, we concentrate on the first possibility. If Δ is contained in N_a, then d_5 admits a reduction of type I, which is not possible. Hence, Δ is not contained in N_a.

Let p' be the subarc of p, which is obtained from p by removing the arcs contained in N_a that connect A and B with the boundary of N_a (Figure 32). By Lemma 7.9 and Remark 7.18, the arc p' of d_5 is in fact

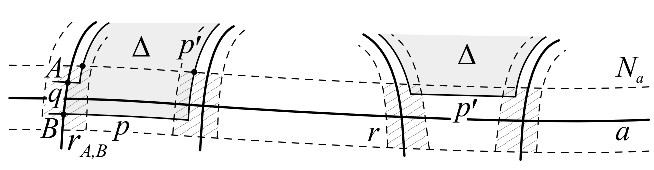

FIGURE 32. Bigon Δ between b and d_5.

an arc of c_3. Hence, the existence of Δ contradicts Lemma 5.1

Counting intersection points between d_5 and b. To finish the proof, we need to show that $I(d_5, b) > I(d_5, a)$. The idea is to show that associated intersection points of $d_5 \cap b$ exist for each intersection point of $d_5 \cap a$.

These arcs of $d \cap N_a$ that intersect a have a one-to-one correspondence to arcs of $c \cap N_a$ that intersect a, hence to arcs of types A–C. Moreover, all reductions we performed on d preserved this bijection because, during the reductions, we did not create any new arcs that intersect a and we did not remove any of the existing ones.

By Lemma 7.20, each arc s of $d_5 \cap N_a$ that corresponds to an arc s' of $c \cap N_a$ of type A–C is parallel to a in at least one rectangle of $N_{a \cap b}$. Hence, it intersects b at least once. Moreover, because d_5 winds strongly around a, some arcs s of $d_5 \cap N_a$ intersect b in at least three points. Therefore $I(d_5, b) > I(d_5, a)$. □

8. Weak Rigidity

The proof of Proposition 7.1 is based on the notion of strong winding: For each three rectangles r_1, r_2, r_3 that are consecutive on a, an arc of $c \cap N_a$ exists, which leads to an arc of $d_5 \cap N_a$ parallel to a in r_1, r_2, and r_3.

This assertion can fail in special cases (S1)–(S3).

Example 8.1. Let a and b be two circles that are in the special position (S1), as shown in Figure 33(i). Segments p_1 and p_2 are adjacent; thus,

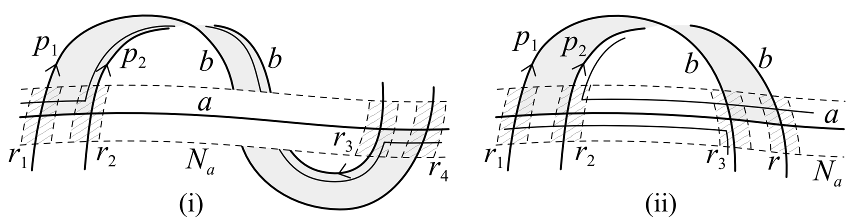

Figure 33. Possible failure in the proof of Proposition 7.1 – Examples 8.1 and 8.2.

the arcs of d, which are obtained from arcs of c of type A in the rectangle r_2, may admit a reduction of type IIIc. Hence, none of these arcs may be parallel to a in r_1. Similarly, all arcs of d that are obtained from arcs of c of type A in r_3 may reduce by reductions of type IIIc. Hence, none of these arcs may be parallel to a in r_4. Therefore, a possibility is that no arc of $d_5 \cap N_a$ which is parallel to a in r_1 and r_4 exists (hence, d_5 does not wind strongly around a).

Example 8.2. Let a and b be two circles that are in the special position (S3) as shown in Figure 33(ii). Segments p_1 and p_2 are adjacent. Thus, the arcs of $c \cap N_a$ which are of type A in r_1 or r_2 give arcs of d that can be reduced so that the resulting arcs of $d_5 \cap N_a$ are not parallel to a in r_1. Similarly, arcs of $c \cap N_a$ that are of type A in r_3 or r yield arcs of $d_5 \cap N_a$ that are not parallel to a in r_3. Therefore, a possibility is that no arc of $d_5 \cap N_a$ which is parallel to a in r_1 and r_3 exists (hence, d_5 does not wind strongly around a).

To overcome the abovementioned problems, we redefine the rigidity of arcs slightly.

Let r_1, r_2, r_3, r_4, r_5, and r_6 be rectangles of $N_{a\cap b}$ that are consecutive vertices of an exterior hexagon Δ (consecutive here means consecutive along $\partial\Delta$). Suppose also that q is an arc of a circle $c \in \mathcal{C}$ as in Figure 35(i); that is, q is of type A in r_1 and r_4, q is of type D between r_2 and r_3, and the segment of b that connects r_5 and r_6 is one-sided. In such a case we say that q is a *one-sided boundary 3-segment* of Δ.

Remark 8.3. Let Δ be an exterior hexagon and let q be an arc of c, which leads to an arc q' of d_3 and allows a reduction of type IIIb across Δ (Figure 35(i)). According to the definition of the reduction of type IIIb, if q enters N_a as an arc of type A (on both ends), then q is a one-sided boundary 3-segment of Δ.

Let q be an arc of $c \in \mathcal{C}$ that is parallel to b in a rectangle r_1 of $N_{a\cap b}$. Some orientation of q is fixed, and q is followed to the rectangles r_2, r_3, and r_4 of $N_{a\cap b}$ following r_1. We say that q is *weakly rigid* in r_1 with respect to b if either

- q is rigid with respect to b in r_1, or
- q does not intersect a in r_2 and r_3, q is parallel to b in r_4, and q is not a one-sided boundary 3-segment of an exterior hexagon (Figure 34).

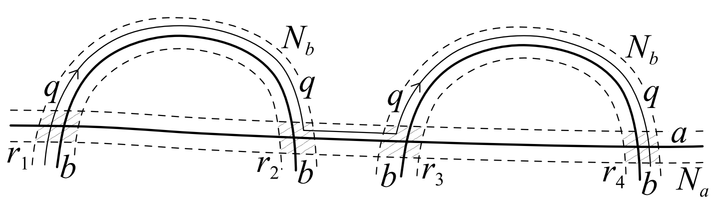

FIGURE 34. Arc q of c which is weakly rigid in a rectangle r_1 of $N_{a\cap b}$.

If c winds around b, then equivalently (from the perspective of q intersecting N_a), q is of type A in r_1 and then either q is of type A in r_2 or q is of type D between r_2 and r_3; q is of type A in r_4; and q is not a one-sided boundary 3-segment of an exterior hexagon.

The following lemma shows that from the point of view of the proof of Proposition 7.1, weakly rigid arcs are as good as rigid arcs.

Lemma 8.4. *Suppose that $c \in \mathcal{C}$ winds around b and let q be an oriented arc of c which is weakly rigid in a rectangle r_1 of $N_{a\cap b}$. Then $t_a^k(q)$ does not admit a reduction of type IIIb.*

Proof. Suppose that q, after leaving r_1, goes around the boundary of an exterior n-gon Δ and then enters N_a in a rectangle r_4 of $N_{a\cap b}$ as an arc of type A (Figure 35(i)). If q leads to an arc that admits a reduction of

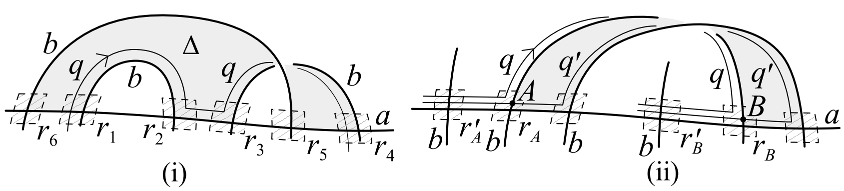

FIGURE 35. Reductions of types IIIb and IIIc.

type IIIb, then q between r_1 and r_4 must follow $\frac{n}{2}-2$ rectangles of $N_{a\setminus b}$. Hence, it must $\frac{n}{2}-2$ times enter N_a as an arc of type D. However, by the definition of weak rigidity, q can intersect N_a as an arc of type D only once. Hence, $n=6$, and by Remark 8.3, q is a one-sided boundary 3-segment of Δ. However, this contradicts the assumption that q is weakly rigid in r_1. $\square$

The next lemma shows that the reductions of types IIIb and IIIc produce arcs that are good candidates for weakly rigid arcs.

Let c, d_3, and d_5 be as in the proof of Proposition 7.1; that is, $c \in \mathcal{C}$ winds around b, d_3 is obtained from $t_a^k(c)$ by all possible reductions of types I–IIIa, and d_5 is obtained from d_3 by all possible reductions of types IIIb and IIIc. Assume also that c is such that the statements of Lemma 7.13 and Remark 7.17 hold true.

