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A NEW ASPECT OF SPACES OF COUNTABLE PSEUDOCHARACTER

VLADIMIR V. TKACHUK

Abstract. We introduce and study the spaces with κ-representative families of pseudo-networks for any infinite cardinal κ. We show that the respective classes are invariant under arbitrary subspaces, countable products, and lifted by condensations. Furthermore, the class of spaces with κ-representative families of pseudo-networks is preserved by σ-products. It turns out that every space of countable pseudocharacter has a representative family of pseudo-networks. If X is a subspace of an ordinal, then X has an ω-representative family of pseudo-networks. We also establish that if a space X has a representative family of countable pseudo-networks, then $L(X) \cdot \psi(X) \leq \omega$ implies that $|X| \leq \mathfrak{c}$. This fact is new for monotonically monolithic spaces; in addition, it generalizes the respective results for spaces of countable tightness, monotonically normal spaces, and spaces of countable Hausdorff pseudocharacter.

1. Introduction

If X is a topological space, and $A \subset X$, then the set $\overline{A}$ can be much larger than A so any method of analyzing the properties of $\overline{A}$ could be useful for understanding the behavior of the topology of X. It is probable that A. V. Arhangel'skiĭ had this idea in mind when he discovered the class of monolithic spaces (see [3]). Recall that, for an infinite cardinal κ, a space X is called *κ-monolithic* if $nw(\overline{A}) \leq \kappa$ for every set $A \subset X$ with $|A| \leq \kappa$. The space X is *monolithic* if it is κ-monolithic for any infinite cardinal κ.

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One cannot avoid dealing with monolithic spaces if function spaces or compact spaces from functional analysis are under consideration. Noting that many important classes of spaces have some stronger versions of monolithity, V. V. Tkachuk introduced in [16] the class of monotonically monolithic spaces and proved that every subspace of a monotonically monolithic space must have the D-property. The class of monotonically monolithic spaces turned out to be reasonably large and with many nice categorical properties. It was also proved in [16] that $C_p(X)$ is monotonically monolithic for any Lindelöf Σ-space X; this gave another good reason to study monotonically monolithic spaces.

Later, monotone κ-monolithity was introduced in [1] for any infinite cardinal κ. One of the results of [1] states that monotone κ-monolithity is preserved by countable products and σ-products.

Gary Gruenhage proved in [9] that every monotonically ω-monolithic compact space must be Corson compact and gave an example of a Corson compact space that fails to be monotonically ω-monolithic. Tkachuk showed in [17] that a monotonically monolithic compact space is not necessarily Gul'ko compact. This great variety of applications that popped up after monotonically monolithic spaces were introduced produced a stable interest in possible modifications and implications of monotone monolithity.

In particular, Liang-Xue Peng defined in [12] the concept of weak monotone monolithity and introduced semi-monotonically monolithic spaces in [13]. He studied the general properties of these two notions and their relationship with the D-property.

In [18], Tkachuk introduced the concept of κ-monotone pseudobase assignment and κ-monotone pseudo-network assignment for any infinite cardinal κ; these notions are obtained by replacing bases with pseudobases and networks with closed pseudo-networks in the definition of monotone monolithity.

The spaces from these new classes need not be κ-monolithic, but they still keep a lot of properties of monotonically monolithic spaces. It was proved in [18] that the classes of spaces with a κ-monotone pseudo-network assignment are invariant under subspaces, countable products and σ-products; in addition, they are inverse invariants for condensations. It was also established that a countably compact space X with an ω-monotone pseudobase assignment is compact and metrizable. If a countably compact space X has an ω-monotone pseudo-network assignment, then X is monotonically monolithic and hence Corson compact. In Lindelöf Σ-spaces, having a κ-monotone pseudo-network assignment is equivalent to being monotonically κ-monolithic.

In this paper we introduce the notion of representative families of pseudo-networks in topological spaces. The class of spaces with such a family is obtained if we omit closedness of pseudo-networks in the definition of the monotone pseudo-network assignments. This gives a much broader class which still has nice categorical properties. We prove that any space of countable pseudocharacter has a representative family of pseudo-networks and any subspace of an ordinal has an ω-representative family of pseudo-networks. The class of spaces with a representative family of pseudo-networks is invariant under subspaces, countable products, and σ-products; in addition, it is inversely invariant under condensations.

We also show that a Lindelöf space of countable pseudocharacter with a representative family of countable pseudo-networks has cardinality at most 2^ω; this generalizes practically all known positive answers to a famous question of Arhangel'skiĭ and gives a new result for monotonically monolithic spaces.

2. Notation and Terminology

All spaces are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology and $\tau(x, X) = \{U \in \tau(X) : x \in U\}$ for all $x \in X$; given any set $A \subset X$, let $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. If X is a set, then $\exp(X) = \{Y : Y \subset X\}$; we will also need the subfamilies $[X]^{<\kappa} = \{Y \in \exp(X) : |Y| < \kappa\}$ and $[X]^{\leq\kappa} = \{Y \in \exp(X) : |Y| \leq \kappa\}$ of $\exp(X)$ for any cardinal κ.

