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SOME RESULTS ON THE REPRESENTATION SPACE AND STRONG TRIODS

ENRIQUE CASTAÑEDA-ALVARADO, NORBERTO ORDOÑEZ, AND MARCO A. RUIZ

Abstract. The representation space of nondegenerate continua was studied for the first time by José G. Anaya, et al. (*On representation spaces*, Topology Appl. **164** (2014), 1–13). In this paper we show that the character of the representation space of all nondegenerate continua is equal to $\aleph_0$. We introduce the concept of strong triod, which helps us to characterize the interior of the class of triods in the family of locally connected continua under surjective mappings which is a partial answer to Problem 3.16 in the above paper. Also, we characterize the interior of the class of triods under surjective confluent mappings, which answers affirmatively Problem 4.11(2) in the same paper.

1. Introduction

A *continuum* is a nonempty compact connected metric space. In [2] the authors define the closure operator Cl_α on the set $\mathcal{N}$ (where $\mathcal{N}$ denotes the class of nondegenerated continua and α is a class of mappings with the composition property, see Definition 2.2), and so the concept of representation started to be a subject of interest to several authors. In [1], we find a complete study of some properties of $(\mathcal{N}, \tau_\alpha)$; for example, the authors show that $(\mathcal{N}, \tau_\alpha)$ is not a T_0 space and its weight is $\mathfrak{c}$, but they only prove that $\aleph_0$ is an upper bound of the character of $(\mathcal{N}, \tau_\alpha)$ (see [1, Theorem 3.2]).

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José G. Anaya, et al. also investigate the problem of characterizing the interior and closure of some classes of continua. For example, they characterize the interior and closure of the classes of locally connected, hereditarily unicoherent, unicoherent, arc-like, and circle-like continua (see [1, propositions 3.9, 3.10, and 3.11]). However, there are still open problems about how to characterize the interior and closure of many classes of continua.

In [5], the authors show that $(\mathcal{N}, \tau_\alpha)$ is pathwise connected. They also show the existence of a continuum L such that its closure in $(\mathcal{N}, \tau_\alpha)$ is $\mathcal{N}$ (in other words, the density of $(\mathcal{N}, \tau_\alpha)$ is one).

Here we continue with the study of the representation space of continua. In §2, we describe the topology of $(\mathcal{N}, \tau_\alpha)$ and we mention some of its known properties. In §3, we prove that the character of $(\mathcal{N}, \tau_\alpha)$ is equal to $\aleph_0$. In §4, we introduce the definition of strong triod, and, using this concept, we characterize the interior of the class of triods for locally connected continua, which is a partial answer to [1, Problem 3.16]. At the end of this section, we present some interesting open problems and conjectures related to the class of strong triods and their relationship to indecomposable continua. In §5, we characterize the interior of triods under confluent mappings, thereby answering [1, Problem 4.11(2)].

2. Preliminaries

In this section we describe the topology of the representation space and we introduce some basic definitions and general results that we are going to use throughout this paper.

Definition 2.1. Given two topological spaces X and Y and a cover $\mathcal{U}$ of X, we say that a mapping $f : X \to Y$ is a *$\mathcal{U}$-mapping* if there is an open cover $\mathcal{V}$ of Y such that $\{f^{-1}(V) : V \in \mathcal{V}\}$ refines $\mathcal{U}$.

Definition 2.2. Let $\mathcal{C}$ be a class of topological spaces and let α be a class of mappings between elements of $\mathcal{C}$. We say that α has the *composition property* if

(1) for every $X \in \mathcal{C}$, the identity mapping $id_X : X \to X$ is in α,
(2) if $f : X \to Y$ and $g : Y \to Z$ are in α, then $g \circ f$ is in α.

Definition 2.3. Let $\mathcal{C}$ be a class of topological spaces, let $\mathcal{P}$ be a subset of $\mathcal{C}$, and let α be a class of mappings having the composition property. Given $X \in \mathcal{C}$, we write $X \in \mathrm{Cl}_\alpha(\mathcal{P})$ if, for every open cover $\mathcal{U}$ of X, there is a space $Y \in \mathcal{P}$, and a $\mathcal{U}$-mapping $f : X \to Y$ that belongs to α.