Lemma 8.5. *Suppose that each arc of $c \cap N_a$ of types A–C gives an arc of $d_5 \cap N_a$ that is parallel to a in at least one rectangle of $N_{a\cap b}$. Let q be an oriented arc of d_5 which starts in a rectangle t of $N_{a\cap b}$ as an arc parallel to a; then it follows a to the next rectangle of $N_{a\cap b}$ and then it* *follows an arc obtained from an arc of* $d_3 \setminus N_a$ *by reductions of types IIIb and/or IIIc (Figure 35(ii)). If* q *is not a one-sided boundary 3-segment of an exterior hexagon, then* q *is weakly rigid in* t *with respect to* a*.*

Proof. Let q' be an arc of d_3, which can be reduced to the arc q of d_5 by reductions of types IIIb and/or IIIc. Assume that the endpoints of q' are $A, B \in d_3 \cap b$ and let r_A and r_B be rectangles of $N_{a\cap b}$ that contain A and B, respectively. By the general properties of reductions of type III (see Lemma 7.13 and Remark 7.17), both ends of q' were formed from the arcs of c intersecting a (hence, not arcs of type D). Therefore, by our assumption, both ends of q after entering N_a must run parallel to a in at least one rectangle of $N_{a\cap b}$. Orient q from r_A to r_B and extend q on both ends so that it starts and terminates in rectangles r'_A and r'_B of $N_{a\cap b}$, which, respectively, precede and follow r_A and r_B along q. From the point of view of q intersecting N_b, this arc is of type A in r'_A; then it intersects N_b as an arc of type D, and then it is again an arc of type A (in r'_B). This finding means that if this arc is not a one-sided boundary 3-segment of an exterior hexagon, then q is weakly rigid in r'_A. □

We say that $c \in \mathcal{C}$ is *weakly rigid* with respect to b if each arc p of c that is parallel to b in a rectangle r of $N_{a\cap b}$ is weakly rigid with some choice of orientation.

Remark 8.6. If c is weakly rigid with respect to b, then by Lemma 8.4, c satisfies the statement of Lemma 7.12. Hence, the assumption that c is weakly rigid with respect to b serves as a replacement for Lemma 7.12.

We say that $c \in \mathcal{C}$ *winds weakly* around b if, for every rectangle r of $N_{a\cap b}$ and each choice of orientation for the collection P of arcs of c that are parallel to b in r, a weakly rigid arc exists in P.

Remark 8.7. In the proof of Proposition 7.1, the assumption that c winds strongly around b was used to conclude that in each rectangle r of $N_{a\cap b}$ an arc s of c exists, which is of type A in r and which leads to an arc of $d_3 \cap N_a$ that does not allow a reduction of type IIIb on either side of r (Remark 7.11). By Lemma 8.4, weak winding provides a slightly weaker conclusion: For each a-side of r, an arc s of c exists, which is of type A in r and which yields an arc of d_3 that does not allow a reduction of type IIIb on the chosen side of r (for such s, choose an arc of c that is weakly rigid on the chosen side of r).

Remark 8.8. Remark 8.7 implies that the proof of Lemma 7.13 remains valid if we replace the notion of strong winding with the notion of weak winding. In fact, that proof was based on the fact that if r is a rectangle r of $N_{a\cap b}$, then for each a-side of r, an arc s of c exists, which is of type

A in r and which yields an arc of d_3 that does not allow a reduction of type IIIb on the chosen side of r.

Let $\widehat{X}_a$ be the set of isotopy classes of circles c in N which satisfy the following conditions:

(1) $c \in \mathcal{C}$,
(2) $I(c,a) = |c \cap a|$ and $I(c,b) = |c \cap b|$,
(3) $I(c,a) < I(c,b)$,
(4) c is weakly rigid with respect to a,
(5) c winds weakly around a.

Similarly, we define $\widehat{X}_b$ by requiring (1)–(2) above and additionally

(3′) $I(c,b) < I(c,a)$,
(4′) c is weakly rigid with respect to b,
(5′) c winds weakly around b.

Lemma 8.9. *Let $c \in \widehat{X}_b$ and assume that a and b are not in the special position (S1), (S2), or (S3). Let d_5 be as in the proof of Proposition 7.1 and let s' be an arc of $c \cap N_a$ of type A, B, or C. If s is the arc of $d_5 \cap N_a$ which corresponds to s', then s is parallel to a in at least two rectangles of $N_{a\cap b}$.*

Proof. Let s'' be the arc of d_3 that corresponds to s' (that is, s'' is obtained from $t_a^k(s')$ by reductions of types I, II, IIIa), and let p and q be segments of b that correspond to the segments of d_5 that start at the endpoints of s.

Assume first that s' is an arc of type C. Arcs of this type do not allow reductions of type IIIb (Lemma 7.13 and Remark 8.8). Hence, the only type of reduction that could decrease the number of rectangles in which s'' is parallel to a is the reduction of type IIIc. If we assume that no rectangles of $N_{a\cap b}$ exist in which s is parallel to a, then each double segment of b must contain a segment joinable either to p or q (Remark 7.19). But this implies that a and b are in the special position (S1), which contradicts our assumption. Analogously, if we assume that s is parallel to a in only one rectangle r of $N_{a\cap b}$ and R is the double segment of b that corresponds to r, then each double segment of b that is different from R contains an oriented segment joinable to p or q. Moreover, the triple $\{p,q,R\}$ must be positively oriented. Hence, a and b are in the special position (S2) or (S3), which again is a contradiction.

If s' is an arc of type A or B, then the situation is completely analogous. If s' is of type A, then s'' can admit on one side one reduction of type IIIb (Lemma 7.12, remarks 8.6 and 7.14), and if s' is of type B, then s'' can admit reductions of type IIIb on both sides (one on each side – Remark 7.14). Hence, the assumption that no rectangle of $N_{a\cap b}$ exists in which s

is parallel to a implies that a and b are in the special position (S1), and if we assume that s is parallel to a in only one rectangle of $N_{a\cap b}$, then a and b are in the special position (S2) or (S3). □

Lemma 8.10. *Let $c \in \widehat{X}_b$ and assume that a and b are not in the special position (S1), (S2), or (S3). If d_5 is as in the proof of Proposition 7.1, then d_5 winds weakly around a.*

Proof. We need to show that for each rectangle r of $N_{a\cap b}$ and for each orientation of arcs parallel to a in r, an arc of d_5 exists, which is parallel to a in r and is weakly rigid with respect to the chosen orientation of arcs in r.

Fix r and let r_1 be the rectangle of $N_{a\cap b}$ following r along a with respect to the chosen orientation of arcs in r. Without loss of generality, we can assume that the configuration of rectangles is as in Figure 36;

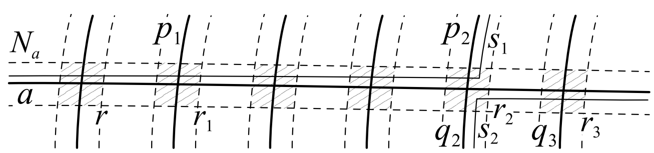

FIGURE 36. Intersection of $t_a^k(s')$ with rectangles of $N_{a\cap b}$.

that is, r_1 is on the right of r. Next, follow a, with the chosen orientation of arcs in r, to the first rectangle r_2 of $N_{a\cap b}$ such that the top oriented segments of b in r_1 and r_2 are not joinable (r_2 exists by Proposition 4.3). If $r_2 = r$, then each double segment of b contains a segment joinable to the top segment of r_1 or to the bottom segment of r. Hence, a and b are in the special position (S1). Therefore, $r_2 \neq r$.

Let r_3 be the rectangle of $N_{a\cap b}$ that follows r_2. If $r_3 = r$, then a and b are in a special position (if the top segment of r_1 is joinable to the top segment of r or if the bottom segment of r_2 is joinable with the bottom segment of r, then we are in the special position (S1); otherwise, we are in the special position (S2) or (S3)). Hence, $r_3 \neq r$.

Let s' be an arc of $c \cap N_a$ that is parallel to b in r_2 and that is weakly rigid on the top side of r_2 (it exists because c winds weakly around b). Under this assumption, the segment s_1 of $t_a^k(c)$ that starts at the top endpoint of s' does not allow a reduction of type IIIb (Lemma 8.4). It may allow reductions of type IIIc, but they cannot reach r_1, because r_2 is not joinable to r_1.