Say that a family $\mathcal{F}$ of subsets of a space X is a network in X if, for any $U \in \tau(X)$, there exists $\mathcal{F}' \subset \mathcal{F}$ such that $U = \bigcup \mathcal{F}'$. The cardinal $nw(X) = \min\{|\mathcal{F}| : \mathcal{F}$ is a network of $X\}$ is called the *network weight of X.*

A family $\mathcal{G}$ of subsets of X is called a *network (base) at a point $x \in X$* if ($\mathcal{G} \subset \tau(X)$ and) for any $U \in \tau(x, X)$ there exists $G \in \mathcal{G}$ such that $x \in G \subset U$. Given a set A in a space X say that a family $\mathcal{N}$ of subsets of X is an external network (base) of A in X if (all elements of $\mathcal{N}$ are open in X and) $\mathcal{N}$ is a network at every $x \in A$.

As usual, $\mathbb{R}$ is the real line with its natural topology. Every ordinal is identified with the set of its predecessors and carries the interval topology.

Suppose that κ is an infinite cardinal and we have sets X and Y. Given a family $\mathcal{A} \subset \exp(X)$, a family $\mathcal{B} \subset \exp(Y)$, and a map $\varphi : \mathcal{A} \to \mathcal{B}$, say that φ is κ-monotone if

(a) $|\varphi(A)| \leq |A| \cdot \omega$ whenever $A \in \mathcal{A}$ and $|A| \leq \kappa$;
(b) if $A, B \in \mathcal{A}$ and $A \subset B$, then $\varphi(A) \subset \varphi(B)$;

(c) if $\lambda \leq \kappa$ is a cardinal, $\{A_\alpha : \alpha < \lambda\} \subset \mathcal{A}$ is a family such that $A_\alpha \subset A_\beta$ whenever $\alpha < \beta$, and $A = \bigcup_{\alpha<\lambda} A_\alpha \in \mathcal{A}$, then $\varphi(A) = \bigcup_{\alpha<\lambda} \varphi(A_\alpha)$.

For an infinite cardinal κ, say that a space X is *monotonically κ-monolithic* if, to any set $A \subset X$ with $|A| \leq \kappa$, we can assign an external network $\mathcal{O}(A)$ of the set $\overline{A}$ in such a way that the assignment $\mathcal{O}$ is κ-monotone. A space X is monotonically monolithic if it is monotonically κ-monolithic for any infinite cardinal κ.

Given a space X say that a family $\mathcal{N}$ is a *pseudo-network (pseudobase) at a point* $x \in X$ if ($\mathcal{N} \subset \tau(X)$ and) $\{x\} = \bigcap \mathcal{N}$. The family $\mathcal{N}$ is a *pseudo-network (pseudobase) in* X if, for any $x \in X$, it contains a pseudo-network (pseudobase) at x. If $x \in X$, then $\psi(x, X)$ is the minimal infinite cardinal κ such that there exists a pseudobase at x of cardinality κ; also, the cardinal $\psi(X) = \sup\{\psi(x, X) : x \in X\}$ is called the *pseudocharacter of* X. We use the Russian term *condensation* for a continuous bijection. A space X *condenses* onto a space Y if there exists a condensation $f : X \to Y$.

The *Lindelöf number* of X is $L(X) = \min\{\kappa :$ every open cover of X has a subcover of cardinality $\leq \kappa\} + \omega$. The *tightness* at a point $x \in X$ is the cardinal $t(x, X) = \min\{\kappa :$ for every $Y \subset X$ with $x \in \overline{Y}$, there is $A \subset Y$ with $|A| \leq \kappa$ and $x \in \overline{A}\}$ and the *tightness* of X is $t(X) = \sup\{t(x, X) : x \in X\} + \omega$.

If X is a space, then a map $N : X \to \tau(X)$ is called a *neighborhood assignment* if $x \in N(x)$ for any $x \in X$. Say that X is a *D-space* if, for every neighborhood assignment $N : X \to \tau(X)$, there exists a closed discrete set $D \subset X$ such that $\bigcup\{N(x) : x \in D\} = X$.

Given an infinite cardinal κ, a space X has a *κ-monotone pseudo-network (pseudobase) assignment* if, to any finite set $K \subset X$, we can assign a countable family $\mathcal{O}(K)$ of closed (open) subsets of X in such a way that for any set $A \subset X$ with $|A| \leq \kappa$, if $x \in \overline{A}$, then the family $\bigcup\{\mathcal{O}(K) : K \in [A]^{<\omega}\}$ contains a pseudo-network (pseudobase) at the point x.

The unexplained topological notions can be found in [6]; R. Hodel's survey in [10] covers everything that is necessary for dealing with cardinal invariants.

3. Representative Families of Pseudo-Networks

We will show that the answer to Arhangel'skiĭ's question about the cardinality of Lindelöf spaces of countable pseudocharacter is positive for spaces that have several kinds of monotone properties.

Definition 3.1. Say that a family $\mathcal{B}$ of subsets of a space X is an *external (closed) pseudo-network* for a set $Y \subset X$ if (all elements of $\mathcal{B}$ are closed in X and), for any $y \in Y$, there exists a subfamily $\mathcal{B}' \subset \mathcal{B}$ with $\{y\} = \bigcap \mathcal{B}'$.