Theorem 2.4 ([1, Theorem 2.4])**.** *The operator Cl_α satisfies the following Kuratowski axioms of the closure operator:*

(1) $\mathcal{A} \subset Cl_\alpha(\mathcal{A})$,

(2) $Cl_\alpha(\mathcal{A}) = Cl_\alpha(Cl_\alpha(\mathcal{A}))$,
(3) $Cl_\alpha(\mathcal{A} \cup \mathcal{B}) = Cl_\alpha(\mathcal{A}) \cup Cl_\alpha(\mathcal{B})$,
(4) $Cl_\alpha(\emptyset) = \emptyset$.

As a consequence of Theorem 2.4, we have that for every family $\mathcal{C}$ of topological spaces and for every class α of mappings between elements of $\mathcal{C}$ with the composition property, we introduce a topology τ_α on $\mathcal{C}$ generated by the operator Cl_α, where open sets in τ_α are the sets $\mathcal{P} \subset \mathcal{C}$ that satisfy $\mathcal{C} - \mathcal{P} = \mathrm{Cl}_\alpha(\mathcal{C} - \mathcal{P})$. We denote τ_α the topology on $\mathcal{C}$ generated by α.

Notice that to define the closure operator, we use topological properties of the considered sets only, so homeomorphic spaces are not distinguishable in our topological spaces. Therefore, it is natural to assume that the set $\mathcal{C}$ does not contain two different homeomorphic spaces.

Using the concept of ϵ-mapping, we give a simpler form to work with the topology of the representation space.

Definition 2.5. Let X and Y be metric spaces and let $f : X \to Y$ be a surjective mapping. If $\epsilon > 0$, then f is called an *ϵ-mapping* provided that f is continuous and the $\mathrm{diam}(f^{-1}(y)) < \epsilon$ for all $y \in Y$.

Proposition 2.6 ([1, Proposition 2.8])**.** *If X is a metric compact space, $\mathcal{P}$ is a class of topological spaces, and α is a set of mappings, then the following two conditions are equivalent:*

(1) *$X \in Cl_\alpha(\mathcal{P})$;*
(2) *for every $\epsilon > 0$, there is a space $Y \in \mathcal{P}$ and an ϵ-mapping $f : X \to Y$ that belongs to α.*

Remark 2.7. Let $\mathcal{P}$ be a class of compact metric spaces and let α be any class of mappings with the composition property. Then by Proposition 2.6, $X \in \mathrm{int}_\alpha(\mathcal{P})$ if and only if there is an $\epsilon > 0$ such that for each ϵ-mapping $f : X \to Y$ where $f \in \alpha$, then $Y \in \mathcal{P}$.

Lemma 2.8 ([8, Lemma 2.4.20])**.** *Let X and Y be continua and $\epsilon < 0$. If $f : X \to Y$ is an ϵ-mapping, then there exists $\delta > 0$ such that $diam(f^{-1}(U)) < \epsilon$ for each subset U of Y with $diam(U) < \delta$.*

For the purpose of this article, we will adopt the following symbols:

- $\mathcal{N}$ - nondegenerated continua.
- $\mathbb{LC}$ - locally connected continua.
- $\mathcal{A}$ - surjective mappings.
- $\mathcal{F}$ - surjective confluent mappings.
- $\mathcal{M}$ - surjective monotone mappings.

3. The Character of $(\mathcal{N}, \tau_{\mathcal{A}})$

In [1, Theorem 3.2], the authors investigate the character of the space $(\mathcal{N}, \tau_\alpha)$ and show that it is less than or equal to $\aleph_0$. The purpose of this section is to show that the character of $(\mathcal{N}, \tau_{\mathcal{A}})$ is equal to $\aleph_0$.

Definition 3.1. The *character of a point* x in a topological space (X, τ) is defined as the smallest cardinal number of the form $|B(x)|$, where $B(x)$ is a base for (X, τ) at the point x, and is denoted by $\chi(x, (X, \tau))$.

The *character* of a topological space (X, τ) is defined as the supremum of all numbers $\chi(x, (X, \tau))$ for $x \in X$ and is denoted by $\chi((X, \tau))$.

The next theorem shows an upper bound of $\chi((\mathcal{N}, \tau_{\mathcal{A}}))$.

Theorem 3.2 ([1, Theorem 3.2])**.** *If α is any class of mappings with the composition property, then $\chi((\mathcal{N}, \tau_\alpha)) \leq \aleph_0$.*

Definition 3.3. Let X be a compactum space. A cover $\mathcal{U}$ of X is said to be *essential* provided that no proper subfamily of $\mathcal{U}$ covers X.