Now, we concentrate on the segment s_2 of $t_a^k(c)$ starting at the bottom endpoint of s'. Let p_i and q_i, for $i = 1, 2, 3$, be segments of b, which go up and down from r_i, respectively. Segment s_2 may admit a reduction of type IIIb and some reductions of type IIIc, but if these reductions reach r, then either all double segments of b contain a segment joinable to p_1 or q_2 (this case happens when s_2 can be reduced by reductions of type IIIc), or only one double segment of b (corresponding to r_2) exists, which does not contain a segment joinable to p_1 or q_3 (this happens when s_2 is reduced by a reduction of type IIIb and then by reductions of type IIIc). According to our assumption that a and b are not special, such a situation is not possible. Hence, the reductions on s_2 cannot reach r, and as a consequence, the arc s of d_5, that corresponds to s', is parallel to a in r and r_1. □

9. The Special Cases (S1) and (S2)

The common feature of cases (S1) and (S2) is the existence of oriented segments p and q starting on different sides of a, such that each double segment (or each double segment except one in case (S2)) contains a segment joinable to p or q. In case (S1) this implies the possibility that some arcs of $c \cap N_a$ of types A–C may lead to arcs of $d_5 \cap N_a$, which do not intersect b (see the proof of Lemma 7.20). We will show below (Lemma 9.13) that such reductions are not possible.

The second problem in cases (S1) and (S2) is that d_5 may not wind strongly around a. We deal with this problem by replacing the notion of strong winding with the notion of weak winding defined in the previous section.

Lemma 9.1. *Let $\pi\colon M \to S^1$ be a bundle over S^1 with fiber $I = [0, 1]$ and which is homeomorphic to a Möbius strip. If a simple oriented arc c in M is monotone with respect to the fixed orientation of S^1 and has endpoints in ∂M, then c intersects every fiber in at most two points.*

Proof. If c intersects some fiber in at least three points, then c winds infinitely many times around the core of M (Figure 37(i)). □

Lemma 9.2. *Let a and b be two generic two-sided circles in N such that $I(a, b) \geq 4$, and assume that oriented segments p and q of b, which start on different sides of a, exist, such that each double segment of b contains an oriented segment joinable to p or q (that is, a and b are in the special position (S1)). Then p is joinable to $-q$.*

Proof. Suppose first that p starts and terminates on the same side of a. In such a case, by Proposition 4.2, segments that start at the terminal points of segments joinable to p must be joinable to q, and vice versa. However,

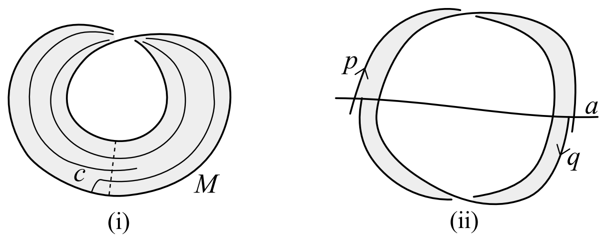

FIGURE 37. Impossible configurations of arcs – lemmas 9.1 and 9.2.

this implies that b is a circle on an annulus (that is, the union of twisted rectangles given by adjacency; see Figure 37(ii)). This contradicts the assumption that $I(a,b) \geq 4$.

Hence, p starts and terminates on two different sides of a. But then the segment starting at the terminal point of p cannot be joinable to q (because p and q begin on two different sides of a), and it cannot be joinable to p (by Proposition 4.2). Hence, $-p$ must be joinable to q. $\square$

Remark 9.3. Let a, b, and p be as in the above lemma and let $p_1, p_2, \ldots, p_n$ be all oriented segments of b joinable to p. The union of adjacency disks between $p_1, p_2, \ldots, p_n$ provides a rectangle Γ. We can assume that p goes up from a and that p_1 and p_n are on the boundary of Γ, where p_n leaves N_a to the left of p_1. By Lemma 9.2, q is equal to one of the segments: $-p_1, -p_2, \ldots, -p_n$. Hence, without loss of generality, we can assume that $q = -p_1$. The other two boundary arcs of Γ are arcs a_1 and a_2 of a. Observe that by Proposition 4.2, $a_1 \cap a_2 = \emptyset$; hence, $a \setminus (a_1 \cup a_2)$ consists of two arcs and the configuration of a and b is as in the left part of Figure 38.

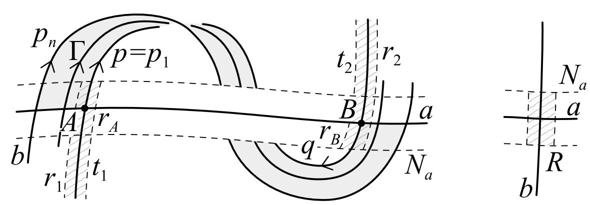

FIGURE 38. Configuration of arcs in cases (S1) and (S2).

Lemma 9.4. *Let a and b be two generic two-sided circles in N such that $I(a,b) \geq 4$, and assume that oriented segments p and q of b and a double* *segment R of b exist, such that a and b are not in the special position (S1), p and q start on different sides of a, p starts and terminates on different sides of a, each double segment of b different from R contains an oriented segment joinable to p or q, and $\{p, q, R\}$ is positively oriented (that is, a and b are in the special position (S2)). Then p is joinable to $-q$.*

Proof. Our first claim is that an oriented segment of b exists, which is joinable to p and such that it neither starts nor terminates in R (that is, neither p nor $-p$ is a segment of R). Given that $I(a, b) \geq 4$, at least three intersection points of $a \cap b$ different from R exist. Hence, at least two arcs s and t, which do not start in R, are joinable to p or joinable to q. If both these arcs are joinable to p, then at least one of them cannot terminate in R (given that they terminate on the same side of a). Hence, our claim follows. Now assume that these two arcs are joinable to q. If q starts and terminates on different sides of a, then the roles of p and q are symmetric, and we can prove our claim by relabeling p to q, and vice versa. Hence, assume that q starts and terminates on the same side of a. Let s' and t' be oriented segments of b that follow s and t, respectively (Figure 39(i)).

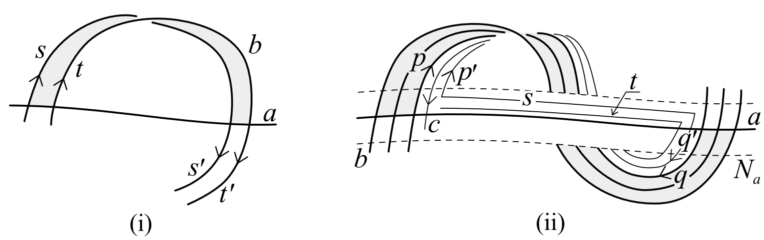

FIGURE 39. Configuration of arcs, Lemmas 9.4 and 9.8.

By Proposition 4.2, none of $-s$, $-t$, s', or t' is joinable to q and by our assumptions $-s$ and $-t$ are not joinable to p (because p and q start on different sides of a). Hence, at least one of s' or t' is joinable to p. If both s' and t' are joinable to p, then our claim follows because only one of them can terminate in R. If only one of them, say s', is joinable to p, then R corresponds to the terminal point of t. Hence, s' cannot terminate in R, which again proves our claim.

Assume that s is a segment of b joinable to p, which does not start or terminate in R. Then the segment that starts at the terminal point of p cannot be joinable to q (because p and q begin on two different sides of a), and it cannot be joinable to p (by Proposition 4.2). Hence, $-p$ must be joinable to q. □

Remark 9.5. Following the lines of the analysis conducted in Remark 9.3, we conclude that the configuration of a and b in case (S2) differs from that in (S1) only by one additional double segment R, which intersects a in one of the arcs of $a \setminus (a_1 \cup a_2)$ (a_1 and a_2 are defined as in Remark 9.3). Hence, the configuration is as in Figure 38. (Note that the mutual position of p, q, and R is determined by the assumption that $\{p, q, R\}$ is positively oriented.)

For the rest of this section, we will use the notation introduced in remarks 9.3 and 9.5; that is, $p_1 = p$ and p_n are segments of b which, together with arcs a_1 and a_2 of a, bound a rectangle Γ that contains all segments of b joinable to p and p_1 goes up from a and leaves N_a to the right of p_n (Figure 38). Moreover, assume that p starts at A in a rectangle r_A of $N_{a\cap b}$ and it terminates in B in a rectangle r_B. Let r_1 and r_2 be rectangles of $N_{b\setminus a}$ which precede and follow p, respectively.