Definition 3.2. Given a space X and an infinite cardinal κ, say that a collection $\mathbb{M} = \{\mathcal{N}(K) : K \in [X]^{<\omega}\}$ is a *κ-representative family of pseudo-networks in* X if every $\mathcal{N}(K) \subset \exp(X)$ is countable and for any set $A \subset X$ with $|A| \le \kappa$, the family $\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$ is an external pseudo-network for $\overline{A}$. The collection $\mathbb{M}$ is a *representative family of pseudo-networks in* X if it is a κ-representative family of pseudo-networks in X for any infinite cardinal κ.

Proposition 3.3. *If a space X has a κ-monotone pseudo-network assignment, then X has a κ-representative family of pseudo-networks. In particular, if X is monotonically κ-monolithic, then X has a κ-representative family of pseudo-networks.*

Proof. Just observe that a space X has a κ-monotone pseudo-network assignment if and only if it has a κ-representative family of *closed* pseudo-networks. To finish the proof, note that monotone κ-monolithity of X implies that X has a κ-monotone pseudo-network assignment by [18, Proposition 3.12]. □

We omit an easy proof of the following fact because it can be carried out by repeating the respective reasoning in [18].

Proposition 3.4. *Given an infinite cardinal κ, a space X has a κ-representative family of pseudo-networks if and only if, for any set $A \subset X$ with $|A| \le \kappa$, there exists a family $\mathcal{N}(A)$ of subsets of X such that $\mathcal{N}(A)$ is an external pseudo-network for the set $\overline{A}$ and the assignment $A \to \mathcal{N}(A)$ is κ-monotone.*

If X is a space and $\mathcal{F} \subset \exp(X)$, then a family $\mathcal{U} \subset \tau(X)$ is called an *open expansion of* $\mathcal{F}$ if $\mathcal{U} = \{U(F) : F \in \mathcal{F}\}$ and $F \subset U(F)$ for each $F \in \mathcal{F}$.

Proposition 3.5. (a) *Every space with a countable pseudo-network has a representative family of pseudo-networks;*

(b) *if X is a space and $|X| \le \mathfrak{c}$, then X has a representative family of pseudo-networks;*

(c) *if X is a space and some pseudo-network $\mathcal{P}$ of X has a point-countable open expansion, then X has a representative family of pseudo-networks.*

Proof. (a) If there exists a countable pseudo-network $\mathcal{P}$ in a space X, then letting $\mathcal{N}(K) = \mathcal{P}$ for any finite $K \subset X$, we obtain a representative family of pseudo-networks in X.

(b) Take a second countable topology μ on the space X. If $\mathcal{P}$ is a countable base of (X, μ), then $\mathcal{P}$ is easily seen to be a pseudo-network in X, so we can apply (a) to see that X has a representative family of pseudo-networks.

(c) Consider a point-countable open expansion $\{O(P) : P \in \mathcal{P}\}$ of the pseudo-network $\mathcal{P}$. Let $\mathcal{N}(K) = \{P \in \mathcal{P} : O(P) \cap K \neq \emptyset\}$; it is clear that the family $\mathcal{N}(K) \subset \exp(X)$ is countable for every finite $K \subset X$. If $A \subset X$ and $x \in \overline{A}$, then take any point $y \neq x$. There exists $P \in \mathcal{P}$ such that $x \in P$ and $y \notin P$. Pick any point $a \in A \cap O(P)$; then $P \in \mathcal{N}(\{a\})$ witnesses the fact that the family $\{\mathcal{N}(K) : K \in [X]^{<\omega}\}$ is a representative family of pseudo-networks in X. $\square$

Proposition 3.6. *If a space X has an ω-representative family of pseudo-networks, then $|\overline{A}| \leq \mathfrak{c}$ for any countable set $A \subset X$.*

Proof. Fix an ω-representative family $\{\mathcal{N}(K) : K \in [X]^{<\omega}\}$ of pseudo-networks in X. If $A \subset X$ is a countable set, then $\mathcal{P} = \bigcup\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$ is a countable external pseudo-network of $\overline{A}$. This implies that the assignment $x \to \{P \in \mathcal{P} : x \in P\}$ is an injection from $\overline{A}$ to $\exp(\mathcal{P})$ so $|\overline{A}| \leq 2^{|\mathcal{P}|} \leq \mathfrak{c}$. $\square$

Proposition 3.7. *Assume that κ is an infinite cardinal and a space X has a κ-representative family of pseudo-networks. Then*

(a) *every $Y \subset X$ has a κ-representative family of pseudo-networks;*
(b) *if $g : Z \to X$ is a condensation, then Z has a κ-representative family of pseudo-networks.*

Proof. If $\mathcal{N} : [X]^{<\omega} \to [X]^{\leq\omega}$ is an operator that witnesses the existence of a κ-representative family of pseudo-networks in X, then letting $\mathcal{N}'(K) = \{F \cap Y : F \in \mathcal{N}(K)\}$ for any finite $K \subset Y$, we obtain an operator that witnesses the existence of a κ-representative family of pseudo-networks in Y; this proves (a).