Definition 3.4. Let G be a connected graph. A continuum X has a *G-structure* if, for every $\epsilon > 0$, there are an (essential) cover $\mathcal{C}_G = \{L_1, \ldots, L_n\}$ of G by connected subgraphs L_i and an essential cover $\mathcal{C}_X = \{C_1, \ldots, C_n\}$ of X by nondegenerated subcontinua C_i satisfying the following conditions for distinct $i, j, k \in \{1, \ldots, n\}$:

(1) $\operatorname{diam}(L_i) < \epsilon$,
(2) $C_i \cap C_j \neq \emptyset$ if and only if $L_i \cap L_j \neq \emptyset$,
(3) $|L_i \cap L_j| \leq 1$,
(4) $L_i \cap L_j \cap L_k = \emptyset$,
(5) $C_i \cap C_j \cap C_k = \emptyset$.

Definition 3.5. Let X and Y be continua. We say that X is *Y-like* if, for any $\epsilon > 0$, there exists an ϵ-mapping $f : X \to Y$.

Theorem 3.6 ([6, Theorem 9])**.** *If X is a locally connected continuum and G is a connected graph, then X has a G-structure if and only if G is X-like.*

Theorem 3.7 ([6, Theorem 27])**.** *If a space X is Y-like and $G_X \subset X$ is a graph (can be non-connected), then there exists a graph $G_Y \subset Y$ such that G_X is G_Y-like.*

In $\mathbb{R}^2$, let $v = (0, 0)$ and $a_n = \frac{1}{n}(\cos(\frac{\pi}{n}), \sin(\frac{\pi}{n}))$ for all $n \in \mathbb{N}$. If va_i denotes the convex segment from v to a_i for all $i \in \mathbb{N}$, then we define

$$F_n = \bigcup_{i=1}^{n} va_i, \text{ for all } n \in \mathbb{N},$$

$$F_\omega = \bigcup_{i=1}^{\infty} va_i.$$

Notice that

- F_n is a simple n-od, for all $n \in \mathbb{N}$ (in the classical sense [9, Definition 9.8]);
- F_ω is a dendrite;
- $\operatorname{diam}(va_i) = \frac{1}{n}$, for all $n \in \mathbb{N}$.

Since $\mathcal{N}$ does not contain two different homeomorphic spaces, in this section, we are going to suppose that F_ω and F_n for all $n \in \mathbb{N}$ are the spaces which represent the homeomorphism classes of these spaces.

Lemma 3.8. *If $n \in \mathbb{N}$, then F_n is not F_{n-1}-like.*

Proof. Let $n \in \mathbb{N}$ and suppose that F_n is F_{n-1}-like. Let $0 < \epsilon < \frac{1}{n}$.

By Theorem 3.6, there exist an essential cover $\mathcal{C}_{F_n} = \{L_1, \ldots, L_k\}$ of F_n by connected subgraphs L_i and an essential cover $\mathcal{C}_{F_{n-1}} = \{C_1, \ldots, C_k\}$ of F_{n-1} by nondegenerated subcontinua C_i satisfying the following conditions for distinct $i, j, l \in \{1, \ldots, k\}$:

(1) $\operatorname{diam}(L_i) < \frac{1}{n}$,
(2) $C_i \cap C_j \neq \emptyset$ if and only if $L_i \cap L_j \neq \emptyset$,
(3) $|L_i \cap L_j| \leq 1$,
(4) $L_i \cap L_j \cap L_l = \emptyset$,
(5) $C_i \cap C_j \cap C_l = \emptyset$.

Without loss of generality, we can suppose that $a_i \in L_i$ for all $i \in \{1, \ldots, n\}$. Since $\operatorname{diam}(va_r) = \frac{1}{r}$ for all $r \in \{1, \ldots, n\}$, we have that $L_i \subset va_i - \{v\}$ for all $i \in \{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$; by (3) and (4), there exists a unique $j_0 \in \{n+1, \ldots, k\}$ such that $L_i \cap L_{j_0} \neq \emptyset$, and by (2), it follows that $C_i \cap C_j \neq \emptyset$ if and only if $j = j_0$; this implies that there exists $t \in \{1, \ldots, n-1\}$ such that $a_t \in C_i$.