Lemma 9.6. *Suppose that a and b are in the special position (S1) and let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be the oriented segments of b which, respectively, go down/up from the initial/terminal points of $p_1, \ldots, p_n$. Then u_i is not joinable to u_j for some $i \neq j$ and v_i is not joinable to v_j for some $i \neq j$.*

Proof. If, for example, all the u_i are mutually joinable, then a and b are circles on the annulus given by adjacency disks between p_i and u_i. This contradicts the assumption that $I(a, b) \geq 4$. □

Lemma 9.7. *Suppose that a and b are in the special position (S2) and let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be the oriented segments of b which, respectively, go down/up from the initial/terminal points of $p_1, \ldots, p_n$. Then the segments that constitute R are joinable neither to u_1 nor to v_1. Moreover, u_i is not joinable to u_j for some $i \neq j$ and v_i is not joinable to v_j for some $i \neq j$.*

Proof. If a segment of R is joinable to u_1 or v_1, then a and b are in the special position (S1), which is not possible. Next, suppose to the contrary that all the u_i are mutually joinable and all the v_i are mutually joinable. Given that $I(a, b) \geq 4$, $v_i = -u_j$ for some i and j. Hence, we can assume that each v_i is joinable to $-u_1$. One of the u_i must terminate in a rectangle r that corresponds to R and also one of the v_i must terminate in r. However, given that v_i is joinable to $-u_i$, this situation contradicts Proposition 4.2. □

Suppose that the component of $N \setminus N_{a\cup b}$, which is determined by $p_1 = p$ and an arc of $a \setminus (a_1 \cup a_2)$, is an exterior n-gon Δ. Let t_1 and t_2 be these boundary sides of r_1 and r_2, respectively, which enter r_A and r_B on the right of $b \cap r_A$ and $b \cap r_B$ (Figure 38).

Now we define two Möbius strips associated with p. The first one M_1 is the union of the rectangle r_p of $N_{b\setminus a}$ that contains p, rectangles r_A and r_B, and a single rectangle r_{AB} of $N_{a\setminus b}$ that connects r_A and r_B. The second one M_2 is the union of r_A, r_B, r_{AB}, and all the rectangles that connect r_A and r_B along the boundary of Δ.

Finally, for the rest of this section, assume that $c \in \mathcal{C}$ and c winds around b.

Lemma 9.8. *Let s be an arc of type C of $c \cap N_a$ which connects the initial points of oriented segments p' and q' of c which run parallel to p_1 and $-p_1$, respectively. Then at least one of the oriented arcs of $c \cap N_a$ following p' and q' is an arc of type D which turns to the left as it enters N_a.*

Proof. Two possible configurations of arcs p' and q' exist (they can pass each other in two different ways). However, these configurations lead to the same conclusions. Hence, assume that we have the configuration shown in Figure 39(ii). Consider the arc t of $c \cap N_a$ following p'. After entering N_a, this arc must turn to the left and is either of type C or of type D. If t were of type C, then c would wind at least three times around the core of the Möbius strip M_1, which contradicts Lemma 9.1. Hence, t is an arc of type D. □

Let t be a common boundary of a rectangle r in $N_{a\setminus b} \cup N_{b\setminus a}$ and an exterior n-gon Δ. We say that an arc q of $c \cap r$ is a *bounding segment* for Δ (with respect to t) if no arcs of c exist between t and q in r. Clearly, $c \cap r$ is either empty or contains exactly one bounding segment for Δ with respect to t.

Lemma 9.9. *Let r_p be the rectangle in $N_{b\setminus a}$ that contains p. If s_1 is an arc of c that passes through r_1, r_A, and r_p and is of type A in r_A, and s_2 is an arc of c which passes through r_2, r_B, and r_p and is of type A in r_B, then either $s_1 \cap r_1$ is not a bounding segment for Δ with respect to t_1 or $s_2 \cap r_2$ is not a bounding segment for Δ with respect to t_2.*

Proof. Two possible configurations of arcs s_1 and s_2 exist (they can pass each other in two different ways). However, these configurations lead to the same conclusions. Hence, assume that we have the configuration shown in Figure 40(i).

Consider the arc s of $c \cap N_a$ following s_1. If s were an arc of type C, then c would wind at least three times along the core of the Möbius strip M_1 and this would contradict Lemma 9.1. Hence, s is either of type A or of type D. In the former case, s_2 is not a bounding segment for Δ with respect to t_2, and in the latter case, s_1 is not a bounding segment for Δ with respect to t_1. □

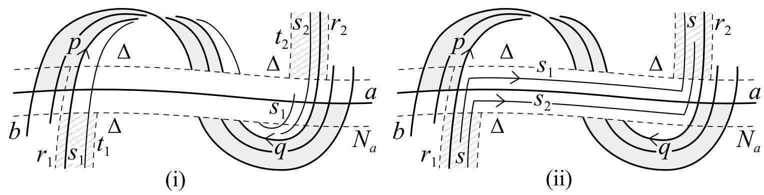

FIGURE 40. Configuration of arcs, Lemmas 9.9, 9.10 and 9.13.

If s is an arc of $c \in \mathcal{C}$ with endpoints in r_A and r_B that connects r_A and r_B in $M_2 \setminus r_p$, then we say that s is a *long bounding segment for* Δ.

Lemma 9.10. *Let s_1 and s_2 be oriented arcs of $c \cap N_a$ of type B such that the arc s of c that starts at the terminal point of s_1 and terminates at the starting point of s_2 is a long bounding segment for Δ (Figure 40(ii)). Then the arcs of c which precede s_1 and follow s_2 are not long bounding segments for Δ.*

Proof. If, for example, the arc that precedes s_1 were a long bounding segment for Δ, then c would be a curve in the Möbius strip M_2 which winds at least three times along the core of M_2; this contradicts Lemma 9.1. □

Lemma 9.11. *Assume that an arc s of $c \cap N_a$ of type B exists, which starts in r_A and terminates in r_B. If s_1 (or s_2) is an arc of c that is of type A in r_A (or r_B), then s_1 (or s_2) is not a bounding segment for Δ with respect to t_1 (or t_2).*

Proof. Given that c cannot intersect itself, s_1 and s_2 must enter N_a on the left of s (our point of view here is along s_1/s_2 towards r_A/r_B; see Figure 41(i)). □

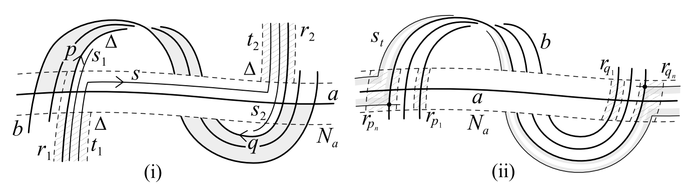

FIGURE 41. Configuration of arcs, Lemma 9.11 and Proposition 9.12.

Now we are ready to adopt the proof of Proposition 7.1 to the special cases (S1) and (S2). However, as we observed in Example 8.1, the notion of rigidity is too strong in these cases. Hence, we replace it with the notion of weak rigidity (see §8).

Proposition 9.12. *Let a and b be two generic two-sided circles in N such that $I(a,b) \geq 4$ and $\{a,b\}$ is not in the special position (S3). Then, for any integer $k \neq 0$, we have*

$$t_a^k(\widehat{X}_b) \subseteq \widehat{X}_a \quad \textit{and} \quad t_b^k(\widehat{X}_a) \subseteq \widehat{X}_b.$$

Proof. As we observed in Remark 8.8, the assumption about weak winding of c around b suffices to prove the statement of Lemma 7.13. As a substitute for Remark 7.11, we have slightly weaker Remark 8.7. Moreover, as observed in Remark 8.6, the assumption that c is weakly rigid with respect to b serves as a replacement for Lemma 7.12. Hence, in most parts, we can copy verbatim the proof of Proposition 7.1, yet there are some differences, which we study in detail below.

Suppose first that a and b are not in the special position (S1) or (S2). In such a case, by lemmas 8.9 and 8.10, d_5 is weakly rigid with respect to a and d_5 winds weakly around a. Moreover, Lemma 8.9 implies that $I(d_5,b) > I(d_5,a)$.

Therefore, for the rest of this section, we can concentrate on circles a and b, which are in the special position (S1) or (S2) and are hence in the situation described in lemmas 9.2 and 9.4 and remarks 9.3 and 9.5.

Counting intersection points between d_5 and b. If we follow the proof of Proposition 7.1, then the first place it can fail in cases (S1) and (S2) is the proof of Lemma 7.20. However, as we will show below, even the stronger statement of Lemma 8.9 holds true in cases (S1) and (S2). The possible failure of the argument in Lemma 8.9 follows from the fact that an arc s' of $c \cap N_a$ of type A, B, or C may exist, which yields an arc s of $d_5 \cap N_a$ that is parallel to a in fewer than two rectangles of $N_{a\cap b}$. Moreover, the proof of Lemma 8.9 provides a specific description of possible configurations of p, q, and s' that may lead to such a failure. Three possibilities exist in the case (S1)/(S2): s' may be an arc of type A in r_A/r_B, s' may be an arc of type B that connects r_A and r_B, or s' may be an arc of type C that connects r_A and r_B. The following lemma shows that in each of these cases, strong restrictions are given for reductions of types IIIb/IIIc.