To settle (b), let $\mathcal{N}_Z(K) = \{g^{-1}(F) : F \in \mathcal{N}(g(K))\}$ for any finite set $K \subset Z$. It is standard to verify that $\mathcal{N}_Z$ is an operator that guarantees the existence of a κ-representative family of pseudo-networks in Z. $\square$

Proposition 3.8. *Given an infinite cardinal κ, suppose that X is a space such that $t(X) \leq \kappa$. Then X has a κ-representative family of pseudo-networks if and only if X has a representative family of pseudo-networks.*

Proof. Let $\mathcal{P}$ denote the fact of existence of a representative family of pseudo-networks in X and denote by $\mathcal{P}_\kappa$ its respective κ-version. Fix an operator $\mathcal{N} : [X]^{<\omega} \to [X]^{\leq\omega}$ that witnesses $\mathcal{P}_\kappa$ and take an arbitrary set $A \subset X$. If $x \in \overline{A}$ and $y \neq x$, then we can find a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$. Since $\mathcal{N}$ witnesses $\mathcal{P}_\kappa$, we can find a finite set $K \subset B$ for which $x \in F \not\ni y$ for some $F \in \mathcal{N}(K)$. Therefore, the family $\bigcup\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$ is an external pseudo-network for the set $\overline{A}$; i.e., $\mathcal{N}$ also witnesses $\mathcal{P}$. $\square$

Theorem 3.9. *For any infinite cardinal κ, if X_n has a κ-representative family of pseudo-networks for every $n \in \omega$, then the space $X = \prod_{n\in\omega} X_n$ also has a κ-representative family of pseudo-networks.*

Proof. Let $\mathcal{P}_\kappa$ denote the fact of existence of a κ-representative family of pseudo-networks in X. Denote by p_n the projection of the space X onto X_n and take an operator $\mathcal{O}_n$ that witnesses the property $\mathcal{P}_\kappa$ in the space X_n for each $n \in \omega$. Given a finite set $K \subset X$, let $\mathcal{O}(K) = \{p_n^{-1}(F) : n \in \omega$ and $F \in \mathcal{O}_n(p_n(K))\}$. It is clear that $\mathcal{O}(K)$ is a countable family of subsets of X.

To see that $\mathcal{O}$ witnesses the property $\mathcal{P}_\kappa$ in X, take a set $A \subset X$ with $|A| \leq \kappa$ and let $A_n = p_n(A)$ for any $n \in \omega$. If $x \in \overline{A}$ and $y \neq x$, then there exists $n \in \omega$ such that $x(n) \neq y(n)$; since $x(n) \in \overline{A}_n$, we can find a finite set $Q \subset A_n$ such that $x(n) \in F \not\ni y(n)$ for some $F \in \mathcal{O}_n(Q)$. Take a finite set $K \subset A$ such that $Q = p_n(K)$ and observe that $G = p_n^{-1}(F) \in \mathcal{O}(K)$. Since also $x \in G \not\ni y$, we proved that $\mathcal{O}$ witnesses the property $\mathcal{P}_\kappa$ in X. $\square$

Theorem 3.10. *Given an infinite cardinal κ, every σ-product of spaces with a κ-representative family of pseudo-networks has a κ-representative family of pseudo-networks.*

Proof. Suppose that a space X_t has a κ-representative family of pseudo-networks for every $t \in T$ and fix a point $a \in X = \prod\{X_t : t \in T\}$. We must prove that the σ-product $Y = \{x \in X : |\{t \in T : x(t) \neq a(t)\}| < \omega\}$ also has a κ-representative family of pseudo-networks.

For each $t \in T$, let $p_t : Y \to X_t$ be the projection and choose an operator $\mathcal{O}_t : [X_t]^{<\omega} \to [X_t]^{\leq\omega}$ witnessing the existence of a κ-representative family of pseudo-networks in the space X_t. There is no loss of generality to assume that $X_t \in \mathcal{O}_t(K) \subset \mathcal{O}_t(L)$ for any finite subsets K and L of the space X_t such that $K \subset L$. For every $x \in Y$, denote by $\operatorname{supp}(x)$ the set $\{t \in T : x(t) \neq a(t)\}$. Given a set $S \subset T$, we will need the point $a_S \in \prod_{t\in S} X_t$ defined by $a_S(t) = a(t)$ for any $t \in S$.

For any finite set $K \subset Y$, let $S = \bigcup\{\operatorname{supp}(x) : x \in K\}$ and consider the family $\mathcal{H}(K, S') = \{(\prod_{t\in S'} P_t) \times \{a_{T\setminus S'}\} : P_t \in \mathcal{O}_t(p_t(K))$ for every $t \in S'\}$ for any $S' \subset S$; letting $\mathcal{O}(K) = \{\{a\}\} \cup \bigcup\{\mathcal{H}(K, S') : S' \subset S\}$, we assign a countable family of subsets of Y to any finite set $K \subset Y$.

To see that $\mathcal{O}$ is an operator that witnesses the existence of a κ-representative family of pseudo-networks in Y, take any set $A \subset Y$ with

$|A| \le \kappa$, a point $x \in \overline{A}$, and $y \in Y\backslash\{x\}$. Since $\{a\} \in \mathcal{O}(K)$ for any finite $K \subset Y$, we can assume that $x \neq a$, and hence the set $\mathrm{supp}(x)$ is non-empty. It is easy to see that $\mathrm{supp}(x) \subset S = \bigcup\{\mathrm{supp}(z) : z \in A\}$.

There exists an index $t \in T$ such that $x(t) \neq y(t)$. It follows from $x(t) \in \overline{p_t(A)}$ that there exists a finite set $K \subset A$ such that $\mathrm{supp}(x) \subset K$ and $x(t) \in F \not\ni y(t)$ for some set $F \in \mathcal{O}_t(p_t(K))$.