Let $\{C_1, \ldots, C_n\}$. Since $L_i \cap L_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, by (2), we have that $C_i \cap C_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. But by the pigeonhole principle, there exists $t \in \{1, \ldots, n-1\}$ such that $a_t \in C_i \cap C_j$ for some $i, j \in \{1, \ldots, n\}$ with $i \neq j$, which is a contradiction. This ends the proof of this lemma. □

Corollary 3.9. *If $n, m \in \mathbb{N}$ and $m < n$, then F_n is not F_m-like.*

Proof. This is a consequence of Lemma 3.8 □

Lemma 3.10. *If $n \in \mathbb{N}$, then F_ω is not F_n-like.*

Proof. Let $n \in \mathbb{N}$. Since $F_{n+1} \subset F_\omega$ and from the fact that if G is a subgraph of F_n, then G is homeomorphic to an arc or F_k for some

$k \in \{1, \ldots, n\}$, by Corollary 3.9, there is no graph $G \subset F_n$ such that F_{n+1} is G-like. By Theorem 3.7, we conclude the result. $\square$

Remark 3.11. Let $n, m \in \mathbb{N}$ with $m \geq n$, and let $f_n^m : F_\omega \to F_m$ be defined as

$$f_n^m(x) = \begin{cases} x & \text{if } x \in \bigcup_{i=1}^{m} va_i. \\ \\ v & \text{if } x \in \bigcup_{i=m+1}^{\infty} va_i. \end{cases}$$

It is clear that f_n^m is a $\frac{1}{n}$-mapping.

Proposition 3.12. *$\{F_\omega\}$ is not an open set in $(\mathcal{N}, \tau_{\mathcal{A}})$.*

Proof. It is enough to show that $F_\omega \in \mathrm{Cl}_{\mathcal{A}}(\mathcal{N} - F_\omega)$. Let $\epsilon > 0$ and let $n, m \in \mathbb{N}$ be such that $\frac{1}{n} < \epsilon$ and $m > n$. Notice that the mapping $f_n^m : F_\omega \to F_m$ as defined in Remark 3.11 is an ϵ-mapping and by Proposition 2.6, we have that $F\omega \in \mathrm{Cl}_{\mathcal{A}}(\mathcal{N} - F_\omega)$. $\square$

Let $X \in \mathcal{N}$ and $n \in \mathbb{N}$, we denote

$$N(X, n) = \{f(X) : f \in \mathcal{A} \text{ and } f \text{ is an } \tfrac{1}{n}\text{-mapping}\}$$

and

$$N_X = \{N(X, n) : n \in \mathbb{N}\}.$$

Lemma 3.13. *If $X \in \mathcal{N}$, then N_X is a local base of X in $(\mathcal{N}, \tau_{\mathcal{A}})$.*

Proof. It is clear that $X \in N(X, n)$ for each $n \in \mathbb{N}$.

We shall prove that $N(X, n)$ is an open set of $(\mathcal{N}, \tau_{\mathcal{A}})$ for all $n \in \mathbb{N}$.

It is enough to show that $\mathrm{Cl}_{\mathcal{A}}(\mathcal{N} - N(X, n)) \subset \mathcal{N} - N(X, n)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $Y \in \mathrm{Cl}_{\mathcal{A}}(\mathcal{N} - N(X, n))$. Suppose that $Y \in N(X, n)$; then there exists $f : X \to Y$, a $\frac{1}{n}$-mapping with $f \in \mathcal{A}$. By Lemma 2.8, there exists $\delta > 0$ such that if $U \subset Y$ with $\mathrm{diam}(U) < \delta$, then $\mathrm{diam}(f^{-1}(U)) < \frac{1}{n}$. Since $Y \in \mathrm{Cl}_{\mathcal{A}}(\mathcal{N} - N(X, n))$, there exist $Z \in \mathcal{N} - N(X, n)$ and $g : Y \to Z$, a δ-mapping with $g \in \mathcal{A}$. Notice that $g \circ f : X \to Z$ belongs to $\mathcal{A}$ and it is a $\frac{1}{n}$-mapping. This shows that $Z \in N(X, n)$, which is a contradiction. Therefore, $N(X, n)$ is an open set for every $n \in \mathbb{N}$.