Lemma 9.13. *Let c be as in the proof of Proposition 7.1.*

(1) *Let s be an arc of $d_3 \cap N_a$ that corresponds to an arc s' of $c \cap N_a$. If s' is an arc of type C with endpoints in r_A and r_B, then s can admit a reduction of type IIIc only on one side of s'.*

(2) *Let s be an arc of c, which is the innermost long bounding segment for Δ (that is, s is a bounding segment for Δ in rectangles of $N_{(a\cup b)\setminus(a\cap b)}$) and which leads to a reduction of type IIIb across Δ. Then the arcs of $c \cap N_a$ that precede/follow s are arcs of type B.*

(3) *Suppose that a long bounding segment of c exists, which leads to a reduction of type IIIb across Δ. Then no arc of c which is of type A in r_A or r_B leads to a reduction of type IIIb across Δ.*

(4) *No arc of $c \cap N_a$ that is of type A in r_A or r_B leads to a reduction of type IIIb across Δ.*

(5) *If s_1 is an arc of c of type B with endpoints in r_A and r_B, then s_1 can lead to a reduction of type IIIb only on one side of s_1.*

Proof.

(1) If s' is an arc of type C, then by Lemma 9.8, at least one of the arcs that precedes/follows s' cannot lead to a reduction of type IIIc (because it turns to the left as it enters N_a).

(2) By Lemma 9.9, the arcs that precede/follow s cannot both be of type A, and by Lemma 9.11, they are not arcs of types A and B. Hence, both arcs must be of type B.

(3) If we choose the arc s of c that admits a reduction of type IIIb across Δ and is the innermost long bounding segment for Δ, then by the previous point, we know that the arcs s_1 and s_2 of c that precede/follow s are of type B in r_A and r_B (Figure 40(ii)). But then, by Lemma 9.10, we know that the arcs that, respectively, precede s_1 and follow s_2 do not admit reductions of type IIIb. Hence, they are the obstacles for the arcs of type A in r_A and r_B to admit such a reduction.

(4) The conclusion is a direct consequence of the previous point.

(5) At this point, we know that if an arc s of c admits a reduction of type IIIb across Δ, then this must be a long bounding segment that connects two arcs s_1 and s_2 of $c \cap N_a$ of type B. But then, by Lemma 9.10, the arc that precedes s_1, which is different from $\pm s$, does not admit reductions of type IIIb. □

The above lemma implies that even in cases (S1)/(S2), none of the arcs s' of $c \cap N_a$ can be reduced so that the resulting arc s of $d_5 \cap N_a$ is parallel to a in fewer than two rectangles of $N_{a\cap b}$. Hence, Lemma 8.9 remains valid in these cases. In particular, d_5 is weakly rigid with respect to a and $I(d_5, b) > I(d_5, a)$.

Weak winding of d_5. Finally, we need to show that d_5 winds weakly around a. Denote by $r_{p_1} = r_A, r_{p_2}, \ldots, r_{p_n}$ and $r_{q_1} = r_B, r_{q_2}, \ldots, r_{q_n}$ the rectangles of $N_{a\cap b}$, which contain the starting points of $p_1, p_2, \ldots, p_n$ and

$-p_1, \dots, -p_n$, respectively (Figure 38). In case (S2), denote the rectangle of $N_{a\cap b}$ that corresponds to R by r_R .

Let s' be an arc of c that is of type A in r_{p_1}, and let s'' and s be the corresponding arcs of d_3 and d_5, respectively. By Lemma 9.13, the bottom part of s'' does not allow a reduction of type IIIb. The top part of s'' may admit reductions of type IIIc which can reach at most r_{p_n}. Hence, s is parallel to a in all the rectangles $r_{q_1}, \dots, r_{q_n}$ and r_R in case (S2). Moreover, if the reductions on the top part s''_t of s'' reach r_{p_n}, then the corresponding arc s_t of d_5 is two-sided with respect to b; that is, s_t, together with an arc of b that connects the intersection points of s_t and b is a two-sided circle (Figure 41(ii)). Hence, s_t is not a one-sided boundary 3-segment of an exterior hexagon, and by Lemma 8.5, s is weakly rigid with respect to a on both sides of r_{q_n} in case (S1) and on both sides of r_R in case (S2). Moreover, s is rigid with respect to a on both sides of $r_{q_2}, \dots, r_{q_{n-1}}$.

In exactly the same way, by considering the arc of c which is of type A in r_{q_1}, we prove that d_5 is weakly rigid with respect to a on both sides of rectangles $r_{p_2}, \dots, r_{p_n}$. Hence, to finish the proof, we need to show the existence of an arc of $d_5 \cap N_a$ that is weakly rigid on both sides of r_{q_1} and an arc of $d_5 \cap N_a$ that is weakly rigid on both sides of r_{p_1}.

In case (S1), let s' be an arc of c that is of type A in r_{p_n} and that is weakly rigid on both sides of r_{p_n}. Let s'' and s be the corresponding arcs of d_3 and d_5, respectively. By Lemma 8.4, the bottom part of s'' does not allow a reduction of type IIIb. It may admit reductions of type IIIc, but by Lemma 9.6, they cannot reach r_{p_1}. The top part of s'' does not admit a reduction of type IIIb (Lemma 8.4) and it does not allow a reduction of type IIIc (Proposition 4.2 and Remark 7.19). Hence, s is parallel to a in r_{p_1}, r_{q_1}, and r_{q_2}. Similarly, by taking an arc s' of c that is weakly rigid on both sides of r_{q_1}, we construct an arc s of d_5 that is parallel to a in r_{p_2}, r_{p_1}, and r_{q_1}.

In case (S2), by Lemma 9.7, we know that either the segments that go down from the starting points of $p_1, \dots, p_n$ or the segments that go up from the terminal points of $p_1, \dots, p_n$ are not mutually joinable. The argument in both cases is completely analogous. Hence, the latter case is assumed. Let s' be an arc of c that is of type A in r_{q_n} and is weakly rigid on both sides of r_{q_n}. Let s'' and s be the corresponding arcs of d_3 and d_5, respectively. By Lemma 8.4, the top part of s'' does not allow a reduction of type IIIb. It may admit reductions of type IIIc, but by our assumption they cannot reach r_{q_1}. The bottom part of s'' does not admit a reduction of type IIIb. Hence, s is parallel to a in r_{p_2}, r_{p_1}, and r_{q_1}. Similarly, if s'' is an arc of c that is of type A in r_{p_n} and s is the corresponding arc of d_5, then s is parallel to a in r_{q_1} and r_{q_2}, and is either parallel to a in r_{p_1}

or weakly rigid on the r_{p_1} side of r_{q_1} (Lemma 8.5). This completes the proof that d_5 winds weakly around a. $\square$

10. The Special Case (S3)

Lemma 10.1. *Let a and b be two generic two-sided circles in N such that $I(a,b) \geq 4$, and assume that oriented segments p and q of b and a double segment R of b exist such that p starts and terminates on one side of a; q starts and terminates on the other side of a; each double segment of b, which is different from R, contains an oriented segment joinable to p or q; and $\{p,q,R\}$ is positively oriented (that is, a and b are in the special position (S3)). Then the oriented arcs that constitute R are joinable neither to p nor to q.*

Proof. Suppose that in each double segment of b, an oriented segment joinable either to p or to q exists. By Proposition 4.2, all segments that start at endpoints of segments joinable to p and are not joinable to $-p$ are joinable to q. And vice versa, all segments that start at endpoints of segments joinable to q and are not joinable to $-q$ are joinable to p. Hence, b is a circle on an annulus (which is a union of adjacency disks between the segments joinable to p and q – see Figure 37(ii)). However, this contradicts the assumption that $I(a,b) \geq 4$. $\square$

Remark 10.2. Let a and b be circles in the special position (S3), let $p_1, p_2, \ldots, p_n$ be all oriented segments of b joinable to p, and let $q_1, \ldots, q_m$ be all oriented segments joinable to q. The union of adjacency disks between $p_1, p_2, \ldots, p_n$ gives a rectangle Δ_p and the union of adjacency disks between $q_1, q_2, \ldots, q_m$ gives a rectangle Δ_q. By Lemma 10.1, we know that one of the segments $p_1, \ldots, p_n$ or $q_1, \ldots, q_m$ ends in R and is followed by an arc that is joinable neither to p nor to q. Without loss of generality, we can assume that p is above a, p_n ends in R, p_1 and p_n are on the boundary of Δ_p, and q_1 and q_m are on the boundary of Δ_q. Given that all oriented segments $p_1, \ldots, p_{n-1}$ must be followed by segments joinable to q, and all segments $q_1, \ldots, q_m$ must be followed by segments joinable to p, we can also assume that $p_1, \ldots, p_{n-1}$ are followed by $q_1, \ldots, q_m$, respectively. Hence, the configuration is as in Figure 42. (Note that the mutual position of p, q, and R is determined by the assumption that $\{p,q,R\}$ is positively oriented.)

