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AN INTEGRAL WEIGHT REALIZATION THEOREM FOR SUBSET CURRENTS ON FREE GROUPS

ILYA KAPOVICH

Abstract. We prove that if $N \geq 2$ and $\alpha : F_N \to \pi_1(\Gamma)$ is a marking on F_N, then, for any integer $r \geq 2$ and any F_N-invariant collection of non-negative integral "weights" associated to all subtrees K of $\widetilde{\Gamma}$ of radius $\leq r$ satisfying some natural "switch" conditions, there exists a finite cyclically reduced folded Γ-graph Δ realizing these weights as numbers of "occurrences" of K in Δ. As an application, we give a new, direct, and explicit proof of one of the main results of our paper with Tatiana Nagnibeda (*Subset currents on free groups*, Geom. Dedicata **166** (2013), 307–348) stating that, for any $N \geq 2$, the set $\mathcal{S}\mathrm{Curr}_r(F_N)$ of all rational subset currents is dense in the space $\mathcal{S}\mathrm{Curr}(F_N)$ of subset currents on F_N. (The proof given in the above-cited paper was indirect and omitted significant details. The proof given here is complete and, we hope, more accessible to the $Out(F_N)$ community.)

We also answer one of the questions (Problem 10.11) posed in the above-mentioned paper. Thus, we prove that if a nonzero $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ has all weights with respect to some marking being integers, then μ is the sum of finitely many "counting" currents corresponding to nontrivial finitely generated subgroups of F_N.

1. Introduction

The main purpose of this paper is to give a proof of Theorem B below (originally established in [24] via an indirect argument) which is self-contained, direct, explicit, and can be relatively easily understood by the

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$Out(F_N)$ community. We start with some history and motivation for the topic of subset currents on F_N.

1.1. Shift-invariant measures and measures supported on periodic orbits.

Let A be a finite alphabet consisting of ≥ 2 letters. One of the main objects studied in symbolic dynamics is the space $\mathcal{M}_A$ of all finite shift-invariant positive Borel measures on the two-sided shift space $A^{\mathbb{Z}}$. Here $A^{\mathbb{Z}}$ is the space of all bi-infinite words $\xi = \ldots x_{-2}x_{-1}x_0x_1x_2\ldots x_n\ldots$, where $x_i \in A$. We also think of elements of $A^{\mathbb{Z}}$ as functions $\xi : \mathbb{Z} \to A$ where $\xi(i)$ is the i^{th} letter of ξ.

The space $A^{\mathbb{Z}}$ is endowed with the standard topology where two sequences $\xi_1, \xi_2 \in A^{\omega}$ are "close" if $\xi_1|_{[-n,\ldots,n]} = \xi_2|_{[-n,\ldots,n]}$ for a large $n \geq 1$. With this topology, $A^{\mathbb{Z}}$ is homeomorphic to the Cantor set. The *shift map* $T : \mathcal{A}^{\mathbb{Z}} \to A^{\mathbb{Z}}$ consists in shifting every bi-infinite word one letter to the left. Thus, for an element $\xi : \mathbb{Z} \to A$ of $A^{\mathbb{Z}}$, we have $(T\xi)(i) = \xi(i+1)$, where $i \in \mathbb{Z}$. This map T is easily seen to be a homeomorphism of $A^{\mathbb{Z}}$. Now $\mathcal{M}_A$ consists of all T-invariant positive Borel measures on $A^{\mathbb{Z}}$ with $0 \leq \mu(A^{\mathbb{Z}}) < \infty$, that is, of all finite positive Borel measures μ on $A^{\mathbb{Z}}$ such that, for every Borel subset $S \subseteq A^{\mathbb{Z}}$, we have $\mu(S) = \mu(T^{-1}S)$.

The space $\mathcal{M}_A$, endowed with the weak-* topology, is a locally compact infinite-dimensional Hausdorff space. For $\xi \in A^{\mathbb{Z}}$ the orbit $\mathcal{O}_T(\xi) = \{T^i\xi | i \in \mathbb{Z}, i \geq 0\}$ is finite if and only if the word ξ is periodic, that is, has the form $\xi = \overset{\infty}{w} = \ldots wwww\ldots$ for some nontrivial word w over A which is not a proper power in the set A^* of all finite words over A. More precisely, if $w \in A^*$ is as above and $m = |w| \leq 1$ is the length of w, the word $\xi = \overset{\infty}{w}$ is defined so that for any $n \in \mathbb{Z}$ and $j \in \{1, \ldots, m\}$ with $n \equiv j$ mod m, the element $\xi(n) \in A$ is the j^{th} letter of w. Then the orbit $\mathcal{O}_T(\xi)$ has cardinality m and $\mathcal{O}_T(\xi) = \{\overset{\infty}{z} \mid z \text{ is a cyclic permutation of } w\}$. Such finite T-orbits are also called *periodic* orbits for T.

Denote by $Z(A)$ the set of all nontrivial words $w \in A^*$ which are not proper powers in A^*. For every $w \in Z(A)$, there is an associated T-invariant measure $\mu_w \in \mathcal{M}_A$ supported on $\mathcal{O}_T(\overset{\infty}{w})$ and defined as $\mu_w = \sum_z \delta_{\overset{\infty}{z}}$, where the summation is taken over all cyclic permutations z of w. Now if $w \in A^*$ is an arbitrary nontrivial word, there exist unique $k \geq 1$ and $w_1 \in Z(A)$ such that $w = w_1^k$. We then define $\mu_w := k\mu_{w_1}$. For convenience for the empty word $\varepsilon \in A^*$, we put $\mu_\varepsilon = 0 \in \mathcal{M}_A$. Then, for every $w \in A^*$ and every $k \geq 1$, we have $\mu_{w^k} = k\mu_w$.

A key basic result of symbolic dynamics says that the set

$$\mathcal{R}_A := \{c\mu_w | w \in Z(A), c \geq 0\} = \{c\mu_w | w \in A^*, c \geq 0\}$$

is a dense subset of $\mathcal{M}_A$; see [25], [27], and [28]. A similar fact also holds for irreducible subshifts of finite type in $A^{\mathbb{Z}}$. There are many different proofs of the fact that $\mathcal{R}_A$ is dense in $\mathcal{M}_A$ but, in combinatorial terms, most of these proofs rely (directly or indirectly) on a certain "weight realization" theorem. Since a generalization of this theorem for the case of subset currents on a free group is the main result of the present paper, we need to explain the classic case of the theorem here in more detail. To every nonempty finite word $v \in A^*$ with $|w| = m \geq 1$ and an integer $n \in \mathbb{Z}$, we associate a *cylinder* set $C_{v,n} \subseteq A^{\mathbb{Z}}$ consisting of all semi-infinite words $\xi \in A^{\mathbb{Z}}$ such that $\xi|_{[n,\ldots,n+m-1]} = v$. The sets $C_{v,n}$ are compact and open in $A^{\mathbb{Z}}$ and the collection of all such cylinder sets is a basis for the standard topology on $A^{\mathbb{Z}}$ mentioned above. If $\mu \in \mathcal{A}$, then, by shift-invariance of μ, for any $n \in \mathbb{Z}$ and any nontrivial $v \in A^*$ we have $\mu(C_{v,n}) = \mu(C_{v,0})$. Thus, for $\mu \in \mathcal{M}_A$ and a nonempty word $v \in A^*$, we define the *weight* $\langle v, \mu\rangle := \mu(C_{v,0})$. For the empty word $\varepsilon \in A^*$, we also put $\langle \varepsilon, \mu\rangle := \mu(A^{\mathbb{Z}})$.

Any measure $\mu \in \mathcal{M}_A$ is then uniquely determined by its collection of weights $(\langle v, \mu\rangle)_{\mu \in A^*}$. The fact that μ is finitely additive translates into the requirement that the weights satisfy the following "switch" conditions: for every $v \in A^*$, we have

$$(\dagger) \qquad \langle v, \mu\rangle = \sum_{a \in A} \langle va, \mu\rangle = \sum_{b \in A} \langle bv, \mu\rangle$$

and hence, for every $v \in A^*$, we have $\sum_{a \in A} \langle va, \mu\rangle = \sum_{b \in A} \langle bv, \mu\rangle$. Note that, for every $\mu \in \mathcal{M}_A$ and every integer $k \geq 1$, we have $\mu(A^{\mathbb{Z}}) = \sum_{v \in A^k} \langle v, \mu\rangle$. Kolmogorov's measure extension theorem implies that for any collection of nonnegative "weights," indexed by elements of A^*, there exists a unique measure $\mu \in \mathcal{M}_A$ realizing these weights. That is, $\mathcal{M}_A$ has a bijective correspondence with the set of all families $\mathfrak{t} = (t_v)_{v \in A^*}$ of nonnegative real numbers such that, for every $v \in A^*$, we have

$$(\spadesuit) \qquad \sum_{a \in A} t_{va} = \sum_{b \in A} t_{bu}.$$

For $w, v \in A^*$, the weight $\langle v, \mu_w\rangle = \mu_w(C_{v,0})$ has a useful combinatorial interpretation. Namely, for a nontrivial $w \in A^*$, let $\overline{w}$ be the associated *cyclic word*, that is, a directed labelled graph obtained by subdividing a circle into $m = |w|$ edges, with directed edges labelled by elements of A, so that going around this circle counterclockwise once from some vertex on this graph results in reading precisely the word w. The graph $\underline{w}$ does not have a distinguished base-vertex, so that, for any cyclic permutation z of w, the graphs $\overline{w}$ and $\overline{z}$ are isomorphic as directed labelled graphs.