Now, let $\mathcal{U}$ be an open set of $(\mathcal{N}, \tau_{\mathcal{A}})$ such that $X \in \mathcal{U}$. We will show that there exists $n \in \mathbb{N}$ such that $N(X, n) \subset \mathcal{U}$. Suppose that, for every $n \in \mathbb{N}$, there exists $f_n(X) \in N(X, n) - \mathcal{U}$; since $f_n : X \to f_n(X)$ is a $\frac{1}{n}$-mapping for all $n \in \mathbb{N}$, we have that $X \in \mathrm{Cl}_{\mathcal{A}}(\mathcal{N} - \mathcal{U}) = \mathcal{N} - \mathcal{U}$, which is a contradiction. Therefore, there is $n \in \mathbb{N}$ such that $N(X, n) \subset \mathcal{U}$. $\square$

Lemma 3.14. *If $n \in \mathbb{N}$, then there exists $m > n$ such that $N(F_\omega, m) \subsetneq N(F_\omega, n)$ and $|N_{F_\omega}| = \aleph_0$.*

Proof. Let $n \in \mathbb{N}$. By Remark 3.11, we have that $f_n^n : F_\omega \to F_n$ is a $\frac{1}{n}$-mapping and so $F_n \in N(F_\omega, n)$. By Lemma 3.10, there exists $m > n$ such that, for all mapping $f : F_\omega \to F_n$, we have that f is not a $\frac{1}{m}$-mapping; therefore, $F_n \notin N(F_\omega, m)$. Since $m > n$, we conclude that $N(F_\omega, m) \subsetneq N(F_\omega, n)$ and so $|N_{F_\omega}| = \aleph_0$. □

Theorem 3.15. $\chi((\mathcal{N}, \tau_\mathcal{A})) = \aleph_0$

Proof. By Theorem 3.2, we have that $\chi((\mathcal{N}, \tau_\mathcal{A})) \leq \aleph_0$.

To show the other inequality, we shall prove that $\chi(F_\omega, (\mathcal{N}, \tau_\mathcal{A})) = \aleph_0$. Let β be a local base of F_ω in $(\mathcal{N}, \tau_\mathcal{A})$ and $\mathcal{U}_1 \in \beta$. By Lemma 3.13, there exists $n_1 \in \mathbb{N}$ such that $N(F_\omega, n_1) \subset \mathcal{U}_1$, and by Lemma 3.14, there exists $n_2 > n_1$ such that $N(F_\omega, n_2) \subsetneq N(F_\omega, n_1)$. Since β is a local base, we can choose $\mathcal{U}_2 \in \beta$ such that $\mathcal{U}_2 \subset N(F_\omega, n_2)$. Notice that $\mathcal{U}_1 \neq \mathcal{U}_2$. Continuing with this process, we can obtain $\beta_1 = \{U_1, U_2, \ldots\}$, a subset of β such that $U_k \neq U_l$ for every $k, l \in \mathbb{N}$ and $k \neq l$. This shows that $|\beta| \geq |N_{F_\omega}| = \aleph_0$. Therefore, $\chi((\mathcal{N}, \tau_\mathcal{A})) = \aleph_0$. □

4. Strong Triods and the $\mathbf{int}_\mathcal{A}(\mathbb{T})$

For the remainder of the paper, if X is a topological space and $A \subset X$, then $\mathrm{Cl}_X(A)$ denotes the closure of A in X.

Definition 4.1. A continuum X is called a *triod*, provided that there is a subcontinuum M of X such that $X - M = K_1 \cup K_2 \cup K_3$, where $K_j \neq \emptyset$ for every $j \in \{1, 2, 3\}$, and $\mathrm{Cl}_X(K_i) \cap K_j = \emptyset$ for every $j, i \in \{1, 2, 3\}$ and $j \neq i$.

In [1, Problem 3.16], the authors ask the following problem.

Problem 4.2. Characterize the interior of the class of triods under surjective mappings.

In this section we introduce the concept of strong triod and, with the help of this class of spaces, we give a solution of [1, Problem 3.16] in the class of locally connected continua.

Definition 4.3. A continuum X is called a *strong triod* provided that there exists a subcontinuum Z of X such that $X - Z = K_1 \cup K_2 \cup K_3$, where $K_i \neq \emptyset$ for every $i \in \{1, 2, 3\}$, and $\mathrm{Cl}_X(K_j) \cap \mathrm{Cl}_X(K_l) = \emptyset$ for every $j, l \in \{1, 2, 3\}$ and $j \neq l$.

A continuum X is *decomposable* provided that it can be written as the union of two of its proper subcontinua. We say X is *indecomposable* if it is not decomposable. We say X is *hereditarily decomposable* (*indecomposable*) if each nondegenerate subcontinuum of X is decomposable (indecomposable).

Example 4.4. Let $X = K_1 \cup K_2 \cup K_3$, where K_1, K_2, and K_3 are indecomposable continua, and $K_i \cap K_j = \{p\}$ for every $i, j \in \{1, 2, 3\}$ and $i \neq j$. It is easy to verify that X is a triod, but it is not a strong triod.