For the rest of this section, we will use the notation introduced in the above remark. Moreover, by r, r_p, and r_q, we denote the rectangles of $N_{a\cap b}$ that correspond, respectively, to R, the initial point of p_n, and the initial point of q_1 (Figure 42).

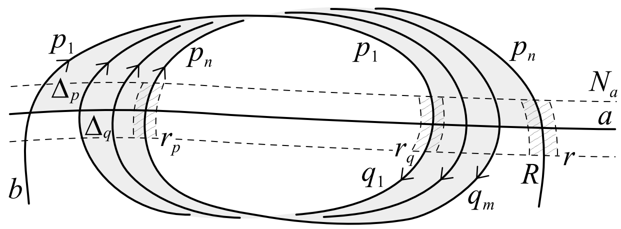

FIGURE 42. Configuration of arcs in case (S3).

Lemma 10.3. *No component of $N \setminus N_{a\cup b}$ is an exterior hexagon.*

Proof. Given that the segment that connects the terminal point of p_n with the initial point of p_1 is one-sided, checking that r can be a vertex of an exterior n-gon only for $n = 4, 8$ is straightforward. All the other exterior n-gons are rectangles. □

Let $\widehat{X}_a$ and $\widehat{X}_b$ be defined as in §8.

Proposition 10.4. *Let a and b be two generic two-sided circles in N such that $I(a, b) \geq 4$. Then for any integer $k \neq 0$, we have*

$$t_a^k(\widehat{X}_b) \subseteq \widehat{X}_a \quad \textit{and} \quad t_b^k(\widehat{X}_a) \subseteq \widehat{X}_b.$$

Proof. Observe first that if a and b are not in the special position (S3), then the proposition coincides with Proposition 9.12. Therefore, we can concentrate on the circles a and b, which are in the special position (S3) and are hence in the situation described in Remark 10.2.

As in special cases (S1) and (S2), we observe that weak winding is sufficient for repeating most of the proof of Proposition 7.1. Hence, we follow the lines of the proof of Proposition 9.12 and we concentrate on places where that proof can fail. The first such place is the proof that d_5 is weakly rigid with respect to a. In the proof of Proposition 9.12 we obtained weak rigidity of d_5 as a consequence of Lemma 8.9, which may not be true in the case (S3).

However, as a replacement we have the following lemma.

Lemma 10.5. *Let s' be an arc of $c \cap N_a$ of type A, B, or C, and let s be the arc of $d_5 \cap N_a$ which corresponds to s'.*

(1) *If s' is an arc of type A in r_p or s' is an arc of type C that connects r_p and r_q, then s is parallel to a in r.*
(2) *If $I(a, b) = 4$ and s' is an arc of type A in r or s' is an arc of type C with the top part in r, then s is parallel to a in r_p.*
(3) *If s' is not as in previous points, then s is parallel to a in at least two rectangles of $N_{a\cap b}$.*

Proof. Observe first that the boundary component of $N_{a\cup b}$ that contains p_n and p_1 cannot bound an exterior rectangle. Otherwise, a would bound a Möbius strip. Hence, the segments of c, which run parallel to p_1, do not lead to reductions of types IIIb nor IIIc. Let s'' be an arc of d_3 that corresponds to s'.

(1) If s' is an arc of type A in r_p, then the bottom part of s'' can admit a reduction of IIIb and then both sides of the obtained arc may admit reductions of type IIIc. By Lemma 10.1, these reductions cannot reach R. Hence, s is parallel to a in r.

The situation is completely analogous if s' is an arc of type C that connects r_p and r_q. (By Lemma 7.13, s'' does not admit reductions of type IIIb in this case.)

(2) If s' is an arc of type A in r, then the top part of s'' may admit some reductions of type IIIc, which can reach at most r_q. The bottom part of s'' may admit a reduction of type IIIb, but then it does not admit any further reductions of type IIIc. Hence, s is parallel to a in r_p.

The situation is completely analogous if s' is an arc of type C with the top part in r. (By Lemma 7.13, s'' does not admit reductions of type IIIb in this case.)

(3) If s' is not as in the previous points, then it is straightforward to check that s is parallel to a either in the rectangles of $N_{a\cap b}$ that contain the initial points of p_{n-1} and p_n or in the rectangles of $N_{a\cap b}$ that contain the terminal points of p_1 and p_2. □

As a consequence of the above lemma, we obtain that every arc s of d_5 that is parallel to a in a rectangle t of $N_{a\cap b}$ is weakly rigid on one side of t. In fact, if s is parallel to a in at least two rectangles of $N_{a\cap b}$, then s is rigid in t. If s is parallel to a in only one rectangle of $N_{a\cap b}$, then s was obtained from an arc of d_3 by reductions of types IIIb and IIIc. In such a case, by lemmas 8.5 and 10.3, s is weakly rigid on one side of t. In particular, d_5 is weakly rigid with respect to a.

Finally, we will show that d_5 winds weakly around a. Let s_1' and s_2' be arcs of c of type A of $N_{a\cap b}$, with s_1' in r_q and with s_2' in the rectangle r_{p_1} that contains the initial point of p_1. Assume also that s_1' and s_2' are weakly rigid below a, and let s_1'' and s_2'' and s_1 and s_2 be arcs of d_3 and d_5 that correspond to s_1' and s_2', respectively. The arc s_2'' does not admit any reductions of types IIIb and IIIc. Hence, s_2 is parallel to a in all rectangles of $N_{a\cap b}$ except r_{p_1}. The bottom part of s_1'' may admit some reductions of type IIIc, but these reductions cannot reach r. Hence, s_1 is parallel to a in r and in all the rectangles of $N_{a\cap b}$ between r_{p_1} and r_p. This proves that d_5 winds weakly around a. □

11. The Case of $I(a,b)=3$ with Nonorientable $N_{a\cup b}$

If $I(a,b)=3$, then we still follow the proofs of propositions 7.1 and 9.12. However, as in the special case (S3), the problem is that, in general, if $I(a,b)=3$, then Lemma 8.9 is not true. Fortunately, this case is quite special because of the following proposition.

Proposition 11.1. *If a and b are two generic two-sided circles in a surface N such that $|a\cap b|=I(a,b)=3$ and $N_{a\cup b}$ is nonorientable, then the exterior n-gons for $N_{a\cup b}$ can exist only if $n=10$ or $n=12$.*

Proof. If all segments of b are two-sided, then $N_{a\cup b}$ is orientable. Hence, one-sided segments of b exist, and two such segments p_1 and p_2 must exist. If we denote the two-sided segment of b as p_3, then we can assume that p_1, p_2, and p_3 are oriented so that p_2 follows p_1 and p_3 follows p_2.

If all segments p_1, p_2, and p_3 start and terminate on different sides of a, then the arc of a that connects the initial point of p_1 with the initial point of p_2 starts and terminates on the same side of b. Hence, we can interchange a with b, and we can always assume that at least one segment of b starts and terminates on the same side of a. In such a case one of the remaining segments must also start and terminate on the same side of a, and the final segment must connect two different sides of a. Hence, we have the configuration of arcs as in Figure 43. We still have two

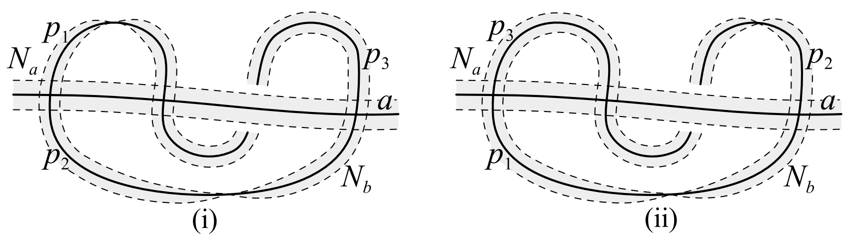

Figure 43. Configurations of segments of b – Proposition 11.1.

possibilities: either p_3 is a segment that connects two different sides of a (Figure 43(i)) or p_3 starts and terminates on the same side of a (Figure 43(ii)). Checking that in the first case the boundary of $N_{a\cup b}$ is connected and it is a 12-gon is straightforward. In the second case, the boundary of $N_{a\cup b}$ has two components: a bigon and a 10-gon. □

The above proposition implies that if $I(a,b)=3$, then no adjacent segments of b exist. Hence, no reductions of type IIIc exist. Moreover, the notion of weakly rigid arcs simplifies, given that no exterior hexagons exist.