For every nontrivial word $v \in A^*$, let $\langle v, \overline{w} \rangle$ be the number of *occurrences* of v in $\overline{w}$, that is, the number of vertices in $\overline{w}$ from which it is possible to "read" the word v in $\overline{w}$ by going counterclockwise and while never leaving the circle $\overline{w}$. (We allow the path corresponding to reading v in $\overline{w}$ to possibly begin and end at different vertices and also to possibly overlap itself.) For example, if $A = \{a, b\}$ and $w = a^2$, then for every $k \geq 1$, we have $\langle a^k, \overline{w} \rangle = 2$. A key basic observation shows that for any nontrivial words $v, w \in A^*$, we have $\langle v, \overline{w} \rangle = \mu_w(C_v) = \langle v, \mu_w \rangle$. While there are many ways to prove that the set $\mathcal{R}_A \subseteq \mathcal{M}_A$ is dense in $\mathcal{M}_A$, the most explicit proofs of this fact rely on the following "integral weight realization theorem."

Proposition 1.1. *Let A be a finite alphabet consisting of at least two letters and let $m \geq 2$ be an arbitrary integer. Let $\tau = (t_v)_{v \in A^m}$ be a family of non-negative integers $t_v \in \mathbb{Z}, t_v \geq 0$ such that, for some $v \in A^m$, $t_v \neq 0$ and such that, for every $u \in A^{m-1}$, we have*

$$(\diamondsuit) \qquad \sum_{a \in A} t_{ua} = \sum_{b \in A} t_{bv}.$$

Then there exists a finite collection of nontrivial words $w_1, \ldots, w_p \in A^*$ such that, for every $v \in A^m$, we have $\sum_{i=1}^{p} \langle v, \overline{w_i} \rangle = t_v$.

Proposition 1.1 straightforwardly implies that the set of all finite linear combinations $c_1\mu_{w_1} + \cdots + c_k\mu_{w_k}$, where $k \geq 1$, $c_i \geq 0$, and $w_i \in \mathcal{A}^*$, is dense in $\mathcal{M}_A$. From here, with a bit of extra work, one can deduce that $\mathcal{R}_A = \{c\mu_w | w \in A^a st, c \geq 0\}$ is dense in $\mathcal{M}_A$ as well. The proofs of Proposition 1.1 usually rely on, in some form, finding Euler circuits in some directed "Rauzy–de Bruijn graph" Γ_τ (see [4], [15], and [29]) associated to $\tau = (t_v)_{v \in A^m}$ as in the statement of the proposition (assuming, say, that $t_v > 0$ for all $v \in A^m$). The existence of such an Euler circuit in Γ_τ requires checking that the in-degree of every vertex of Γ_w is equal to the out-degree of this vertex, and this condition does hold because of equation ($\diamondsuit$). See, for example, [19] and [20] for the implementation of this approach in the context of (ordinary) geodesic currents on free groups.

This "Euler circuit" approach to proving Proposition 1.1, as well as most other approaches to proving that $\mathcal{R}_A$ is dense in $\mathcal{M}_A$, significantly relies on the "commutative" or "linear" nature of the shift space $A^{\mathbb{Z}}$, that is, on the fact that every finite subword of an element of $A^{\mathbb{Z}}$ can be thought of as written on a segment of $\mathbb{Z}$. A key element of the geometry of $\mathbb{Z}$ utilized in all of these proofs uses the fact that every finite subsegment of $\mathbb{Z}$ has a unique direction of extending it forward in $\mathbb{Z}$ and a unique direction of extending it backwards in $\mathbb{Z}$. These approaches no longer work in the non-commutative and highly branching context of subset currents on free groups, and we will see below that a different tool is needed to prove a version of the integral weight realization theorem there.

1.2. **Subset currents on free groups.**

In a paper with Tatiana Nagnibeda [24] we introduced and studied the notion of a subset current on a free group F_N. This concept is motivated by that of a geodesic current. *Geodesic currents* on F_N are measures that generalize conjugacy classes of nontrivial elements of F_N. The space $Curr(F_N)$ of all geodesic currents on F_N turns out to be highly useful in the study of the dynamics and geometry of $\mathrm{Out}(F_N)$ and of the Culler–Vogtmann Outer space, particularly via the use of the "geometric intersection form" constructed in [21]. See [24] for an extended discussion and [2], [3], [9], [10], [18], [21], and [22] for recent examples of such applications. Similarly, the notion of a subset current is a measure-theoretic analog of the conjugacy class of a nontrivial finitely generated subgroup of F_N. For a free group F_N, let $\mathfrak{C}_N$ be the space of all closed subsets $S \subseteq \partial F_N$ such that S consists of at least two elements. The space $\mathfrak{C}_N$ comes equipped with a natural topology (see §2 below and [24] for details) such that $\mathfrak{C}_N$ is a locally compact totally disconnected Hausdorff topological space. The action of F_N on ∂F_N by translations extends to a natural translation action of F_N on $\mathfrak{C}_N$ by homeomorphisms. A *subset current* on F_N is a positive Borel measure μ on $\mathfrak{C}_N$ such that μ is finite on compact subsets and is F_N-invariant. The space $\mathcal{S}\mathrm{Curr}(F_N)$ of all subset currents on F_N comes equipped with a natural weak-* topology and a natural action of $\mathrm{Out}(F_N)$ by continuous $\mathbb{R}_{\geq 0}$-linear transformations.

Given a nontrivial finitely generated subgroup $H \leq F_N$, there is a naturally associated counting subset current $\eta_H \in \mathcal{S}\mathrm{Curr}(F_N)$. The limit set $\Lambda(H) \subseteq \partial F_N$ is a closed F_N-invariant subset of ∂F_N and, since $H \neq \{1\}$, we have $\Lambda(H) \in \mathfrak{C}_N$. Moreover, for any $g \in F_N$, $\Lambda(gHg^{-1}) = g\Lambda(H)$. If H is equal to its commensurator $Comm_{F_N}(H)$, we define $\eta_H := \sum_{H_1 \in [H]} \delta_{\Lambda(H_1)}$, where $[H]$ is the conjugacy class of H in F_N. For an arbitrary nontrivial finitely generated subgroup $H \leq F_N$, it is known that $m := [Comm_{F_N}(H) : H] < \infty$ and that $Comm_{F_N}(H)$ is equal to its own commensurator in F_N. Then we define $\eta_H := m\,\eta_{Comm_{F_N}(H)}$. It is shown in [24] that η_H is indeed a subset current on F_N. A subset current $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ is called *rational* if $\mu = c\eta_H$ for some $c \geq 0$ and some nontrivial finitely generated $H \leq F_N$. Denote by $\mathcal{S}\mathrm{Curr}_r(F_N)$ the set of all rational subset currents on F_N.

One can also equivalently describe η_H in more combinatorial terms, using Stallings' core graphs (see [24] and Proposition-Definition 2.5 below). Such a combinatorial description exists for any "marking" on F_N (that is, an isomorphism α between F_N and $\pi_1(\Gamma)$ where Γ is a finite connected graph without any degree-1 vertices and with the first Betti number equal to N). For the purposes of stressing the analogy with $A^{\mathbb{Z}}$ described above, we will assume that $A = \{a_1, \ldots, a_N\}$ is a free basis of F_N, that Γ_A is a wedge of N oriented loop-edges labelled by $a_1, \ldots, a_N$ wedged at a vertex p_0, and that F_N is identified with $\pi_1(\Gamma_A, p_0)$ in the natural way according to this labelling. Thus, $X_A = \widetilde{\Gamma}_A$ is the Cayley graph of F_N with respect to A and $\partial F_N = \partial X_A$. The space $\mathfrak{C}_N$ is then canonically identified with the space $\mathfrak{T}_A$ of all infinite subtrees Y of X_A such that Y has no degree-1 vertices. This identification is given by sending a tree $Y \in \mathfrak{T}_A$ to the closed subset $\partial Y \subseteq \partial X_A = \partial F_N$ so that $\partial Y \in \mathfrak{C}_N$. The inverse map is given by taking a closed subset S of ∂X_A consisting of at least two points and putting $T \in \mathfrak{T}_A$ to be the convex hull of S in X_A. Thus, elements of $\mathcal{S}\mathrm{Curr}(F_N)$ can be thought of as locally finite F_N-invariant measures on $\mathfrak{T}_A$. Let $\mathfrak{T}_{A,1}$ be the space of all $Y \in \mathfrak{T}_A$ such that the element $1 \in F_N$ is a vertex of Y; we think of 1 as a base-vertex for every tree $T \in \mathfrak{T}_{A,1}$. Then $\mathfrak{T}_{A,1}$ is a compact subset of $\mathfrak{T}_A$ and one can further identify $\mathcal{S}\mathrm{Curr}(F_N)$ with the set of all finite Borel measures on $\mathfrak{T}_{A,1}$ which are invariant with respect to "root change." This point of view connects $\mathcal{S}\mathrm{Curr}(F_N)$ with the study of "invariant random subgroups" (IRS) and of "unimodular graph measures" [1], [7], [8], [12], [13], [16], [31], [33], [34]. See [24] for a more detailed discussion regarding these connections.