If $t \in S$, then let $F_s = X_s$ for every $s \in \mathrm{supp}(x)\backslash\{t\}$; it is immediate that the set $P = F \times \prod\{F_s : s \in \mathrm{supp}(x)\backslash\{t\}\} \times \{a_{T\backslash\mathrm{supp}(x)}\}$ belongs to $\mathcal{O}(K)$, so it follows from the inclusions $x \in P \subset Y\backslash\{y\}$ that the family $\bigcup\{\mathcal{O}(L) : L \in [A]^{<\omega}\}$ separates x from y.

Now, if $t \notin S$, then let $F_s = X_s$ for any $s \in \mathrm{supp}(x)$. As before, the set $P = \prod\{F_s : s \in \mathrm{supp}(x)\} \times \{a_{T\backslash\mathrm{supp}(x)}\}$ belongs to $\mathcal{O}(K)$ and it follows from $x \in P \subset Y\backslash\{y\}$ that the family $\bigcup\{\mathcal{O}(L) : L \in [A]^{<\omega}\}$ separates x from y. $\square$

Theorem 3.11. *Any space of countable pseudocharacter has a representative family of pseudo-networks.*

Proof. Choose a pseudobase $\mathcal{B}_x = \{U_n^x : n < \omega\}$ at the point x such that $\mathrm{cl}(U_{n+1}^x) \subset U_n^x$ for all $x \in X$ and $n \in \omega$. For every $x \in X$ and $n \in \omega$, consider the set $G_n^x = \{y \in X : x \in U_n^y\}$; if $K \subset X$ is a finite set, then the family $\mathcal{N}(K) = \{G_n^x : x \in K \text{ and } n < \omega\}$ is countable. We will prove that the operator $\mathcal{N}$ gives us a representative family of pseudo-networks in X.

Take any set $A \subset X$ and a point $x \in \overline{A}$. If $y \neq x$, then there exists a number $n \in \omega$ such that $x \notin \mathrm{cl}(U_n^y)$. There exists a point $z \in (U_n^x \backslash \mathrm{cl}(U_n^y)) \cap A$. Then $x \in G_n^z$ and $y \notin G_n^z$ while $G_n^z \in \bigcup\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$. Therefore, the family $\bigcup\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$ is an external pseudo-network for $\overline{A}$. $\square$

Theorem 3.12. *Any subspace of an ordinal has an ω-representative family of pseudo-networks.*

Proof. It follows from Proposition 3.7 that it suffices to prove our theorem for ordinals, so fix an ordinal μ and denote by L the set of ω-cofinal non-isolated elements of μ. For any $\alpha \in L$, let $S_\alpha = \{\xi_n^\alpha : n \in \omega\}$ be a strictly increasing sequence converging to α and denote by I_n^α the set $\{\beta \in \mu : \xi_n^\alpha < \beta \le \alpha\}$ for every $n \in \omega$. For any ordinal $\beta \in \mu$, let $\mathcal{G}_\beta = \{\{\beta\}\} \cup \{G_n^\beta : n \in \omega\}$ where $G_n^\beta = \{\alpha \in L : \beta \in I_n^\alpha\}$ for each $n \in \omega$. If $K \subset \mu$ is a finite set, then let $\mathcal{N}(K) = \bigcup\{\mathcal{G}_\beta : \beta \in K\}$.

To see that the operator $\mathcal{N}$ gives us an ω-representative family of pseudo-networks take any countable set $A \subset \mu$ and $\alpha \in \overline{A}$. If $\alpha \in A$, then the set $\{\alpha\}$ belongs to $\bigcup\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$ and separates α from any other point of μ.

If $\alpha \in \overline{A}\setminus A$, then $\alpha \in L$; take any point $\beta \neq \alpha$. If $\beta \notin L$, then pick a point $\gamma \in A \cap I_0^\alpha$. Then $\alpha \in G_0^\gamma$ and $\beta \notin G_0^\gamma$, so the set $G_0^\gamma \in \bigcup\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$ separates α from β.

If $\beta \in L$, then it is easy to find $n \in \omega$ such that the sets I_n^α and I_n^β are disjoint. Pick a point $\gamma \in A \cap I_n^\alpha$ and observe that $\alpha \in G_n^\gamma$, but $\beta \notin G_n^\gamma$, so the set $G_n^\gamma \in \bigcup\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$ separates α from β. $\square$

Example 3.13. It follows from Proposition 3.6 that the space $\beta\omega$ does not have an ω-representative family of pseudo-networks because it is separable and has cardinality greater than $\mathfrak{c}$.

Example 3.14. The ordinal ω_1 with its interval topology is a first countable space so it has a representative family of pseudo-networks by Theorem 3.11. However, it does not have an ω-monotone pseudo-network assignment because it is a countably compact non-compact space (see [18, Theorem 3.19]).

Example 3.15. Countable tightness of a compact space X is not sufficient for having an ω-representative family of pseudo-networks, at least consistently, because under $V = L$, V. V. Fedorchuk [8] constructed an example of a hereditarily separable compact space X such that $|X| > \mathfrak{c}$. It follows from Proposition 3.6 that X is a compact space of countable tightness that does not have an ω-representative family of pseudo-networks.