Throughout this paper

- $\mathbb{T}$ denotes the class of the triods.
- $\mathbb{T}_F$ denotes the class of the strong triods.

Proposition 4.5. *Let X be a locally connected continuum. Then $X \in \mathbb{T}$ if and only if $X \in \mathbb{T}_F$.*

Proof. We need only to prove that if $X \in \mathbb{T}$, then $X \in \mathbb{T}_F$.

Let $X \in \mathbb{T}$; then there exists a subcontinuum Z of X such that $X - Z = K_1 \cup K_2 \cup K_3$ where $K_j \neq \emptyset$ for every $j \in \{1, 2, 3\}$ and $\mathrm{Cl}_X(K_i) \cap K_j = \emptyset$ for every $j, i \in \{1, 2, 3\}$ and $j \neq i$.

Let $p_i \in K_i - Z$ for every $i \in \{1, 2, 3\}$ and let U be an open set of X such that $Z \subset U \subset \mathrm{Cl}_X(U) \subset X - \{p_1, p_2, p_3\}$. Since X is locally connected, for every $x \in Z$, there is a connected open set U_x of X such that $x \in U_x \subset U$. Let $W = \mathrm{Cl}_X(\cup\{U_x : x \in Z\})$. Notice that W is a subcontinuum of X and $Z \subset \mathrm{int}(W)$. Finally, it is easy to verify that $X - W = (K_1 - W) \cup (K_2 - W) \cup (K_3 - W)$, where $K_i - W \neq \emptyset$ for every $i \in \{1, 2, 3\}$, and $\mathrm{Cl}_X(K_i - W) \cap \mathrm{Cl}_X(K_j - W) = \emptyset$ for every $i, j \in \{1, 2, 3\}$ and $i \neq j$. □

Theorem 4.6. $\mathbb{T}_F = int_{\mathcal{A}}(\mathbb{T}_F)$.

Proof. We need only to prove that $\mathbb{T}_F \subset \mathrm{int}_{\mathcal{A}}(\mathbb{T}_F)$.

Let $X \in \mathbb{T}_F$. By definition, there exists a subcontinuum Z of X such that $X - Z = K_1 \cup K_2 \cup K_3$, where $K_j \neq \emptyset$ for each $j \in \{1, 2, 3\}$, and $\mathrm{Cl}_X(K_i) \cap \mathrm{Cl}_X(K_j) = \emptyset$ for each $j, i \in \{1, 2, 3\}$ and $j \neq i$. For every $i \in \{1, 2, 3\}$, let $k_i \in K_i - Z$ and let $\epsilon > 0$ be such that

$$\epsilon < \min\{d(\mathrm{Cl}_X(K_i), \mathrm{Cl}_X(K_j)), d(k_i, Z) : i, j \in \{1, 2, 3\} \text{ and } i \neq j\}\}.$$

Let $Y \in \mathcal{N}$ be such that there is an ϵ-mapping $f : X \to Y$ with $f \in \mathcal{A}$. We are going to prove that $Y \in \mathbb{T}_F$. Since $f \in \mathcal{A}$ and by the election of ϵ, it is easy to verify that

(1) $f(Z)$ is a subcontinuum of Y,
(2) $Y - f(Z) = (f(K_1) - Z) \cup (f(K_2) - Z) \cup (f(K_3) - Z)$, and
(3) $f(k_i) \in f(K_i)$, for every $i \in \{1, 2, 3\}$.

Therefore, we need only to prove that $\mathrm{Cl}_X(f(K_i) - f(Z)) \cap \mathrm{Cl}_X(f(K_j) - f(Z)) = \emptyset$ for every $i, j \in \{1, 2, 3\}$ and $i \neq j$.

Let $i, j \in \{1, 2, 3\}$ with $i \neq j$ and suppose that there is $y \in \mathrm{Cl}_X(f(K_i) - f(Z)) \cap \mathrm{Cl}_X(f(K_j) - f(Z))$. Then there exist sequences $\{x_n\}_{n=1}^{\infty}$ in $K_i - Z$ and $\{w_n\}_{n=1}^{\infty}$ in $K_j - Z$ such that $\lim f(x_n) = y$ and $\lim f(w_n) = y$; and we can suppose, without loss of generality, that $\lim x_n = x$ for some $x \in \mathrm{Cl}_X(K_i)$ and $\lim w_n = w$ for some $w \in \mathrm{Cl}_X(K_j)$. By continuity of f, we have that $x, w \in f^{-1}(y)$ and so $d(x, y) < \epsilon$, which contradicts the election of ϵ. This shows that $Y \in \mathbb{T}_F$.