The standard topology on $\mathfrak{T}_A$ (and thus on $\mathfrak{C}_N$) can be described in terms of suitable "cylinder" sets. For any finite non-degenerate subtree K of X_A, the cylinder $Cyl_A(K)$ is defined to be the set of all $Y \in \mathfrak{T}_A$ such that $K \subseteq Y$ and such that for every $\xi \in \partial T$, there exists a terminal edge e of K (oriented "from" K) such that the geodesic ray from the initial vertex of e to ξ in X_A starts with e.

We denote by $\mathcal{B}_A$ the set of all finite non-degenerate subtrees of X_A and by $\mathbf{B}_A$ the set of all F_N-translation classes $[K]$ of trees $K \in \mathcal{B}_A$. The cylinder $Cyl_A(K)$, where $K \in \mathcal{B}_A$, is compact and open in $\mathfrak{T}_A$ and forms a basis for the standard topology on $\mathfrak{T}_A$ (and hence, via the identification of $\mathfrak{T}_A$ with $\mathfrak{C}_N$, of the standard topology on $\mathfrak{C}_N$). If $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ and $K \in \mathcal{B}_A$, then, by F_N-invariance of μ, the value $\mu(Cyl_A(K))$ depends only on μ and the F_N-translation class $[K] \in \mathbf{B}_A$ of K. For $K \in \mathcal{B}_A$ and $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$, we define the *weights* $\langle [K], \mu \rangle_A = \langle K, \mu \rangle_A := \mu(Cyl_A(K))$. Then any $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ is uniquely determined by its collection of weights $(\langle K, \mu \rangle_A)_{K \in \mathcal{B}_A}$.

There are natural disjoint union "splitting" formulas for the cylinders which, in view of finite additivity of generalized currents, imply corresponding "switch condition" equations that are satisfied by the weights of subset currents. These switch conditions have somewhat unexpected form and that is why we discuss them here in more detail. Let $K \subseteq X_A$ be a finite non-degenerate tree and let e be a terminal (oriented) edge of K. Let $q(e)$ be the set of all edges e' of X_A such that ee' is a reduced edge-path in X_A. Since X_A is a $(2N)$-regular tree, we have $\#(q(e)) = 2N - 1$. Denote by $P_+(q(e))$ the set of all nonempty subsets of $q(e)$. Then $Cyl_A(K) = \sqcup_{K' \in P_+(q(e))} Cyl_A(K \cup K')$. Therefore, for every $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$, for every $K \in \mathcal{B}_A$, and for every terminal edge e of K, we have

$$\langle K, \mu \rangle_A = \sum_{K' \in P_+(q(e))} \langle K \cup K', \mu \rangle_A. \tag{!}$$

Equations (!) for elements of $\mathcal{S}\mathrm{Curr}(F_N)$ play the role of the switch conditions (†) for shift-invariant measures on a two-sided shift space discussed above. As noted earlier, if $H \leq F_N$ is a nontrivial finitely generated subgroup, then the counting subset current $\eta_H \in \mathcal{S}\mathrm{Curr}(F_N)$ can be described more explicitly in combinatorial terms. Namely, let $\widehat{X}_A$ be the cover of Γ_A corresponding to H and let $\Delta_H \subseteq \widehat{X}_A$ be the *core* of $\widehat{X}_A$, that is, the smallest connected subgraph of $\widehat{X}_A$ whose inclusion into $\widehat{X}_A$ is a homotopy equivalence. Then the oriented edges of Δ_H are naturally labelled by elements of $A^{\pm 1}$ and this labelling makes Δ_H into a *folded A-graph* in the sense of the theory of Stallings' folds ([23], [32]). That is, for any vertex of Δ_H and any letter $a \in A^{\pm 1}$, there is at most one edge labelled a in Δ_H originating from this vertex. Moreover, Δ_H is also *cyclically reduced*; that is, every vertex of this graph has degree ≥ 2. There is a natural bijective correspondence between the set of conjugacy classes of finitely generated nontrivial subgroups of F_N and the set of labelled isomorphism types of finite connected cyclically reduced folded A-graphs.

Given any folded (and possibly disconnected) cyclically reduced A-graph Δ and a finite tree $K \in \mathcal{B}_A$, an *occurrence* of K in Δ is a label-preserving graph map $f : K \to \Delta$ (sending vertices to vertices and edges to edges, respecting labels) which is an immersion and which is a local homeomorphism at every point of K other than terminal vertices of K. Thus, if x is a non-terminal vertex of K, then f maps a small neighborhood of x in K homeomorphically onto a small neighborhood of $f(x)$ in Δ. In particular, it follows that the degree of x in K is equal to the degree of $f(x)$ in Δ. We denote by $\langle K, \Delta \rangle_A$ the number of all occurrences of K in Δ. Then it turns out that, for every nontrivial finitely generated subgroup $H \leq F_N$ and every $K \in \mathcal{B}_A$, we have $\langle K, \eta_H \rangle_A = \langle K, \Delta_H \rangle_A$.

In order to approximate elements of $\mathcal{S}\mathrm{Curr}(F_N)$ by rational subset currents, we need to understand how to realize finite collections of non-negative integral weights (where, instead of all trees $K \in \mathcal{B}_A$, we take a suitable finite subset of $\mathcal{B}_A$) satisfying the switch conditions (!) by finite cyclically reduced graphs Δ.

Theorem A below (cf. Theorem 4.3) provides the requisite realizability result. Theorem A is stated for an arbitrary marking on F_N, but the case of the marking $F_N = \pi_1(\Gamma_A)$ given by an "N-rose" corresponding to a free basis A of F_N already conveys the essence of Theorem A. To set up the relevant notation for the general case, let $N \geq 2$, let $\alpha : F_N \to \Gamma$ be a marking on F_N, and let $X = \widetilde{\Gamma}$. Let $r \geq 2$ be an arbitrary integer and let $\mathcal{B}_{\Gamma,r}$ be the set of all finite non-degenerate subtrees K of X such that, for some vertex p of K, the distance from p to every terminal vertex of K is equal to r. Let $\mathcal{B}'_{\Gamma,r}$ be the set of all finite non-degenerate subtrees J of X such that, for some edge of J, the distance from the midpoint p of this edge to every terminal vertex of K is equal to $r - \frac{1}{2}$. Now let $J \in \mathcal{B}'_{\Gamma,r}$ and let p be the midpoint of an edge e of J such that the distance from p to every terminal vertex of J is equal to $r - \frac{1}{2}$. Let J_0 be the connected component of $J - \{p\}$ containing the origin of e and let J_1 be the connected component of $J - \{p\}$ containing the terminus of e. Let $e_1, \dots, e_n$ be all the terminal edges of J contained in J_0 and let $f_1, \dots, f_m$ be all the terminal edges of J contained in J_1. We say that $\{e_1, \dots, e_n\} \sqcup \{f_1, \dots, f_m\}$ is the *geometric partition* of the set of terminal edges of J.

Theorem A (Integral Weight Realization Theorem)**.** *Let $N \geq 2$, let $\alpha : F_N \to \Gamma$ be a marking on F_N, let $X = \widetilde{\Gamma}$, and let $r \geq 2$ be an arbitrary integer.*

Let $\vartheta : \mathcal{B}_{\Gamma,r} \to \mathbb{Z}_{\geq 0}$ be a function such that

(1) *for every $K \in \mathcal{B}_{\Gamma,r}$ and every $g \in F_N$, we have $\vartheta(gK) = \vartheta(K)$;*
(2) *for every $J \in \mathcal{B}'_{\Gamma,r}$ with the geometric partition $\{e_1, \dots, e_n\} \sqcup \{f_1, \dots, f_m\}$ of the set of terminal edges of J, we have*
$$\sum_{(U_1,\dots,U_n)} \vartheta(J \cup U_1 \cup \dots \cup U_n) = \sum_{(V_1,\dots,V_m)} \vartheta(J \cup V_1 \cup \dots \cup V_m).$$
Here, the first sum is taken over all $(U_1, \dots, U_n)$ such that $U_i \in P_+(q(e_i))$ and the second sum is taken over all $(V_1, \dots, V_m)$ such that $V_j \in P_+(q(f_j))$;
(3) *there exists $K \in \mathcal{B}_{\Gamma,r}$ such that $\vartheta(K) > 0$.*

Then there exists a finite folded (possibly disconnected) cyclically reduced Γ-graph Δ such that, for every $K \in \mathcal{B}_{\Gamma,r}$, we have
$$\vartheta(K) = \langle K, \Delta \rangle_\Gamma.$$

Note that in (2), constructing $J \cup U_1 \cup \dots \cup U_n$ amounts to adding new edges to each terminal vertex of J on one side of the geometric partition, while $J \cup V_1 \cup \cdots \cup V_m$ is obtained from J by adding edges to the terminal vertices of the other side of the geometric partition. Note also that if J is as in (2), then, for every $(U_1, \ldots, U_m)$ and every $(V_1, \ldots, V_m)$ as in the theorem, we automatically have that the trees $J \cup U_1 \cup \cdots \cup U_n$ and $J \cup V_1 \cup \cdots \cup V_m$ belong to $\mathcal{B}_{\Gamma,r}$.

Theorem A is a substitute for Proposition 1.1 in the context of subset currents. However, compared to the context of the two-sided shift $A^{\mathbb{Z}}$ considered in Proposition 1.1, even the statement of Theorem A is considerably more complicated and it requires a significant change of perspective to properly account for the "non-linear" nature of the tree $X = \widetilde{\Gamma}$ which replaces $\mathbb{Z}$ here. For the same reason, the proof of Theorem A does not rely on Euler circuit considerations but, instead, uses certain kinds of perfect matching arguments. Crucially, the proof of Theorem A is constructive, and therefore it can be used to produce explicit approximations of various versions of "random" or "uniform" subset currents considered in [24, §10] by rational subset currents (see further discussion below).