Let $\overline{X}_a$ be the set of isotopy classes of circles c in N, which satisfy the following conditions:

(1) $c \in \mathcal{C}$,
(2) $I(c, a) = |c \cap a|$ and $I(c, b) = |c \cap b|$,
(3) $I(c, a) < I(c, b)$,
(4) c winds weakly around a.

Similarly, we define $\overline{X}_b$ by requiring (1)–(2) above and additionally

(3′) $I(c, b) < I(c, a)$,
(4′) c winds weakly around b.

The main difference between sets $\overline{X}_b$ and $\widehat{X}_b$ is the lack of the assumption that c is weakly rigid with respect to b. As a replacement for this assumption we have the following lemma.

Lemma 11.2. *Let a and b be two generic two-sided circles in N such that $I(a, b) = 3$ and $c \in \overline{X}_b$. Let s be an arc of $c \cap N_a$ of type A and let s' be the arc of d_3 that corresponds to s. Then s' can admit a reduction of type IIIb with only one orientation of s.*

Proof. Suppose to the contrary that s leads to reductions of type IIIb with both orientations of s (that is, on both sides of a) and let r be the rectangle of $N_{a\cap b}$ that contains s. Observe first that both ends of s' must be involved in two different reductions of type IIIb; otherwise, s would intersect a only once, which would contradict the assumption that c winds around b. By Proposition 11.1, at most one exterior n-gon Δ exists. Hence, both reductions on s' must correspond to the same one-sided segment p of b; this segment corresponds to one side of the bigon defining a reduction. Therefore, if we follow s in both directions until we obtain the arcs s_1 and s_2 of $c \cap N_a$ which intersect a, then s_1 and s_2 must enter N_a in r. Moreover, the arcs of c that connect s with s_1 and s_2 run through rectangles of $N_{a\cup b}$, which together with r, constitute a Möbius strip strip M. Hence, s_1 and s_2 enter r on the same side of s (Figure 44(i)). Therefore, by Lemma 7.13 and Remark 8.8, at least one of the arcs s_1 or s_2 must be an arc of type A, which leads to a contradiction with Lemma 9.1. □

Proposition 11.3. *Let a and b be two generic two-sided circles in N such that $I(a, b) = 3$. Then for any integer $k \neq 0$, we have*

$$t_a^k(\overline{X}_b) \subseteq \overline{X}_a \quad \textit{and} \quad t_b^k(\overline{X}_a) \subseteq \overline{X}_b.$$

Proof. As in the special case (S3), the main problem that may lead to the failure of the proof of Proposition 9.12 is the fact that if $I(a, b) = 3$, then

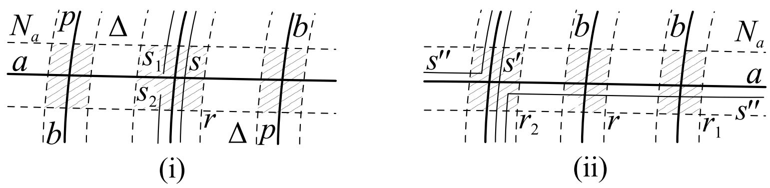

FIGURE 44. Configurations of segments of c and d_3 – Lemma 11.2 and Proposition 11.3.

Lemma 8.9 may not be true. However, as a replacement for that lemma, we have the following slightly weaker result.

Lemma 11.4. *Let s' be an arc of $c \cap N_a$ of type A, B, or C, and let s be the arc of $d_5 \cap N_a$ that corresponds to s'. Then, s is parallel to a in at least one rectangle of $N_{a\cap b}$*

Proof. Let s'' be an arc of $d_3 \cap N_a$ that corresponds to s'. If s' is an arc of type A, then by Lemma 11.2, s'' may admit a reduction of type IIIb only on one side of a. Hence, s is parallel to a in at least one rectangle of $N_{a\cap b}$. If s' is an arc of type B, then s'' may admit reductions of type IIIb on both sides of a, but this leads to same conclusion as above. Arcs of type C do not allow reductions of type IIIb (see Lemma 7.13 and Remark 8.8). Hence, in this case, s is parallel to a in exactly one rectangle of $N_{a\cap b}$. □

As a consequence of the above lemma and Proposition 11.1, we have the following simplified version of Lemma 8.5.

Lemma 11.5. *Let q be an oriented arc of d_5 that starts in a rectangle t of $N_{a\cap b}$ as an arc parallel to a; then it follows a to the next rectangle of $N_{a\cap b}$, and then it follows an arc s obtained from an arc of $d_3 \setminus N_a$ by a reduction of type IIIb. Then q is weakly rigid in t with respect to a.*

Weak rigidity of d_5. Fix a rectangle r in $N_{a\cap b}$ and assume that the orientation of arcs parallel to a in r is such that these arcs point to the rectangle r_1, which is on the right of r (if the orientation is opposite, we can rotate the whole picture by $180°$; see Figure 44(ii)). Let s' be an arc of $c \cap N_a$ of type A in a rectangle r_2 of $N_{a\cap b}$ different from r and r_1, and assume that s' is weakly rigid in r_2 with respect to b and with the orientation pointing down (we use the assumption that c winds weakly around b). The arc s'' of d_3 that corresponds to s' is parallel to a in r and r_1. If this arc does not admit a reduction of type IIIb, then s' is, in fact, an arc of d_5 and this arc is rigid in r. If, on the other hand, s'' admits a reduction of type IIIb, then this reduction must be on the top part of s''.

Hence, by Lemma 11.5, the arc s of d_5 that corresponds to s'' is weakly rigid in r.

Counting intersection points between d_5 and b. Lemma 11.4 guarantees that for each intersection point of c and a, we have at least one intersection point of d_5 and b. To prove that $I(d_5, b) > I(c, a)$, we need to show that for some arcs of $c \cap N_a$ the corresponding arcs of $d_5 \cap N_a$ intersect b at least twice.

In fact, by Proposition 11.1, at most one exterior n-gon exists. Hence, all reductions of type IIIb must correspond to the same segment p of b; this segment corresponds to one side of the bigon that defines a reduction. Let r_A and r_B be the rectangles of $N_{a\cap b}$ that contain the endpoints A and B of p. Let r_1 and r_2 be rectangles of $N_{a\cap b}$ that precede r_A and r_B, respectively, with respect to the twisting direction in N_a; that is, r_1 is the right (left) neighbor of r_A if p approaches a from above (from below), similarly for r_2 (see Figure 45). All arcs of $c \cap N_a$ of type A that lead to

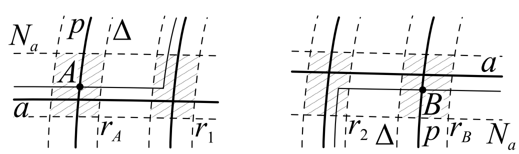

FIGURE 45. Configuration of rectangles of $N_{a\cap b}$, Proposition 11.3.

the reduction of type IIIb must be contained in either r_1 or r_2. Hence, if we choose an arc s' of $c \cap N_a$ that is of type A in a rectangle r of $N_{a\cap b}$ different from r_1 and r_2, then the corresponding arc s'' of d_3 does not admit reductions of type IIIb. Therefore, the arc s of $d_5 \cap N_a$ that corresponds to s' is parallel to a in two rectangles of $N_{a\cap b}$. $\square$

12. The Case of $I(a, b) = 2$ with Nonorientable $N_{a\cup b}$

Checking that propositions 7.1 and 10.4 are false if $I(a, b) = 2$ and $N_{a\cup b}$ is nonorientable is not difficult. Hence, we need a slightly more sophisticated analysis in that case.

The case in question is special because of the following proposition.