We have shown that there exists $\epsilon > 0$ such that if $f \in \mathcal{A}$ and $f : X \to Y$ is an ϵ-mapping, then $Y \in \mathbb{T}_F$. By Remark 2.7, we have that $X \in \mathrm{int}_{\mathcal{A}}(\mathbb{T}_F)$. □

Corollary 4.7. $\mathbb{T}_F \subset int_{\mathcal{A}}(\mathbb{T})$.

Proof. This follows from Theorem 4.6. □

Corollary 4.8. $int_{\mathcal{A}}(\mathbb{T} \cap \mathbb{LC}) = \mathbb{T} \cap \mathbb{LC}$.

Proof. This follows directly from Proposition 4.5 and Corollary 4.7. □

To finish this section we are going to present some interesting open problems and conjectures about the concept of strong triods. In particular, the following conjecture implies a complete solution to Problem 4.2.

Conjecture 4.9. $int_{\mathcal{A}}(\mathbb{T}) = \mathbb{T}_F$.

Since in locally connected continua the concepts of strong triod and triod are the same, it is natural to ask for a classification of the triods in this class of spaces. In this way, the following conjecture will be a complete classification of triods in locally connected continua.

Conjecture 4.10. *Let X be a locally connected continuum. Then $X \in \mathbb{T}$ if and only if X is not homeomorphic to any of the following spaces:*

(1) $I = \{(x, 0) \in \mathbb{R}^2 : x \in [-1, 1]\}$,
(2) $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$,
(3) $\theta = S^1 \cup I$,
(4) $C = S^1 \cup \{(x, 0) \in \mathbb{R}^2 : x \in [1, 2]\}$,
(5) $W = C \cup \{(x, y) + (3, 0) \in \mathbb{R} : (x, y) \in S^1\}$.

Keeping in mind the structure of strong triods and the examples that we can construct, we can see that when a continuum is not locally connected, then there exists a very strong relation between containing a decomposable continuum and being a strong triod but no triod. In this sense, we present the follow.

Conjecture 4.11. *If $X \in \mathbb{T} - \mathbb{T}_F$, then X contains at least two indecomposable subcontinua.*

Problem 4.12. Let X be a hereditarily decomposable continuum. Is it true that $X \in \mathbb{T}$ if and only if $X \in \mathbb{T}_F$?

5. **Confluent Map**

The purpose of this section is to answer affirmatively the following question.

Problem 5.1 ([1, Problem 4.11(2)])**.** Is it true that $\mathrm{int}_{\mathcal{F}}(\mathbb{T}) = \mathbb{T}$?

Theorem 5.2. *$int_{\mathcal{F}}(\mathbb{T}) = \mathbb{T}$.*

Proof. We need only to prove that $\mathbb{T} \subset \mathrm{int}_{\mathcal{F}}(\mathbb{T})$. Let $X \in \mathbb{T}$. By definition, there exists a subcontinuum Z of X such that $X - Z = K_1 \cup K_2 \cup K_3$, where $K_i \neq \emptyset$ for every $i \in \{1, 2, 3, \}$, and $\mathrm{Cl}_X(K_i) \cap K_j = \emptyset$ for every $i, j \in \{1, 2, 3\}$ and $j \neq i$. Let $x_i \in K_i$ for every $i \in \{1, 2, 3\}$, and let $\epsilon = \min\{d(x_i, X - K_i) : i \in \{1, 2, 3\}\} > 0$.

Let $Y \in \mathcal{N}$ and let $f : X \to Y$ be an ϵ-mapping with $f \in \mathcal{F}$. We are going to show that $Y \in \mathbb{T}$.

For every $i \in \{1, 2, 3\}$, we define $B_i = Z \cup K_i$.

Claim. $f(B_i) \cap f(B_j)$ is a connected subset of Y for every $i, j \in \{1, 2, 3\}$ and $i \neq j$.

Proof. Without loss of generality, we are going to prove that $f(B_1) \cap f(B_2)$ is a connected subset of Y.

Suppose on the contrary. Then there are two disjoint open subsets V and W of Y such that $f(B_1) \cap f(B_2) \subset V \cup W$, $(f(B_1) \cap f(B_2)) \cap V \neq \emptyset$, and $(f(B_1) \cap f(B_2)) \cap W \neq \emptyset$. Since $f(Z)$ is a subcontinuum of Y and $f(Z) \subset f(B_1) \cap f(B_2)$, we can assume that $f(Z) \subset V$; therefore, we have that $f^{-1}(f(Z)) \cap f^{-1}(W) = \emptyset$ and $f^{-1}(W) \subset \bigcup_{i=1}^{3} K_i$.