Using Theorem A, we obtain a new proof of one of the main results of [24].

Theorem B. *Let $N \geq 2$ be an integer. Then $\mathcal{S}Curr_r(F_N)$ is a dense subset of $\mathcal{S}Curr(F_N)$.*

Theorem B generalizes a similar result (see [20], [26]) for $\mathrm{Curr}(F_N)$, but the case of $\mathcal{S}\mathrm{Curr}(F_N)$ is considerably more difficult. The proof of Theorem B in [24] is indirect and relies on deep work of Lewis Bowen and Gábor Elek about "unimodular graph measures," that is, measures on spaces of rooted graphs that are invariant, in the appropriate sense, with respect to root-change. Given a free basis A of F_N and the Cayley graph X_A of F_N with respect to A, in [24] we relate subset currents to root-change invariant measures on the space $\mathcal{T}_1(X_A)$ of all infinite subtrees Y of X_A without degree-1 vertices such that Y contains the vertex 1 of X_A. For studying $\mathcal{T}_1(X_A)$, one can use the results of Bowen [5], [6] and Elek [14] about weakly approximating these measures by sequences of finite graphs and eventually conclude that $\mathcal{S}\mathrm{Curr}_r(F_N)$ is dense in $\mathcal{S}\mathrm{Curr}(F_N)$. Here we give a direct proof of Theorem B, bypassing the "unimodular graph measures" results. The proof shares some similarities with the approaches of Bowen and Elek, but is more combinatorial and explicit.

As another application of Theorem A, we solve [24, Problem 10.11] and obtain the following.

Theorem C (cf. Theorem 4.6)**.** *Let $N \geq 2$, let $\alpha : F_N \to \Gamma$ be a marking on F_N, and let $X = \widetilde{\Gamma}$. Let $\mu \in \mathcal{S}Curr(F_N)$ be a nonzero subset current such that, for every $K \in \mathcal{B}_\Gamma$, we have $\langle K, \mu \rangle_\Gamma \in \mathbb{Z}$.*

Then there exist $k \geq 1$ and nontrivial finitely generated subgroups $H_1, \ldots, H_k \leq F_N$ such that

$$\mu = \eta_{H_1} + \cdots + \eta_{H_k}.$$

One of the main reasons for writing this paper was to provide a proof of Theorem B that is complete and relatively easily understandable by the $Out(F_N)$ community. The technology developed here has already proved useful in [30], a new paper by Dounna Sasaki who solved a problem posed in [24] and related subset currents to the Strengthened Hanna Neumann Conjecture. Other potential applications include, for example, studying the "generic volume distortion factors" for free group automorphisms. In [24, §9] we constructed several versions of "uniform subset currents" on F_N corresponding to a free basis A of F_N, including the "absolute uniform current" $m_A^{\mathcal{S}} \in \mathcal{S}\mathrm{Curr}(F_N)$. If $T_A \in \mathrm{cv}_N$ is the Cayley tree of F_N with respect to A and if $\varphi \in \mathrm{Out}(F_N)$, it should be possible to interpret the geometric intersection number (as defined in [24]) $\langle T_A, \varphi m_A^{\mathcal{S}} \rangle$ as the "generic volume distortion" $\mathrm{vol}(\Delta_A(\varphi(H)))/\mathrm{vol}(\Delta_A(H))$. Here H is a "random subgroup," in the sense of projectively approximating $m_A^{\mathcal{S}}$ finitely generated subgroup of F_N, and $\Delta_A(H)$ is the core Stallings subgroup graph for H with respect to A. In order to provide such a characterization of $\langle T_A, \varphi m_A^{\mathcal{S}} \rangle$, we need to create a random process at step n outputting a subgroup $H_n \leq F_N$ such that almost surely $\lim_{n\to\infty}[\eta_{H_n}] = [m_A^{\mathcal{S}}]$ in $\mathbb{P}\mathcal{S}\mathrm{Curr}(F_N)$. (Here, for a subset current $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$, we denote by $[\mu] \in \mathbb{P}\mathcal{S}\mathrm{Curr}(F_N)$ the *projective class* of μ.) To do that, we need an explicit procedure for how to projectively approximate $m_A^{\mathcal{S}}$ by rational subset currents, and Theorem A provides such a procedure.

In addition, there is work in progress by Sasaki on developing the theory of subset currents for surface groups. Obtaining an analog of Theorem B remains an open problem in that context, and we hope that an explicit proof of Theorem B for free groups may prove useful there.

2. Background

We will use the same notation, conventions, and definitions as in [24] and only briefly recall some of them here. If Y is a graph, we denote by EY the set of oriented edges of Y. For $e \in EY$, $o(e)$ is the initial vertex of e, $t(e)$ is the terminal vertex of e, and $e^{-1} \in EY$ is the inverse edge of e.

2.1. The space $\mathfrak{C}_N$.

Let F_N be a free group of finite rank $N \geq 2$. The space $\mathfrak{C}_N$ consists of all closed subsets $S \subseteq \partial F_N$ such that S consists of at least two points. We topologize $\mathfrak{C}_N$ by choosing a visual metric d on ∂F_N and then using the

Hausdorff distance between closed subsets of ∂F_N to metrize $\mathfrak{C}_N$. This metric topology on $\mathfrak{C}_N$ does not depend on the choice of a visual metric on ∂F_N and turns $\mathfrak{C}_N$ into a locally compact totally disconnected Hausdorff topological space. The topology on $\mathfrak{C}_N$ can be described more explicitly in terms of the "subset cylinders." Given a marking $\alpha : F_N \xrightarrow{\sim} \pi_1(\Gamma)$ (where Γ is a finite connected graph without degree-1 and degree-2 vertices), let $X = \widetilde{\Gamma}$, taken with the simplicial metric, where every edge has length 1. Then α induces a quasi-isometry between F_N and X and hence gives an identification, via an F_N-equivariant homeomorphism, between ∂F_N and ∂X. As in [24], we denote by $\mathcal{K}_\Gamma$ the set of all finite non-degenerate subtrees $K \subseteq X$. If e is an oriented edge of X, we denote by $Cyl_X(e)$ the set of all $\xi \in \partial F_N$ such that the geodesic from $o(e)$ to ξ in X starts with e. Thus, $Cyl_X(e) \subseteq \partial F_N$ is a compact-open subset of ∂F_N. Now let $K \in \mathcal{K}_\Gamma$. Let $e_1, \dots, e_n \in EX$ be all the terminal edges of K (oriented "from" K). We define the *subset cylinder* $\mathcal{S}Cyl_\alpha(K) \subseteq \mathfrak{C}_N$ as the set of all $S \in \mathfrak{C}_N$ such that $S \subseteq \cup_{i=1}^n Cyl_X(e_i)$ and such that, for each $i = 1, \dots, n$, $S \cap Cyl_X(e_i) \neq \emptyset$. Then $\mathcal{S}Cyl_\alpha(K)$ is a compact-open subset of $\mathfrak{C}_N$ and the family $\{\mathcal{S}Cyl_\alpha(K) | K \in \mathcal{K}_\Gamma\}$ forms a basis for the topology on $\mathfrak{C}_N$ defined above.

Denote by $q(e)$ the set of all oriented edges e' in X such that ee' is a reduced edge-path in X. For any set B, we denote by $P_+(B)$ the set of all nonempty subsets of B. The following basic fact plays a key role in the theory of subset currents.

Lemma 2.1 (cf. [24, Lemma 3.5]). *Let $K \in \mathcal{K}_\Gamma$ and let $e_1, \dots, e_n$ be all the terminal edges of K. Then, for every $i = 1, \dots, n$, we have $\mathcal{S}Cyl_\alpha(K) = \sqcup_{U \in P_+(q(e_i))} \mathcal{S}Cyl_\alpha(K \cup U)$.*

2.2. Subset currents.

A *subset current* on F_N is a positive Borel measure μ on $\mathfrak{C}_N$ which is F_N-invariant and locally finite, that is, finite on all compact subsets of $\mathfrak{C}_N$.

The set of all subset currents on F_N is denoted $\mathcal{S}\mathrm{Curr}(F_N)$. The space $\mathcal{S}\mathrm{Curr}(F_N)$ is endowed with the natural weak-* topology of point-wise convergence of integrals of continuous functions. The weak-* topology on $\mathcal{S}\mathrm{Curr}(F_N)$ can be described in more concrete terms:

Let $\mu, \mu_n \in \mathcal{S}\mathrm{Curr}(F_N)$. Then $\lim_{n\to\infty} \mu_n = \mu$ in $\mathcal{S}\mathrm{Curr}(F_N)$ if and only if, for every finite non-degenerate subtree K of X, we have

$$\lim_{n\to\infty} \mu_n(\mathcal{S}Cyl_\alpha(K)) = \mu(\mathcal{S}Cyl_\alpha(K)).$$

For $K \in \mathcal{K}_\Gamma$ and $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$, denote $\langle K, \mu\rangle_\alpha := mu(\mathcal{S}Cyl_\alpha(K))$ and call this quantity the *weight of K in μ*. If $\mu, \mu' \in \mathcal{S}\mathrm{Curr}(F_N)$ satisfy

$\langle K, \mu\rangle_\alpha = \langle K, \mu'\rangle_\alpha$ for all $K \in \mathcal{K}_\Gamma$, then $\mu = \mu'$. Note that if $K \in \mathcal{K}_\Gamma$ and $g \in F_N$, then $g\mathcal{S}Cyl_\alpha(K) = \mathcal{S}Cyl_\alpha(gK)$. Hence, for any $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$, $g \in F_N$, and $K \in \mathcal{K}_\Gamma$, we have $\mu(\mathcal{S}Cyl_\alpha(K)) = \mu(g\mathcal{S}Cyl_\alpha(K))$, so that $\langle K, \mu\rangle_\alpha = \langle gK, \mu\rangle_\alpha$. For a given finite subtree K of X, we denote the F_N-translation class of K by $[K]$ (so that $[K]$ consists of all the translates of K by elements of F_N). We put $\langle [K], \mu\rangle_\alpha := \langle K, \mu\rangle_\alpha$ and call it the *weight of* $[K]$ *in* μ.

