http://topology.auburn.edu/tp/

TOPOLOGY
PROCEEDINGS

Volume 50, 2017

Pages 277–280

http://topology.nipissingu.ca/tp/

Nonconnected Inverse Limits

by

Hussam Abobaker

Electronically published on February 20, 2017

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: (Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS

Volume 50 (2017)
Pages 277-280

E-Published on February 20, 2017

NONCONNECTED INVERSE LIMITS

HUSSAM ABOBAKER

ABSTRACT. In this paper we give an example of an inverse limit sequence on $[0,1]$ with a single upper semi-continuous set-valued bonding function f such that $G(f^n)$ is an arc for each positive integer n, but the inverse limit is not connected. This answers a question posed by W. T. Ingram.

1. Introduction

In [1] Iztok Banič and Judy Kennedy pose a question: If $f : [0,1] \to 2^{[0,1]}$ is an upper semi-continuous function such that $G(f)$ is an arc and $G(f^n)$ is connected for each positive integer n, is $\varprojlim f$ connected? In [2] W. T. Ingram answers their question in the negative (see Example 1) and asked whether f produces a connected inverse limit in case $G(f^n)$ is an arc for each positive integer n. In this paper we give a negative answer to this question.

2. Definitions and Notation

A *continuum* is a non-empty compact connected metric space. If X is a continuum, $2^X = \{A \subseteq X : A \text{ is non-empty closed in } X\}$ denotes the *hyperspace* of X. If X and Y are continua, a function $f : X \to 2^Y$ is said to be *upper semi-continuous* if for every $x_0 \in X$ and every open subset U of Y such that $f(x_0) \subset U$, the set $\{x \in X : f(x) \subset U\}$ is an open subset of X. The *graph* of the function $f : X \to 2^Y$ is $G(f) = \{(x, y) : y \in f(x)\}$, and for a subset A of X, we define $f(A) = \{y \in Y : y \in f(x) \text{ for some } x \in A\}$. If $f : X \to 2^X$, then we denote the composition $f \circ f$ by f^2 and, for any integer $n > 2$, $f^n = f^{n-1} \circ f$.

2010 *Mathematics Subject Classification.* 54F15, 54C60.
Key words and phrases. generalized inverse limits, set-valued functions.

Let $\boldsymbol{X}$ be a sequence $\{X_i\}_{i=1}^{\infty}$ of continua, and let $\boldsymbol{f}$ be a sequence $\{f_i\}_{i=1}^{\infty}$ of functions such the $f_n : X_{i+1} \to 2^{X_i}$, then the subspace $\varprojlim \boldsymbol{f} = \{x \in \Pi_{i=1}^{\infty} X_i : x_i \in f_i(x_{i+1})$ for each positive integer $i\}$ of the product topology $\Pi_{i=1}^{\infty} X_i$ is called the *inverse limit* of $\boldsymbol{f}$. The functions f_i are called *bonding functions*. In this paper we will use inverse limits with a single upper semi-continuous set-valued bonding function. More information about inverse limits can be found in [3] and [4].

The following lemma is known (see [6, Lemma 3.2]).

Lemma 2.1. *Suppose X is a Hausdorff continuum, $f : X \to 2^X$ is an upper semi-continuous set-valued function, and, for each n, G_n is the set of all $(x_1, x_2, ..., x_n) \in \Pi_{i=1}^{n} X$ such that $x_i \in f(x_{i+1})$ for $i = 1, ..., n-1$. Then $\varprojlim \boldsymbol{f}$ is connected if and only if G_n is connected for each n.*

3. Examples

The following example by Ingram answers the question of Banič and Kennedy. We recall it here for completeness.

Example 3.1. Let $f : [0,1] \to 2^{[0,1]}$ be the function whose graph consists of five straight line intervals, one from $(1/4, 1/4)$ to $(0,0)$, one from $(0,0)$ to $(1/2, 0)$, one from $(1/2, 0)$ to $(1, 1/2)$, one from $(1, 1/2)$ to $(1, 1)$, and one from $(1, 1)$ to $(3/4, 3/4)$ (see Figure 1). Then $G(f)$ is an arc and $G(f^n)$ is connected for each positive integer n, but $\varprojlim f$ is not connected.

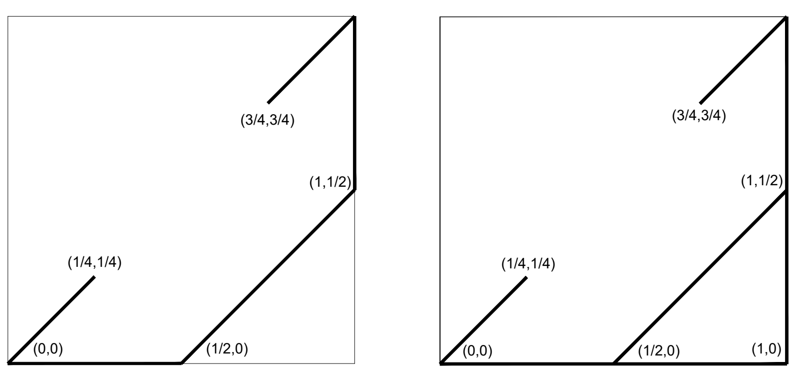

FIGURE 1. The graph of the bonding function f (left) and f^2 (right).