We will see that $f^{-1}(f(B_1)) \subseteq [K_1 \cup K_3 \cup f^{-1}(V)] \cup [f^{-1}(W) \cap K_2]$. Let $x \in f^{-1}(f(B_1))$ and suppose that $x \notin K_1 \cup K_3 \cup f^{-1}(V)$. Therefore, $x \in K_2$, and we have that $f(x) \in f(B_1) \cap f(B_2)$ and so $f(x) \in V \cup W$. Since $x \notin f^{-1}(V)$, we obtain that $f(x) \in W$. Thus, $x \in K_2 \cap f^{-1}(W)$. This shows that $f^{-1}(f(B_1)) \subseteq [K_1 \cup K_3 \cup f^{-1}(V)] \cup [f^{-1}(W) \cap K_2]$.

It is clear that $[K_1 \cup K_3 \cup f^{-1}(V)] \cap [f^{-1}(W) \cap K_2] = \emptyset$, $Z \subset [K_1 \cup K_3 \cup f^{-1}(V)]$, and $f^{-1}(f(B_1)) \cap [f^{-1}(W) \cap K_2] \neq \emptyset$.

Let C be a component of $f^{-1}(f(B_1))$ contained in $f^{-1}(W) \cap K_2$; then $f(C) \subset W$, which implies that $f(C) \cap f(Z) = \emptyset$, but this is a contradiction because f is confluent. Therefore, $f(B_1) \cap f(B_2)$ is connected, and this ends the proof of the claim.

Let $Z' = [f(B_1) \cap f(B_2)] \cup [f(B_1) \cap f(B_3)] \cup [f(B_2) \cap f(B_3)]$.

Notice that Z' is a subcontinuum of Y for which $Y - Z' = \bigcup_{i=1}^{3}(f(K_i) - Z')$, where, by the election of ϵ, we have that $f(K_i) - Z' \neq \emptyset$ for every $i \in \{1, 2, 3\}$.

We shall prove that $\mathrm{Cl}_X(f(K_i) - Z') \cap (f(K_j) - Z') = \emptyset$ for every $i, j \in \{1, 2, 3\}$ with $i \neq j$. Suppose that there is $y \in \mathrm{Cl}_X(f(K_i) - Z') \cap$

$(f(K_j) - Z')$ for some $i, j \in \{1, 2, 3\}$ and $i \neq j$. Then we can choose a sequence $\{x_n\}_{i=1}^{\infty}$ in $K_i - f^{-1}(Z')$ such that $\lim f(x_n) = y$; and without loss of generality, we can suppose that $\lim x_n = x$ for some $x \in \mathrm{Cl}_X(K_i - f^{-1}(Z')) \subset B_i$. Notice that $y = f(x) \in f(B_i)$. Since $y \in f(K_j) - Z'$, there is $w \in K_j - f^{-1}(Z') \subset B_j$ such that $f(w) = y$, and so $y = f(w) \in f(B_j)$. This shows that $y \in Z'$, which is a contradiction. Therefore, $\mathrm{Cl}_X(f(K_i) - Z') \cap (f(K_j) - Z') = \emptyset$, and we conclude that $Y \in \mathbb{T}$.

We have shown that there is $\epsilon > 0$ for which, if $Y \in \mathcal{N}$ and $f : X \to Y$ is an ϵ-mapping with $f \in \mathcal{F}$, then $Y \in \mathbb{T}$. Therefore, by Remark 2.7, we have that $X \in \mathrm{int}_{\mathcal{F}}(\mathbb{T})$. □

Corollary 5.3. *$int_{\mathcal{F}}(\mathbb{T}_F) = \mathbb{T}_F$.*

Proof. Since $\mathcal{F} \subset \mathcal{A}$, the result follows from Theorem 4.6 and [1, Theorem 2.10(5)]. □

Corollary 5.4. *The following equalities hold:*
(1) *$int_{\mathcal{M}}(\mathbb{T}) = \mathbb{T}$.*
(2) *$int_{\mathcal{M}}(\mathbb{T}_F) = \mathbb{T}_F$.*

Proof. Since $\mathcal{M} \subset \mathcal{A}$, the results follow from Theorem 4.6 and [1, Theorem 2.10(5)]. □

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