Lemma 2.1 immediately implies the following (cf. [24, Proposition 3.11]).

Proposition 2.2 (Kirchhoff formulas for weights)**.** *Let K be a finite non-degenerate subtree of X. Let e be one of the terminal edges of K and let $\mu \in \mathcal{S}Curr(F_N)$. Then*

$$(\bigstar) \qquad \langle K, \mu\rangle_\alpha = \sum_{U \in P_+(q(e))} \langle K \cup U, \mu\rangle_\alpha.$$

2.3. Γ-graphs.

Let $\alpha : F_N \xrightarrow{\sim} \pi_1(\Gamma)$ be a marking. A Γ-*graph* is a graph Δ together with a graph morphism $\tau : \Delta \to \Gamma$. For a vertex $x \in V\Delta$, we say that the *type of* x is the vertex $\tau(x) \in V\Gamma$. Similarly, for an oriented edge $e \in E\Gamma$, the *type of* e or the *label of* e is the edge $\tau(e)$ of Γ. Every covering of Γ has a canonical Γ-graph structure. In particular, Γ itself is a Γ-graph and so is the universal cover $\widetilde{\Gamma}$ of Γ. Also, every subgraph of a Γ-graph is again a Γ-graph.

Let $\tau_1 : \Delta_1 \to \Gamma$ and $\tau_2 : \Delta_2 \to \Gamma$ be Γ-graphs. A graph-map $f : \Delta_1 \to \Delta_2$ is called a Γ-*map*, or Γ-*morphism*, if it respects the labels of vertices and edges, that is, if $\tau_1 = \tau_2 \circ f$. A Γ-graph Δ is *folded* if the labeling map $\tau : \Delta \to \Gamma$ is an immersion, that is, if τ is locally injective.

Definition 2.3 (Link of a vertex)**.** Let Δ be a Γ-graph. For a vertex $x \in V\Delta$, denote by $Lk_\Delta(x)$ (or just by $Lk(x)$) the function

$$Lk_\Delta(x) : E\Gamma \to \mathbb{Z}_{\geq 0}$$

where, for every $e \in E\Gamma$, the value $(Lk_\Delta(x))(e)$ is the number of edges of Δ with origin x and label e.

Thus, a Γ-graph Δ is folded if and only if for every vertex $x \in V\Delta$ and every $e \in E\Gamma$, we have

$$(Lk_\Delta(x))(e) \leq 1.$$

If Δ is folded, we will also think of $Lk_\Delta(x)$ as a subset of $E\Gamma$ consisting of all those $e \in E\Gamma$ with $(Lk_\Delta(x))(e) = 1$, that is, of all $e \in E\Gamma$ such that there is an edge in Δ with origin x and label e.

We say that a nonempty finite Γ-graph Δ is *cyclically reduced* if Δ is folded and every vertex of Δ has degree ≥ 2. If $\tau : \Delta \to \Gamma$ is a cyclically reduced Γ-graph, then $W := \tau_{\#}(\pi_1(\Delta)) \leq \pi_1(\Gamma)$ is a finitely generated subgroup of $\pi_1(\Gamma)$. Recall that we also have a marking $\alpha : F_N \xrightarrow{\sim} \pi_1(\Gamma)$. We say that the subgroup $H := \alpha^{-1}(W) \leq F_N$ is *represented* by Δ. The conjugacy class of $[H]$ in F_N does not change if we replace α by an equivalent marking.

Definition 2.4 (Occurrence)**.** Let $K \subseteq \widetilde{\Gamma}$ be a finite non-degenerate subtree (recall that $\widetilde{\Gamma}$ and all of its subgraphs have canonical Γ-graph structure).

Let Δ be a finite cyclically reduced Γ-graph. An *occurrence* of K in Δ is a Γ-morphism $\mathfrak{O} : K \to \Delta$ such that for every vertex x of K of degree at least 2 in K, we have $Lk_K(x) = Lk_\Delta(\mathfrak{O}(x))$.

We denote the number of all occurrences of K in Δ by $\langle K; \Delta\rangle_\Gamma$, or just $\langle K; \Delta\rangle$.

In topological terms, a Γ-morphism $\mathfrak{O} : K \to \Delta$ is an occurrence of K in Δ if $\mathfrak{O}$ is an immersion and if $\mathfrak{O}$ is a covering map at every point $x \in K$ (including interior points of edges) except for the degree-1 vertices of K. That is, for every $x \in K$, other than a degree-1 vertex of K, $\mathfrak{O}$ maps a small neighborhood of x in K homeomorphically *onto* a small neighborhood of $\mathfrak{O}(x)$ in Δ. We need the following key fact from [24].

Proposition 2.5. *Let $\alpha : F_N \to \pi_1(\Gamma)$ be a marking on F_N and let $X = \widetilde{\Gamma}$. Recall that $\mathcal{K}_\Gamma$ is the set of all non-degenerate finite simplicial subtrees of X. Let $\tau : \Delta \to \Gamma$ be a finite cyclically reduced Γ-graph.*

Then there is a unique generalized current $\mu_\Delta \in \mathcal{S}Curr(F_N)$ such that for every $K \in \mathcal{K}_\Delta$

$$\langle K, \mu_\Delta\rangle_\alpha = \langle K; \Delta\rangle_\Gamma.$$

Moreover, if Δ is also connected, then $\mu_\Delta = \eta_H$, where $H \leq F_N$ is the finitely generated subgroup of F_N represented by Δ.

3. More on Cylinders and Kirchhoff-type Formulas

Convention 3.1. For the remainder of this paper, unless specified otherwise, we fix a marking $\alpha : F_N \to \pi_1(\Gamma)$. Put $X = \widetilde{\Gamma}$. We also equip X with the simplicial metric d by giving each edge of X length 1.

Let $K \subseteq X$ be a nondegenerate finite subtree and let e be a terminal edge of K. For an integer $m \geq 1$, we say that a finite nondegenerate subtree $U \subseteq X$ is (K, e, m)-*admissible* if

(1) we have $K \cap U = \{t(e)\}$.
(2) For every terminal vertex v of U such that $v \neq t(e)$, we have $d(t(e), v) = m$.

For $m = 0$, we also say that the degenerate tree $U = \{t(e)\}$ is $(K, e, 0)$-*admissible*.

For $m \geq 1$, we denote by $\mathcal{B}(K, e, m)$ the set of all U such that U is (K, e, m)-admissible. Thus, $P_+(q(e)) = \mathcal{B}(K, e, 1)$. Lemma 2.1 easily implies the following.

Corollary 3.2. *Let $K \subseteq X$ be a nondegenerate finite subtree and let e be a terminal edge of K. Then, for every integer $m \geq 1$, we have*

$$\mathcal{S}Cyl_\alpha(K) = \sqcup_{U \in \mathcal{B}(K,e,m)} \mathcal{S}Cyl_\alpha(K \cup U).$$

Definition 3.3 (Round graph)**.** For an integer $r \geq 1$, we say that a finite subtree K of X is a *round graph* of *grade* r in X if there exists a (necessarily unique) vertex v of K such that, for every terminal vertex u of K, we have $d(v, u) = r$.

Let $K \subseteq X$ be a nondegenerate finite subtree and let v be a vertex of K (possibly a terminal vertex). We denote by $R(K, v)$ the maximum of $d(v, v')$ where v' varies over all terminal vertices of K. The fact that K is nondegenerate means that $R(K, v) \geq 1$.

Let $e_1, \dots, e_n$ be the terminal edges of K and let $r \geq R(K, v)$ be an integer. We say that an n-tuple $\mathcal{T} = (U_1, \dots, U_n)$ of finite subtrees U_i of X is (K, v, r)-*admissible* if, for each $i = 1, \dots, n$, the tree U_i is (K, e_i, m_i)-admissible, where $m_i = r - d(v, t(e_i))$. Note that if $\mathcal{T} = (U_1, \dots, U_n)$ is (K, v, r)-admissible and $K' = K \cup U_1 \cdots \cup U_n$, then for every terminal vertex u of K', we have $d(v, u) = r$. Thus, K' is a round graph of grade r with center v. Corollary 3.2 directly implies the following.

Corollary 3.4. *Let $K \subseteq X$ be a nondegenerate subtree with terminal edges $e_1, \dots, e_n$. Let v be a vertex of K and let $r \geq R(K, v)$ be an integer. Denote by $\mathcal{B}(K, v, r)$ the set of all (K, v, r)-admissible n-tuples. Then*

$$\mathcal{S}Cyl_\alpha(K) = \bigsqcup_{(U_1,\dots,U_n) \in \mathcal{B}(K,v,r)} \mathcal{S}Cyl_\alpha(K \cup U_1 \cup \dots \cup U_n).$$

For a finite nondegenerate subtree $K \subseteq X$, we put $r(K)$ to be the minimum of $R(K, v)$ where v varies over all vertices of K. We refer to $r(K)$ as the *radius* of K.