Example 3.2. Let $f : [0,1] \to 2^{[0,1]}$ be a function defined by $f(0) = [0,1]$, $f(x) = \{x, 1-x\}$ for $0 < x < 1/4$, $f(x) = \{1/4, 3/4\}$ for $1/4 \leq x \leq 3/4$, and $f(x) = \{x\}$ if $3/4 < x \leq 1$. Then $G(f^n) = G(f)$, so $G(f^n)$ is an arc for any positive integer n (see Figure 2), but $\varprojlim f$ is not connected.

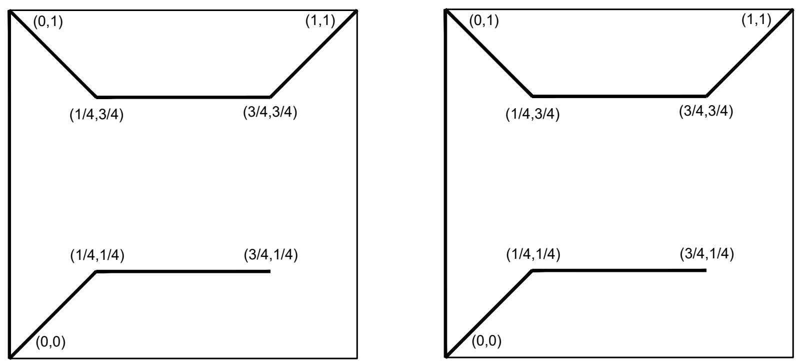

FIGURE 2. Graphs of the bonding functions f (left) and f^n (right).

Proof. It is not hard to verify that $G(f^n) = G(f)$, so $G(f^n)$ is an arc for any positive integer n. To show that $\varprojlim f$ is not connected, we will use Lemma 2.1. Let $A = \{1/4\} \times \{3/4\} \times \{1/4\} \times [1/4, 3/4]$, then A is a closed subset of G_4. We will show that A is a clopen subset of G_4. Let $0 < \epsilon < 1/4$ and put $U = (1/4-\epsilon, 1/4+\epsilon) \times (3/4-\epsilon, 3/4+\epsilon) \times (1/4-\epsilon, 1/4+\epsilon) \times (1/4-\epsilon, 3/4+\epsilon)$, and let $(x_1, x_2, x_3, x_4) \in G_4 \cap U$.

Case 1: If $1/4 - \epsilon < x_4 < 1/4$, then $x_3 \in f(x_4) = \{x_4, 1-x_4\}$. Since $x_3 \in (1/4-\epsilon, 1/4+\epsilon)$, it follows that $x_3 = x_4$. So $x_2 \in f(x_3) = f(x_4) = \{x_4, 1-x_4\}$, but $x_2 \in (3/4-\epsilon, 3/4+\epsilon)$; therefore, $x_2 = 1 - x_4$ and $x_1 \in f(x_2) = \{1 - x_4\}$. But this contradicts $x_1 \in (1/4-\epsilon, 1/4+\epsilon)$. So $x_4 \notin (1/4-\epsilon, 1/4)$.

Case 2. If $3/4 < x_4 < 3/4 + \epsilon$, then $x_3 \in f(x_4) = \{x_4\}$, but this contradicts $x_3 \in (1/4-\epsilon, 1/4+\epsilon)$. So, $x_4 \notin (3/4, 3/4+\epsilon)$.

It follows that if $(x_1, x_2, x_3, x_4) \in G_4 \cap U$, then $x_4 \in [1/4, 3/4]$. Since for any $x \in [1/4, 3/4]$ we have $f(x) = \{1/4, 3/4\}$, we can conclude that $G_4 \cap U = A$. Therefore, A is a clopen subset of G_4. Thus, G_4 is not connected and by Lemma 2.1, $\varprojlim f$ is not connected. $\square$

References

[1] Iztok Banič and Judy Kennedy, *Inverse limits with bonding functions whose graphs are arcs*, Topology Appl. **190** (2015), 9–21.

[2] W. T. Ingram, *Concerning inverse limits on* $[0, 1]$ *with set-valued functions having graphs that are arcs.* Submitted.

[3] W. T. Ingram, *An Introduction to Inverse Limits with Set-valued Functions.* Springer Briefs in Mathematics. New York: Springer, 2012.

[4] W. T. Ingram and William S. Mahavier, *Inverse Limits. From Continua to Chaos.* Developments in Mathematics, 25. New York: Springer, 2012

[5] Sam B. Nadler, Jr. *Continuum Theory. An Introduction.* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 158. New York: Marcel Dekker, Inc., 1992.

[6] Van Nall, *Connected inverse limits with a set-valued function*, Topology Proc. **40** (2012), 167–177.

Department of Mathematics & Statistics; Missouri University of Science and Technology; Rolla, Missouri 65409-0020

E-mail address: haq3f@mail.mst.edu