Since Theorem 3.11 gives new information about spaces of countable pseudocharacter, it is a must to find out what it can contribute to a solution of the famous problem of Arhangel'skiĭ on whether any space X with $L(X) \cdot \psi(X) \leq \omega$ must have cardinality at most $\mathfrak{c}$. There are several consistent counterexamples, but the only one that does not use forcing was recently constructed by Alan Dow in [5]. We will show that inside a natural subclass of spaces with representative pseudo-networks, we have a positive answer to Arhangel'skiĭ's problem.

Definition 3.16. Given an infinite cardinal κ, we will say that a space X has a κ-representative family of *countable* pseudo-networks if, to any finite set $K \subset X$, we can assign a countable family $\mathcal{N}(K)$ of subsets of X such that, for any $A \subset X$ with $|A| \leq \kappa$, if $x \in \overline{A}$, then there exists a countable family $\mathcal{A} \subset \bigcup\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$ such that $\bigcap \mathcal{A} = \{x\}$. The space X has a representative family of countable pseudo-networks if it has a κ-representative family of countable pseudo-networks for every infinite cardinal κ.

Proposition 3.17. *If X is a space such that $t(X) \cdot \psi(X) \leq \omega$, then X has a representative family of countable pseudo-networks.*

Proof. Choose a pseudobase $\mathcal{B}_x = \{U_n^x : n < \omega\}$ at the point x such that $\mathrm{cl}(U_{n+1}^x) \subset U_n^x$ for all $x \in X$ and $n \in \omega$. For every $x \in X$ and $n \in \omega$, consider the set $G_n^x = \{y \in X : x \in U_n^y\}$; if $K \subset X$ is a finite set, then the family $\mathcal{N}(K) = \{G_n^x : x \in K \text{ and } n < \omega\}$ is countable. We will prove that the operator $\mathcal{N}$ gives us a representative family of countable pseudo-networks in X.

Take any set $A \subset X$ and a point $x \in \overline{A}$. There exists a countable set $B \subset A$ such that $x \in \overline{B}$. If $y \neq x$, then there exists a number $n \in \omega$ such that $x \notin \mathrm{cl}(U_n^y)$. There exists a point $z \in (U_n^x \backslash \mathrm{cl}(U_n^y)) \cap B$. Then $x \in G_n^z$ and $y \notin G_n^z$, while $G_n^z \in \bigcup\{\mathcal{N}(K) : K \in [B]^{<\omega}\}$. Therefore, the family $\bigcup\{\mathcal{N}(K) : K \in [B]^{<\omega}\}$ is a countable external pseudo-network for $\overline{A}$. □

Recall that a space X is said to have a *countable Hausdorff pseudocharacter* if, for every $x \in X$, we can find a countable pseudobase $\mathcal{B}_x$ at the point x such that, for any distinct $x, y \in X$, there exist $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ such that $U \cap V = \emptyset$.

Proposition 3.18. *If X is a space of countable Hausdorff pseudocharacter, then X has a representative family of countable pseudo-networks.*

Proof. For every $x \in X$, take a pseudobase $\mathcal{B}_x = \{U_n^x : n \in \omega\}$ at the point x such that $U_{n+1}^x \subset U_n^x$ for all $n \in \omega$ and the family $\{\mathcal{B}_x : x \in X\}$ witnesses that X has countable Hausdorff pseudocharacter.

For every $x \in X$ and $n \in \omega$, consider the set $G_n^x = \{y \in X : x \in U_n^y\}$; if $K \subset X$ is a finite set, then the family $\mathcal{N}(K) = \{G_n^x : x \in K \text{ and } n < \omega\}$ is countable. We will prove that the operator $\mathcal{N}$ gives us a representative family of countable pseudo-networks in X.

Take any set $A \subset X$ and $x \in \overline{A}$. Pick a point $a_n \in A \cap U_n^x$ for every $n \in \omega$. The family $\mathcal{G} = \{G_k^{a_n} : n, k \in \omega\} \subset \bigcup\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$ is countable. Take any $y \in X \backslash \{x\}$ and observe that there exists $m \in \omega$ such that $U_m^y \cap U_m^x = \emptyset$. It follows from $a_m \in U_m^x \backslash U_m^y$ that $x \in G_m^{a_m} \in \mathcal{G}$ and $y \notin G_m^{a_m}$; i.e., we proved that $\mathcal{G}$ contains a pseudo-network at the point x. □

It is easy to see that any first countable space has a countable Hausdorff pseudocharacter so we have the following fact.

Corollary 3.19. *Any first countable space has a representative family of countable pseudo-networks.*

Recall that a space X is *monotonically normal* if there exists a map

$$G : \{(x, U) : x \in U \in \tau(X)\} \to \tau(X)$$

such that $x \in G(x, U)$ for every $x \in X$ and $G(x, U) \cap G(y, V) \neq \emptyset$ implies that $y \in U$ or $x \in V$. It is an immediate consequence of the definition of the operator G that $G(x, U) \subset U$.