In view of finite additivity of subset currents, Corollary 3.2 and Corollary 3.4 immediately imply the following.

Corollary 3.5. *Let $K \subseteq X$ be a finite non-degenerate subtree of X and let $\mu \in \mathcal{S}Curr(F_N)$.*

(1) *Then, for any terminal edge e of K and any integer $m \geq 1$, we have*

$$\langle K, \mu \rangle_\alpha = \sum_{U \in \mathcal{B}(K,e,m)} \langle K \cup U, \mu \rangle_\alpha.$$

(2) *Let v be a vertex of K, let $e_1, \dots, e_n$ be the terminal edges of K, and let $r \geq R(K, v)$ be an integer. Then we have*

$$\langle K, \mu\rangle_\alpha = \sum_{(U_1,\dots,U_n)\in\mathcal{B}(K,v,r)} \langle K \cup U_1 \cdots \cup U_n, \mu\rangle_\alpha.$$

Recall that, as noted earlier, for (2), if $(U_1, \dots, U_n) \in \mathcal{B}(K, v, r)$ and $K' = K \cup U_1 \cdots \cup U_n$, then, for any terminal vertex u of K', we have $d(v, u) = r$ so that K' is a round graph of grade r in X. Thus, Corollary 3.5(2) implies that, for $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ and an integer $r \geq 1$, knowing the μ-weights of all round graphs of grade r uniquely determines the μ-weights of all the subtrees of radius $\leq r$.

Definition 3.6 (Semi-round graph)**.** Let p be the mid-point of an edge e of X and let $r \geq 2$ be an integer. We say that a finite subtree J of X is a *semi-round graph* of *grade* r with *center* p if $e \in J$ and if, for every terminal vertex u of J, we have $d(p, u) = r - \frac{1}{2}$. Thus, for every terminal vertex u of J belonging to the connected component of $J - \{p\}$ containing $o(e)$, we have $d(o(e), v) = r - 1$. Similarly, for every terminal vertex u of J belonging to the connected component of $J - \{p\}$ containing $t(e)$, we have $d(t(e), v) = r - 1$.

Definition 3.7 (Child of a round graph)**.** Let $r \geq 2$ and let $K \subseteq X$ be a round graph of grade r in X centered at a vertex v of X. Let e be an edge of K with $o(e) = v$. Let p be the mid-point of e. We define a semi-round graph of grade r, centered at p, called the *e-child of K* and denoted K_e, as follows:

The graph K_e consists of all points $q \in K$ with $d(p, q) \leq r - \frac{1}{2}$. In other words, K_e is obtained from K by removing all those terminal vertices u of K and the terminal edges of K adjacent to these vertices such that the geodesic $[v, u]$ does not pass through the edge e. The definition of K_e is illustrated in Figure 1.

If H is a semi-round graph of grade r with center at the midpoint p of an edge e, then H can be enlarged to round graphs of grade r in two different "directions," namely to round graphs centered at $o(e)$ and at $t(e)$. This yields the following.

Proposition 3.8. *Let $J \subseteq X$ be a semi-round graph of grade $r \geq 2$ centered at the midpoint p of an edge e. Let J_0 and J_1 be the connected components of $J - \{p\}$ containing $o(e)$ and $t(e)$ accordingly. Let $e_1, \dots, e_n$ be all the terminal edges of J contained in J_0 and let $f_1, \dots, f_k$ be all the terminal edges of J contained in J_1. Let $\mathcal{B}_0$ be the set of all n-tuples of the form $(U_1, \dots, U_n)$ where each $U_i \in P_+(q(e_i))$, and let $\mathcal{B}_1$ be the set of all k-tuples of the form $(V_1, \dots, V_k)$ where each $V_j \in P_+(q(f_j))$.*

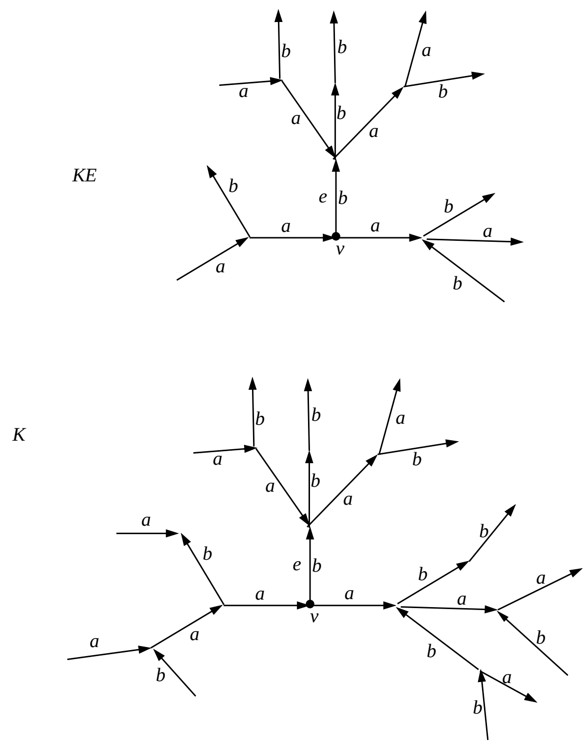

Figure 1. Child K_e of a round graph K of grade 3 with center v. Here $N = 2$, $F_2 = F(a, b)$, and Γ is the standard rose corresponding to the free basis $\{a, b\}$ of $F(a, b)$.

Then, for any $\mu \in \mathcal{S}Curr(F_N)$, we have

$$\langle J, \mu\rangle_\alpha = \sum_{(U_1,\ldots,U_n)\in\mathcal{B}_0} \langle J \cup U_1 \cdots \cup U_n, \mu\rangle_\alpha =$$

$$\sum_{(V_1,\ldots,V_k)\in\mathcal{B}_1} \langle J \cup V_1 \cdots \cup V_k, \mu\rangle_\alpha.$$

(Note that in the above summation each $J \cup U_1 \cdots \cup U_n$ is a round graph of grade r centered at $o(e)$ and each $J \cup V_1 \cdots \cup V_k$ is a round graph of grade r centered at $t(e)$.)

Proof. This statement is a direct corollary of Proposition 2.2. $\square$

4. Finite-Dimensional Polyhedral Approximations of $\mathcal{S}\text{Curr}(F_N)$ and the Integral Weight Realization Theorem

Let $r \geq 2$ be an integer. Denote by $\mathcal{B}_{\Gamma,r}$ the set of all finite subtrees $K \subseteq X$ such that K is a round graph of grade r in X. Also, denote by $\mathbf{B}_{\Gamma,r}$ the set of all F_N-translation classes $[K]$ of trees $K \in \mathcal{B}_{\Gamma,r}$.

Denote by $\mathbf{J}_{\Gamma,r}$ the set of all F_N-translation classes $[J]$ of semi-round graphs $J \subseteq X$ of grade r.

Definition 4.1 (Approximating polyhedra)**.** Denote by $\mathcal{Q}_{\Gamma,r}$ the set of all functions $\vartheta : \mathcal{B}_{\Gamma,r} \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

(1) For every $K \in \mathcal{B}_{\Gamma,r}$ and every $g \in F_N$, we have $\vartheta(K) = \vartheta(gK)$.
(2) For every semi-round graph $J \subseteq X$ of grade r, in the notation of Proposition 3.8, we have

$$\sum_{(U_1,\ldots,U_n)\in\mathcal{B}_0} \vartheta(J \cup U_1 \cdots \cup U_n) = \sum_{(V_1,\ldots,V_k)\in\mathcal{B}_1} \vartheta(J \cup V_1 \cdots \cup V_k).$$

We call $\mathcal{Q}_{\Gamma,r}$ an *approximating polyhedron.*

Note that since X is locally finite, there are only finitely many F_N-translation classes $[K]$ of trees $K \in \mathcal{B}_{\Gamma,r}$. Thus, a point $\theta \in \mathcal{Q}_{\Gamma,r}$ can be viewed as a function from a finite set $\mathbf{B}_{\Gamma,r}$ to $\mathbb{R}_{\geq 0}$. Namely, if m is the cardinality of $\mathbf{B}_{\Gamma,r}$, we can view $\mathcal{Q}_{\Gamma,r}$ as a subset of $\mathbb{R}^m_{\geq 0}$, given by finitely many linear equations with integer coefficients coming from Definition 4.1(2).

The following lemma is a straightforward inductive corollary of Definition 2.4.

Lemma 4.2. *Let Δ be a finite cyclically reduced Γ-graph. Then, for every integer $r \geq 1$,*

$$\#V(\Delta) = \sum_{[K]\in\mathbf{B}_{\Gamma,r}} \langle K, \Delta\rangle_\alpha.$$

Recall that, by definition, any Γ-graph Υ comes equipped with a "labelling" graph-map $\tau : \Upsilon \to \Gamma$.

The following statement is equivalent to Theorem A from the Introduction.