Proposition 12.1. *If a and b are two generic two-sided circles in a surface N such that $|a \cap b| = I(a, b) = 2$ and $N_{a\cup b}$ is nonorientable, then no component of $N \setminus (a \cup b)$ is a disk.*

Proof. Observe that $N_{a\cup b}$ is a Klein bottle with two boundary components (Figure 46(i)). Hence, if one of the components of $N \setminus (a \cup b)$ is a disk, then

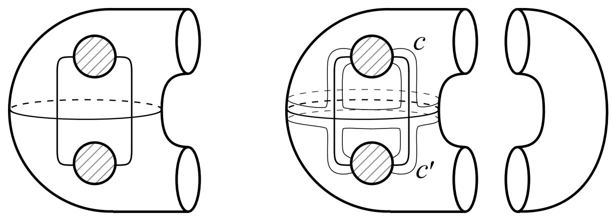

FIGURE 46. Klein bottle with two holes as a regular neighborhood of $a \cup b$.

one of the circles a or b bounds a Möbius strip , which is a contradiction. □

For a circle $c \in \mathcal{C}$ define

$$J(c,a) = \text{number of connected components of } c \setminus N_a$$
$$J(c,b) = \text{number of connected components of } c \setminus N_b.$$

Proposition 12.2. *Let a and b be two generic two-sided circles in a surface N such that $|a \cap b| = I(a,b) = 2$ and $N_{a\cup b}$ is nonorientable. If $c, c' \in \mathcal{C}$ such that c is isotopic to c', then $J(c,a) = J(c',a)$ and $J(c,b) = J(c',b)$.*

Proof. Suppose first that $|c \cap c'| > 0$ and let Δ be a bigon formed by c and c'. By taking the innermost bigon, we can assume that the interior of Δ is disjoint from $c \cup c'$. Given that the boundary of $N_{a\cup b}$ is disjoint from $c \cup c'$, if a component of $N \setminus N_{a\cup b}$ intersects Δ, then this component must be a disk. By Proposition 12.1, this case is not possible. Hence, Δ is contained in $N_{a\cup b}$. Moreover, we can assume that the vertices of Δ are in the interior of rectangles of $N_{a\cup b}$.

Fix a rectangle r in $N_{a\setminus b} \cup N_{b\setminus a}$, and then let Δ_r be a connected component of $\Delta \cap r$. Given that $\Delta \subset N_{a\cup b}$ and c and c' do not turn back in any of the rectangles of $N_{a\cup b}$, Δ_r must be a rectangle, a triangle, or a bigon, with two sides being arcs of c and c' (Figure 47). In any case, if

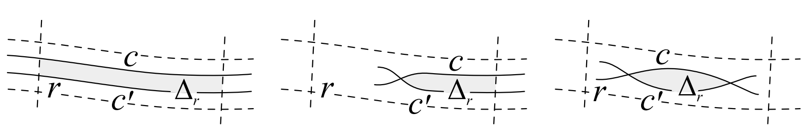

FIGURE 47. Arcs of c and c' in a rectangle r.

we remove the bigon Δ, that is, if we replace c' with the circle c'' isotopic to c' which is obtained by pushing c' across Δ (Figure 48(i)), then

$$J(c'', a) = J(c', a) \text{ and } J(c'', b) = J(c', b).$$

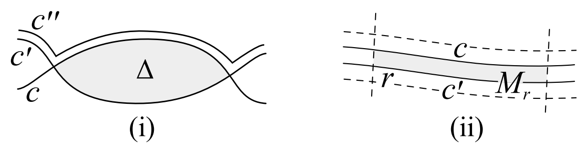

FIGURE 48. Arcs of c and c' in a rectangle r.

Therefore, we can assume that c and c' are disjoint. This assumption means that an annulus M in N with boundary curves c and c' exists.

If a component of $N \setminus N_{a\cup b}$ intersects M, then this component must be an annulus with boundary curves isotopic to c and c'. In such a case, N is a nonorientable surface of genus 4 (Figure 46(ii)) and

$$J(c, a) = J(c', a) = 2 \text{ and } J(c, b) = J(c', b) = 2.$$

Finally, assume that M is contained in $N_{a\cup b}$. As in the case of a bigon formed by c and c', if r is a rectangle in $N_{a\setminus b} \cup N_{b\setminus a}$ and M_r is a connected component of $M \cap r$, then M_r must be a rectangle with two sides being arcs of c and c' that connect opposite sides of r (Figure 48(ii)).

This implies that M_r gives in r exactly one arc of c and one arc of c'. This result means that $J(c, a) = J(c', a)$ and $J(c, b) = J(c', b)$. □

For two generic two-sided circles a and b in N such that $|a \cap b| = I(a, b) = 2$, we define $\widetilde{X}_a$ as the set of isotopy classes of circles in N which satisfy the following conditions:

(1) $c \in \mathcal{C}$,
(2) $J(c, a) < J(c, b)$,
(3) c winds around a.

Similarly, we define $\widetilde{X}_b$ by requiring (1) above and additionally

(2′) $J(c, b) < J(c, a)$,
(3′) c winds around b.

As an analog of Proposition 7.1, we have the following proposition.

Proposition 12.3. *Let a and b be two generic circles in N such that $I(a, b) = 2$ and $N_{a\cup b}$ is nonorientable. Then for any integer $k \neq 0$, we have*

$$t_a^k(\widetilde{X}_b) \subseteq \widetilde{X}_a \quad \textit{and} \quad t_b^k(\widetilde{X}_a) \subseteq \widetilde{X}_b.$$

Proof. As in the proof of Proposition 7.1, we concentrate on the inclusion $t_a^k(\widetilde{X}_b) \subseteq \widetilde{X}_a$. We begin by constructing the circle $d = t_a^k(c)$ and as before, we assume that t_a^k twists to the right in N_a. We perform reductions of type I on d, and as a result, we obtain a circle $d_1 \in \mathcal{C}$ which winds around a. Observe that by Proposition 12.2, showing that $J(d_1, b) > J(d_1, a)$ is sufficient (we do not need to focus on reductions of types II–III).

Let n_A, n_B, n_C, and n_D be numbers of arcs of $c \cap N_a$ of types A, B, C, and D, respectively (Figure 10). In particular,

$$J(d_1, a) = n_A + n_B + n_C + n_D.$$

To determine the number $J(d_1, b)$, suppose first that $|k| = 1$. Each arc of $c \cap N_a$ of type A gives an arc of d_1 which goes once around a and therefore gives $I(a, b) = 2$ in $J(d_1, b)$. An arc of $c \cap N_a$ of type B gives $I(a, b) + 1 = 3$ in $J(d_1, b)$, and an arc of type C gives $I(a, b) - 1 = 1$. An arc of $c \cap N_a$ of type D does not change after the twist and gives 1 in $J(d_1, b)$. Finally, if $|k| > 1$, then for each arc of $c \cap N_a$ of types A–C, we have additional $(|k| - 1) \cdot I(a, b) = 2(|k| - 1)$ arcs of $d_1 \setminus N_b$. Hence, we proved the following formula:

$$J(d_1, b) = 2n_A + 3n_B + n_C + n_D + 2(|k| - 1) \cdot I(a, c).$$

Given that c winds around b, we have $n_A > 0$. Hence, $J(d_1, b) > J(d_1, a)$. □

Remark 12.4. The proof of Proposition 12.3 can be repeated with minimal changes when $I(a, b) \geq 2$ and no component of $N \setminus N_{a\cup b}$ is a disk or an annulus (for example if $N = N_{a\cup b}$). However, if disks are present in the complement of $N_{a\cup b}$, then Proposition 12.2 is not true and the situation becomes difficult.

13. Twists Generating a Free Group

Recall the so-called "Ping Pong Lemma" (see, for example, [2, Lemma 3.15]).

Lemma 13.1. *Suppose that a group G acts on a set Y, and $Y_1, Y_2 \subseteq Y$ are nonempty and disjoint. Let $g_1, g_2 \in G$ such that for every nonzero integer k,*

$$g_1^k(Y_2) \subseteq Y_1 \quad \text{and} \quad g_2^k(Y_1) \subseteq Y_2.$$

Then the group generated by g_1 and g_2 is a free group of rank 2.

Theorem 13.2. *Let a and b be two generic two-sided circles in a nonorientable surface N. If $I(a, b) \geq 2$, then the group generated by t_a and t_b is isomorphic to the free group of rank 2.*

Proof. If $I(a,b) \in \{2,3\}$ and $N_{a\cup b}$ is orientable, then we can repeat Atsushi Ishida's proof [3] without any changes.

Let (Y_a, Y_b) be equal to $(\widehat{X}_a, \widehat{X}_b)$ or $(\overline{X}_a, \overline{X}_b)$ or else $(\widetilde{X}_a, \widetilde{X}_b)$, where the sets $\widehat{X}_a$, $\widehat{X}_b$, $\overline{X}_a$, $\overline{X}_b$, $\widetilde{X}_a$, and $\widetilde{X}_b$ are defined in sections 8, 11, and 12. Observe that Y_a and Y_b satisfy

$$Y_a \cap Y_b = \emptyset,\ a \in Y_a,\ b \in Y_b.$$

Hence, if $I(a,b) \in \{2,3\}$ and $N_{a\cup b}$ is nonorientable, or $I(a,b) \geq 4$, then the theorem follows from Lemma 13.1 and propositions 10.4, 11.3, and 12.3. □

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