Corollary 3.20. *If X is a monotonically normal space and $\psi(X) \le \omega$, then X has a representative family of countable pseudo-networks.*

Proof. For every $x \in X$, fix a countable pseudobase $\mathcal{B}_x$ at the point $x \in X$ and let G be an operator that witnesses monotone normality of X. It is clear that $\mathcal{C}_x = \{G(x,U) : U \in \mathcal{B}_x\}$ is also a pseudobase at x for each $x \in X$. Given distinct $x, y \in X$, take $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ such that $x \notin V$ and $y \notin U$; then $G(x,U) \cap G(y,V) = \emptyset$. This shows that the collection $\{\mathcal{C}_x : x \in X\}$ witnesses countable Hausdorff pseudocharacter of X, so we can apply Proposition 3.18 to see that X has a representative family of countable pseudo-networks. $\square$

Theorem 3.21. *For any infinite cardinal κ, if $L(X) \cdot \psi(X) \le \omega$ and the space X has a κ-monotone pseudo-network assignment, then X has a κ-representative family of countable pseudo-networks.*

Proof. For any finite $K \subset X$, fix a countable family $\mathcal{N}(K)$ of closed subsets of X such that $\{\mathcal{N}(K) : K \in [X]^{<\omega}\}$ is a κ-representative family of pseudo-networks. If $x \in \overline{A}$ for some $A \subset X$ such that $|A| \le \kappa$, then take a family $\mathcal{F} \subset \{\mathcal{N}(K) : K \in [A]^{<\omega}\}$ such that $\bigcap \mathcal{F} = \{x\}$. There exists a sequence $\{U_n : n \in \omega\} \subset \tau(x, X)$ such that $\bigcap_{n\in\omega} U_n = \{x\}$.

Given any $n \in \omega$, the Lindelöf property of $X \backslash U_n$, together with the fact that $\bigcap \mathcal{F} = \{x\} \subset U_n$, implies that there exists a countable family $\mathcal{F}_n \subset \mathcal{F}$ such that $\bigcap \mathcal{F} \subset U_n$. Therefore, $\mathcal{F}' = \bigcup_{n\in\omega} \mathcal{F}_n$ is a countable pseudo-network at x contained in $\{\mathcal{N}(K) : K \in [A]^{<\omega}\}$. $\square$

Theorem 3.22. *Given a space X, suppose that $L(X) \cdot \psi(X) \le \omega$ and X has a $\mathfrak{c}$-representative family of countable pseudo-networks. Then $|X| \le \mathfrak{c}$.*

Proof. Fix an operator $\mathcal{N}$ that provides a $\mathfrak{c}$-representative family of countable pseudo-networks in X. For every $x \in X$, take a countable family $\mathcal{B}_x \subset \tau(x, X)$ such that $\{x\} = \bigcap \mathcal{B}_x$. Pick a point $x_0 \in X$ arbitrarily and let $F_0 = \{x_0\}$. Proceeding inductively, assume that $\beta < \omega_1$ and we have constructed subsets $\{F_\alpha : \alpha < \beta\}$ of the space X with the following properties:

(1) $|F_\alpha| \le 2^\omega$ for each $\alpha < \beta$;
(2) $F_\alpha \subset F_\gamma$ whenever $\alpha < \gamma < \beta$;
(3) if $\gamma < \beta$ and $\mathcal{U} \subset \bigcup\{\mathcal{B}_x : x \in \bigcup_{\alpha<\gamma} F_\alpha\}$ is a countable family such that $X \backslash \bigcup \mathcal{U} \ne \emptyset$, then $F_\gamma \backslash \bigcup \mathcal{U} \ne \emptyset$;
(4) if $\gamma < \beta$ and, for some countable family $\mathcal{F} \subset \bigcup\{\mathcal{N}(B) : B \in [\bigcup_{\alpha<\gamma} F_\alpha]^{<\omega}\}$, we have $F = \bigcap \mathcal{F} \ne \emptyset$, then $F_\gamma \cap F \ne \emptyset$.

The cardinality of the set $F'_\beta = \bigcup_{\alpha<\beta} F_\alpha$ does not exceed 2^ω, and hence the cardinality of the family $\mathbb{A} = \{\mathcal{V} \subset \bigcup\{\mathcal{B}_x : x \in F'_\beta\} : |\mathcal{V}| \le \omega$ and $X \backslash \bigcup \mathcal{V} \ne \emptyset\}$ is at most 2^ω; choose a point $x(\mathcal{V}) \in X \backslash \bigcup \mathcal{V}$ for any $\mathcal{V} \in \mathbb{A}$.

The family $\mathbb{B} = \{\mathcal{G} : \mathcal{G} \subset \bigcup\{\mathcal{N}(B) : B \in [F'_\beta]^{<\omega}\},\ |\mathcal{G}| \leq \omega \text{ and } \bigcap \mathcal{G} \neq \emptyset\}$ has cardinality at most 2^ω; take a point $y(\mathcal{G}) \in \bigcap \mathcal{G}$ for any $\mathcal{G} \in \mathbb{B}$. If we let

$$F_\beta = F'_\beta \cup \{x(\mathcal{V}) : \mathcal{V} \in \mathbb{A}\} \cup \{y(\mathcal{G}) : \mathcal{G} \in \mathbb{B}\},$$

then it is immediate that the conditions (1)–(4) are still satisfied for all $\alpha \leq \beta$, so our inductive procedure can be continued to construct a family $\{F_\beta : \beta < \omega_1\}$ such that (1)–(4) hold for all $\beta < \omega_1$.