Theorem 4.3. *Let $r \geq 2$ and let $\vartheta \in \mathcal{Q}_{\Gamma,r}$ be such that, for some $K_0 \in \mathcal{B}_{\Gamma,r}$, we have $\vartheta(K_0) > 0$. Suppose also that, for every $K \in \mathcal{B}_{\Gamma,r}$, we have $\vartheta(K) \in \mathbb{Z}$. Then there exists a cyclically reduced (and possibly disconnected) finite Γ-graph Δ such that, for every $K \in \mathcal{B}_{\Gamma,r}$, we have $\vartheta(K) = \langle K, \Delta\rangle_\alpha$.*

Proof. For each $[K] \in \mathbf{B}_{\Gamma,r}$, we choose a representative $K \in [K]$, so that $K \in \mathcal{B}_{\Gamma,r}$ and let $v = v_{[K]}$ be the center vertex of K. Thus, K is a round graph of rank r centered at v. Denote $n_{[K]} := \vartheta(K)$. By assumption, every $n_{[K]} \geq 0$ is an integer and there exists $K_0 \in \mathcal{B}_{\Gamma,r}$ such that $n_{[K_0]} \geq 0$.

For every $[K] \in \mathbf{B}_{\Gamma,r}$, we make $n_{[K]}$ copies $v_{[K],i}$ (where $i = 1, \ldots, n_{[K]}$) of the vertex $v_{[K]}$, together with "half-links" of $v_{[K]}$ in K. That is, for each $v_{[K],i}$ and for each edge e of K with $o(e) = v_{[K]}$, we attach a closed half-edge $[v_{[K],i}, p_{e,i}]$ at $v_{[K],i}$ representing a copy of the initial half of the edge e.

We refer to the points $p_{e,i}$ as *subvertices* and to the segments $[v_{[K],i}, p_{e,i}]$ as *sub-edges*. We endow each subvertex $p_{e,i}$ with a *decoration*, which is an ordered pair $(\tau(e), [K_e])$, where K_e is the e-child of K at v.

Let Ω_ϑ be the collection of all the decorated "half-links" obtained in this way. Thus, Ω_ϑ consists of $M := \sum_{[K]\in\mathbf{B}_{\Gamma,r}} \vartheta(K) = \sum_{[K]\in\mathbf{B}_{\Gamma,r}} n_{[K]}$ "half-links."

Property (2) in Definition 4.1 implies that for every semi-round graph $J \subseteq X$ of grade r with center p being a mid-point of an oriented edge e_J of X, the number of subvertices with decoration $(\tau(e_J), [J])$ is equal to the number of subvertices with decoration $(\tau(e_J^{-1}), [J])$.

For each $[J] \in \mathbf{J}_{\Gamma,r}$, as above, we choose a matching (i.e., a bijection) between the set of subvertices in Ω_ϑ with decoration $(\tau(e_J), [J])$ and the set of subvertices with decoration $(\tau(e_J^{-1}), [J])$

We then identify each subvertex with decoration $(\tau(e_J), [J])$ with the corresponding to it under this matching subvertex with decoration $(\tau(e_J^{-1}), [J])$. We perform these identifications simultaneously for all $[J] \in \mathbf{J}_{\Gamma,r}$. This gluing procedure is illustrated in Figure 2.

The resulting object Δ has a natural structure of a graph, where every vertex is of the form $v_{[K],i}$ and every edge is obtained by gluing two sub-edges along a subvertex; thus, subvertices become mid-points of edges in Δ. Moreover, Δ inherits a natural Γ-graph structure as well. Indeed, an oriented edge f in Δ arises as the result of gluing a sub-edge $[v_{[K],i}, p_{e,i}]$ and a sub-edge $[v_{[K'],j}, p_{e',j}]$ by identifying the subvertices $p_{e,i}$ and $p_{e',j}$ where $p_{e,i}$ is decorated by $(\tau(e), [K_e])$ and $p_{e',j}$ is decorated by $(\tau(e'), [K_{e'}])$ such that $[K_e] = [K_{e'}]$ and such that $\tau(e') = \tau(e)^{-1}$. In Δ, we have $o(f) = v_{[K],i}$ and $t(f) = v_{[K'],j}$. We put $\tau(f) := \tau(e) \in E\Gamma$ and $\tau(f^{-1}) := \tau(e') = \tau(e)^{-1}$. Also, for each vertex $v_{[K],i}$ of Δ, put $\tau(v_{[K],i}) := \tau(v_{[K]})$. This turns Δ into a nonempty Γ-graph. Moreover, by construction Δ is finite, folded, and cyclically reduced and the number of vertices in Δ is equal to $M = \sum_{[K]\in\mathbf{B}_{\Gamma,r}} \vartheta(K) = \sum_{[K]\in\mathbf{B}_{\Gamma,r}} n_{[K]}$.

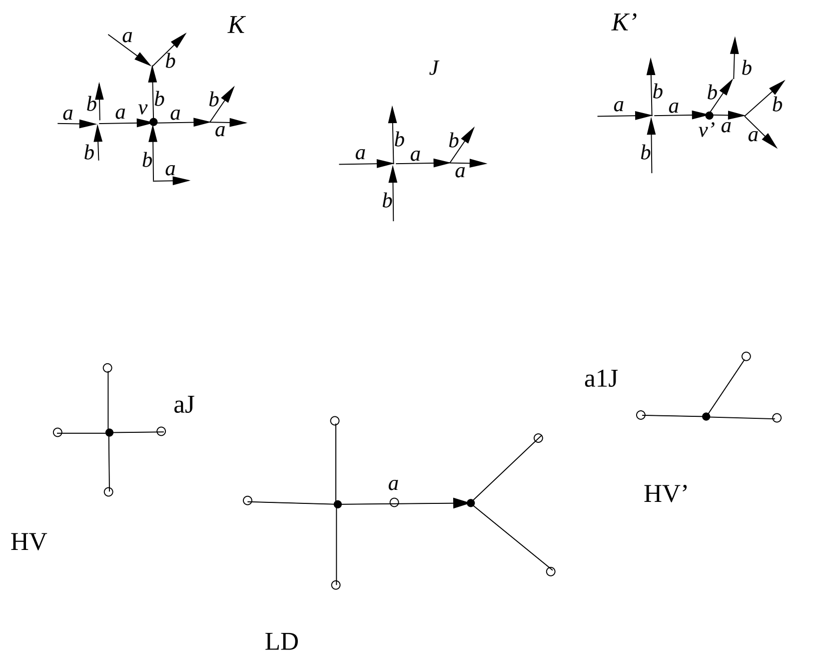

FIGURE 2. Illustration of the gluing procedure for constructing Δ in the proof of Theorem 4.3. Here $N = 2$, $F_2 = F(a, b)$, and Γ is the standard rose corresponding to the free basis $\{a, b\}$ of $F(a, b)$. Filled circles represent vertices and non-filled circles represent subvertices.

Note that by construction, for every vertex $v_{[K],i}$ of Δ, we have $Lk_\Delta(v_{[K],i}) = Lk_K(v_{[K]})$. (Recall that $v_{[K]}$ is the center vertex of the round graph K.) Moreover, by definition of a child of a round graph and using the fact that $r \geq 2$, we see that if $f = [v_{[K],i}, v_{[K'],j}]$ is an edge of Δ as in the preceding paragraph, then $Lk_\Delta(v_{[K'],j}) = Lk_K(t(e))$. Iteratively applying this crucial fact to the spheres of increasing radius around the center vertex in K, we see that, for each vertex $v_{[K],i}$ of Δ as above, sending $v_{[K]}$ to $v_{[K],i}$ extends to a (necessarily unique) morphism of Γ-graphs $\mathfrak{O} : K \to \Delta$ with $\mathfrak{O}(v_{[K]}) = v_{[K],i}$ such that $\mathfrak{O}$ is an occurrence of K in Δ in the sense of Definition 2.4.

Moreover, given a vertex u of Δ, there exists exactly one occurrence of a round graph of grade r in Δ that sends the center of that round graph to u. (This occurrence corresponds to taking the ball of radius r in $\widetilde{\Delta}$ centered at a lift of u.) Thus, by construction, we see that, for every $[K] \in \mathbf{B}_{\Gamma,r}$, the number of occurrences of $[K]$ in Δ is equal to $n_{[K]}$. Hence, for every $K \in \mathcal{B}_{\Gamma,r}$, we have $\vartheta(K) = \langle K, \Delta \rangle_\alpha$, as required. $\square$

Remark 4.4. There is an alternative equivalent description of the graph Δ constructed in the proof of Theorem 4.3. Namely, for every $[K] \in \mathbf{B}_{\Gamma,r}$, we make $n_{[K]} = \vartheta(K)$ copies $[K]_i$ (where $i = 1, \dots, n_{[K]}$) of K and denote the center vertex of $[K]_i$ by $v_{[K],i}$. We then look at the set Ξ of all pairs $([K]_i, e)$ where $[K]_i$ is as above and e is an edge of K with $o(e) = v_{[K]}$, the center vertex of K. We endow each $([K]_i, e)$ with a "decoration" $(\tau(e), [K_e])$. Thus, $[K_e]$ is a semi-round graph of grade r, which comes from the ball of radius $r - \frac{1}{2}$ in K centered at the midpoint of e. Property (2) in Definition 4.1 implies that for every semi-round graph $J \subseteq X$ of grade r with center p being a mid-point of an oriented edge e_J of X, the number of elements of Ξ with decoration $(\tau(e_J), [J])$ is equal to the number of elements of Ξ with decoration $(\tau(e_J^{-1}), [J])$.

For each $[J] \in \mathbf{J}_{\Gamma,r}$ as above we choose a matching between the set of elements of Ξ with decoration $(\tau(e_J), [J])$ and the set of elements of Ξ with decoration $(\tau(e_J^{-1}), [J])$. Then we perform partial gluings on the disjoint union Ω of all $[K]_i$ (where $[K]$ varies over $\mathbf{B}_{\Gamma,r}$) as follows. Whenever $([K]_i, e)$ is matched with $([K']_j, e')$, it follows that the e-child K_e of K is (canonically) isomorphic as a Γ-graph to the e'-child $K'_{e'}$ of K'. (Recall that K_e is the ball of radius $r - \frac{1}{2}$ in K centered at the midpoint of e and that $K'_{e'}$ is the ball of radius $r - \frac{1}{2}$ in K' centered at the midpoint of e'.) We glue the copy of K_e in $[K]_i$ to the copy of $K'_{e'}$ in $[K']_j$ along the Γ-graph isomorphism between K_e and $K'_{e'}$. We perform these gluings simultaneously on the disjoint union Ω of all $[K]_i$, as $[K]$ varies over $\mathbf{B}_{\Gamma,r}$. The result is a cyclically reduced finite Γ-graph which is the same as the Γ-graph Δ constructed in the proof of Theorem 4.3. This alternative gluing procedure is illustrated in Figure 3.

Remark 4.5. Suppose that in Theorem 4.3, $\vartheta \in \mathcal{Q}_{\Gamma,r}$ has the property that whenever $\vartheta(K) > 0$, $K \subseteq X$ is a geodesic segment of length $2r$ in X. Then, for each such K, the center vertex $v_{[K]}$ (which is the mid-point of this segment) has degree 2 in K and the proof of Theorem 4.3 produces a finite cyclically reduced graph Δ where every vertex has degree 2 so that Δ is a disjoint union of finitely many simplicial circles. One can use this fact to adapt the proof of Theorem 5.1 below to the case of ordinary geodesic currents and to produce a new proof (different from those given in [20] and [26]) that the set of rational currents is dense in $\mathrm{Curr}(F_N)$.

The following statement (cf. Theorem C from the Introduction) provides a positive answer to [24, Problem 10.11].

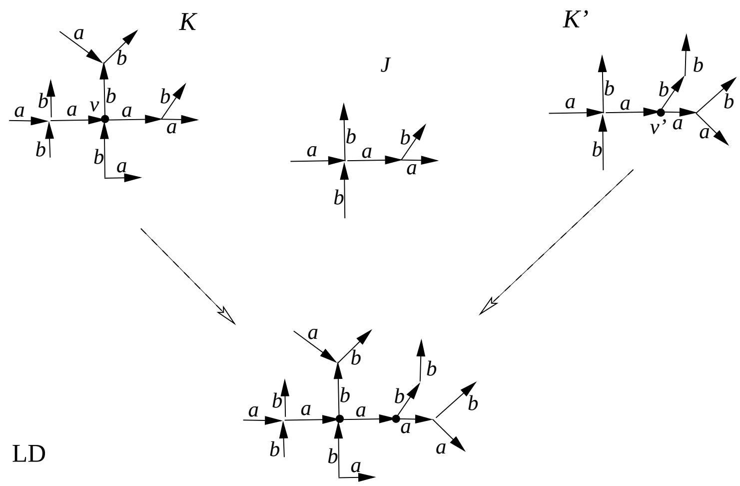

FIGURE 3. Illustration of the alternative gluing procedure for constructing Δ as in Remark 4.4. Here $N = 2$, $F_2 = F(a, b)$, and Γ is the standard rose corresponding to the free basis $\{a, b\}$ of $F(a, b)$.

Theorem 4.6. *Let $\mu \in \mathcal{S}Curr(F_N)$ be a nonzero subset current such that, for every nondegenerate finite subtree $K \subseteq X$, we have $\langle K, \mu \rangle \in \mathbb{Z}$. Then there exists a finite cyclically reduced (possibly disconnected) Γ-graph Δ such that $\mu = \mu_\Delta$.*

Proof. Put $M := \sum_{[K] \in \mathbf{B}_{\Gamma,1}} \langle K, \mu \rangle_\alpha$. For every $r \geq 2$, define the function $\theta_r : \mathcal{B}_{\Gamma,r} \to \mathbb{R}_{\geq 0}$ by $\theta_r(K) := \langle K, \mu \rangle_\alpha$, where $K \in \mathcal{B}_{\Gamma,r}$. Since μ is a nonzero subset current, we have that $\theta_r \in \mathcal{Q}_{\Gamma,r}$ for all $r \geq 2$.

Hence, by Theorem 4.3, for every $r \geq 1$, there exists a finite cyclically reduced Γ-graph Δ_r such that $\langle K, \Delta_r \rangle_\alpha = \langle K, \mu \rangle_\alpha$ for every $K \in \mathcal{B}_{\Gamma,r}$. Corollary 3.5 then implies that for every $r \geq 2$ and every finite nondegenerate subtree K of X of radius $\leq r$, we have $\langle K, \Delta_r \rangle_\alpha = \langle K, \mu \rangle_\alpha$. Hence, by Lemma 4.2, each graph Γ_r has exactly M vertices. There are only finitely many isomorphism types of finite cyclically reduced Γ-graphs with M vertices. Therefore, there exists a finite cyclically reduced Γ-graph Δ such that for some sequence $r_n \to \infty$ as $n \to \infty$ the graph Δ_r is isomorphic, as Γ-graph, to Δ. For any finite nondegenerate subtree K of X, there exists some r_n such that r_n is greater than or equal to the radius of K. Therefore, by construction, for every finite nondegenerate subtree K of X, we have $\langle K, \Delta \rangle_\alpha = \langle K, \mu \rangle_\alpha$. This implies that $\mu_\Delta = \mu$, as required. $\square$

5. Rational Subset Currents Are Dense

Theorem 5.1. *Let $N \geq 2$. Then the set $\mathcal{S}Curr_r(F_N)$ of all rational subset currents is dense in $\mathcal{S}Curr(F_N)$.*

Proof. Let $\mu \in \mathcal{S}\mathrm{Curr}(F_N)$ be a nonzero subset current. To show that μ can be approximated by rational subset currents, it suffices to show that, for every integer $r \geq 1$ and any $\varepsilon > 0$, there exist $c \geq 0$ and a finite connected cyclically reduced Γ-graph Δ such that, for every nondegenerate subtree $K \subseteq X$ of radius $\leq r$, we have $|\langle K, \mu\rangle_\alpha - \langle K, c\mu_\Delta\rangle| < \varepsilon$.

Choose a large integer $r \geq 1$. Define a function $\theta : \mathcal{B}_{\Gamma,r} \to \mathbb{R}_{\geq 0}$ by putting $\theta(K) = \langle K, \mu\rangle_\alpha$. Then $\theta \in \mathcal{Q}_{\Gamma,r}$. Since the polyhedron $\mathcal{Q}_{\Gamma,r}$ is defined by a finite collection of linear equations and inequalities with rational (actually, integer) coefficients, the points with rational coordinates are dense in $\mathcal{Q}_{\Gamma,r}$. Thus, we can find a nonzero $\theta' \in \mathcal{Q}_{\Gamma,r}$ such that, for every $K \in \mathcal{B}_{\Gamma,r}$, $\theta'(K) \in \mathbb{Q}$ and $|\theta'(K) - \theta(K)|$ is arbitrarily small.

In view of Corollary 3.5, if $\mu' \in \mathcal{S}\mathrm{Curr}(F_N)$ is such that $\theta'(K) = \langle K, \mu'\rangle_\alpha$ for every $K \in \mathcal{B}_{\Gamma,r}$, then for every finite subtree $K \subseteq X$ of radius $\leq r$, the value $|\langle \mu, K\rangle_\alpha - \langle \mu', K\rangle_\alpha|$ is also arbitrarily small.

Choose an integer $m \geq 1$ such that, for every $K \in \mathcal{B}_{\Gamma,r}$, we have $m\theta'(K) \in \mathbb{Z}$ and put $\theta'' := m\theta'$. By Theorem 4.3, there exists a finite cyclically reduced Γ-graph Δ such that $\langle K, \mu_\Delta\rangle_\alpha = \theta''(K) = m\theta'(K)$ for every $K \in \mathcal{B}_{\Gamma,r}$.

Let $\Delta_1, \dots, \Delta_s$ be the connected components of Δ. Put $\mu' := \frac{1}{m}\mu_\Delta = \sum_{i=1}^s \frac{1}{m}\mu_{\Delta_i}$. Thus, each $\frac{1}{m}\mu_{\Delta_i}$ is rational and hence, μ' belongs to the linear span of the set of all rational subset currents in $\mathcal{S}\mathrm{Curr}(F_N)$. By [24, Proposition 5.2], the set $\mathcal{S}\mathrm{Curr}_r(F_N)$ of all rational currents is dense in its linear span in $\mathcal{S}\mathrm{Curr}(F_N)$. (Note that the proof of this proposition was based on an explicit combinatorial surgery argument using large finite covers and did not rely on the results of Bowen and Elek about unimodular graph measures.) Therefore, there exists $\mu'' \in \mathcal{S}\mathrm{Curr}_r(F_N)$ such that $|\langle K, \mu'\rangle_\alpha - \langle K, \mu''\rangle|$ is arbitrarily small for all finite subtrees $K \subseteq X$ of radius $\leq r$. It follows that, for every finite subtree $K \subseteq X$ of radius $\leq r$, the value $|\langle \mu, K\rangle_\alpha - \langle \mu,'' K\rangle_\alpha|$ is also arbitrarily small, as required. $\square$

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