It follows from property (1) that the set $F = \bigcup_{\beta<\omega_1} F_\beta$ has cardinality not exceeding 2^ω, so it suffices to prove that $F = X$. Our first step is to show that F is closed in X. Striving for a contradiction, assume that $z \in \overline{F} \backslash F$. Since $\{\mathcal{N}(K) : K \in [X]^{<\omega}\}$ is a $\mathfrak{c}$-representative family of countable pseudo-networks, we can find a countable family $\mathcal{G} \subset \bigcup\{\mathcal{N}(B) : B \in [F]^{<\omega}\}$ such that $\{z\} = \bigcap \mathcal{G}$. There exists an ordinal $\beta < \omega_1$ such that $\mathcal{G} \subset \bigcup\{\mathcal{N}(B) : B \in [F_\beta]^{<\omega}\}$; property (4) shows that $F_{\beta+1} \cap \bigcap \mathcal{G} \neq \emptyset$, and therefore $z \in F_{\beta+1} \subset F$, which is a contradiction. Therefore, the set F is closed in X.

Finally, assume that $F \neq X$, and hence we can take a point $q \in X \backslash F$. For every $z \in F$, pick a set $V_z \in \mathcal{B}_z$ such that $q \notin V_z$. The open cover $\{V_z : z \in F\}$ of the set F has a countable subcover $\mathcal{V}$ and it follows from $q \in X \backslash \bigcup \mathcal{V}$ that $X \backslash \bigcup \mathcal{V} \neq \emptyset$. There exists an ordinal $\beta < \kappa^+$ such that $\mathcal{V} \subset \bigcup\{\mathcal{B}_t : t \in F_\beta\}$, so property (3) guarantees that $F_{\beta+1} \backslash \bigcup \mathcal{V} \neq \emptyset$, which contradicts the fact that $F_{\beta+1} \subset F \subset \bigcup \mathcal{V}$. Therefore, $F = X$, and hence $|X| = |F| \leq 2^\omega$, as promised. □

Corollary 3.23 ([2]). *If X is a space with $L(X) \cdot \psi(X) \cdot t(X) \leq \omega$, then $|X| \leq \mathfrak{c}$.*

Corollary 3.24 ([11]). *If a Lindelöf space X has countable Hausdorff pseudocharacter, then $|X| \leq \mathfrak{c}$.*

Corollary 3.25 ([7]). *If a Lindelöf space X is monotonically normal and $\psi(X) \leq \omega$, then $|X| \leq \mathfrak{c}$.*

Corollary 3.26. *If a space X has a $\mathfrak{c}$-monotone pseudo-network assignment and $L(X) \cdot \psi(X) \leq \omega$, then $|X| \leq \mathfrak{c}$.*

Corollary 3.27. *If a Lindelöf space X of countable pseudocharacter is monotonically $\mathfrak{c}$-monolithic, then $|X| \leq \mathfrak{c}$.*

Example 3.28. In [5], Dow, under Jensen's principle $\diamondsuit^*$, constructed an example of a Lindelöf space X such that $\psi(X) = \omega$ and $|X| > \omega_1 = \mathfrak{c}$. Applying Theorem 3.11 and Theorem 3.22, we conclude that X has a representative family of pseudo-networks but does not have a representative family of countable pseudo-networks.

4. Open Problems

The class of spaces with a representative family of pseudo-networks appears for the first time in this work, but the author hopes that the obtained results show that it is a nice and useful class. To convince the reader that the topic is by no means exhausted, we list some interesting open questions below.

Question 4.1. Suppose that X is a Lindelöf monotonically ω_1-monolithic space of countable pseudocharacter. Is it true in ZFC that $|X| \leq \mathfrak{c}$?

Question 4.2. Suppose that X is a sequential space. Must X have a representative family of pseudo-networks?

Question 4.3. Suppose that X is a Fréchet–Urysohn space. Must X have a representative family of pseudo-networks?

Question 4.4. Suppose that X is a sequential compact space. Must X have a representative family of pseudo-networks?

Question 4.5. Suppose that X is a compact Fréchet–Urysohn space. Must X have a representative family of pseudo-networks?

Question 4.6. Suppose that X is a compact ω-monolithic space of countable tightness. Must X have a representative family of pseudo-networks?

Question 4.7. Suppose that X is a Corson compact space. Must X have a representative family of pseudo-networks?

Question 4.8. Suppose that X is a linearly ordered compact space. Must X have an ω-representative family of pseudo-networks?

Question 4.9. Suppose that X is a monotonically normal compact space. Must X have an ω-representative family of pseudo-networks?

Question 4.10. Suppose that X is a compact space with a representative family of pseudo-networks. Must any continuous image of X have a representative family of pseudo-networks?

Question 4.11. Suppose that X is a first countable compact space. Must any continuous image of X have a representative family of pseudo-networks?

Question 4.12. Suppose that X is a monotonically Sokolov space of countable pseudocharacter. This implies that $L(X) \cdot \psi(X) = \omega$ (see [15, Corollary 4.20]). Is it true that $|X| \leq \mathfrak{c}$?

Question 4.13. Suppose that X is a Lindelöf Sokolov space of countable pseudocharacter. Is it true that $|X| \leq \mathfrak{c}$?

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