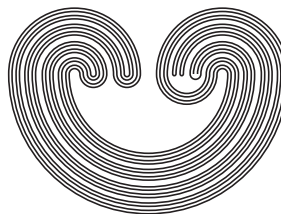


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ALGEBRAIC DYNAMICS

by

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ALGEBRAIC DYNAMICS

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ABSTRACT. Dynamical notions are introduced in arbitrary tight categories. The enveloping semigroup of X is the free object on one generator in the variety generated by X . Two new examples are dynamical systems in which all spaces are countably tight and compact spaces which are homeomorphic to their square. All dynamic varieties have a universal minimal object. Comfort types are identified with certain singly-generated submonads of the ultrafilter monad.

1. INTRODUCTION

Topological dynamics studies the action of a group on a locally compact Hausdorff space. It has long been known [2] that a point is almost periodic if and only if its orbit closure is minimal and compact. Beginning with the work of Robert Ellis [7] and continued by many others, the study of almost periodicity and the proximal relation for compact group actions was formulated in terms of very simple structure with the device of the enveloping semigroup.

The category of topological spaces and continuous maps is “loose” in that a subset of a space admits many topologies with the inclusion continuous, and a product of spaces admits many topologies making the projections continuous. Other familiar categories such as groups, or compact Hausdorff spaces are more “tight.” The notion of a “tight category” is defined in 1.4 below. A “dynamic category” is a tight category with extra structure to allow the ideas originated by Ellis [7] to develop dynamical notions in a more general setting. Ellis showed that when the theory of almost periodicity and proximality of section 4 below is applied to the

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tight category of topological group actions on compact Hausdorff spaces, his new definitions were the same as the original ones found, for example, in [11]. It is our thesis that this theory has application in the broader context of dynamic categories.

Classical examples of compact dynamical systems are metrizable. One cannot assume all spaces are metrizable because the enveloping semigroup is rarely metrizable. The theory of the final two sections stays closer to this traditional setting, but relaxes the compact Hausdorff assumption with the gain that all spaces (including the enveloping semigroup) are countably tight.

There are many dynamic structures which would not usually be regarded as dynamical systems. One such example is a compact Hausdorff space which is homeomorphic to its square.

Definition 1.1. A *category of sets with structure* is defined by the following data and axioms.

For each set, X is given a set σ_X whose elements are called *structures* on X . This defines the objects of a category \mathcal{C} which are pairs (X, δ) with $\delta \in \sigma_X$ and these are also called *structures*. Additionally, for each $\delta \in \sigma_X$ and $\epsilon \in \sigma_Y$, we are given a subset of *admissible maps* from X to Y . We indicate that f is admissible by the notation $f : (X, \delta) \rightarrow (Y, \epsilon)$. The axioms are exactly those required to make \mathcal{C} into a category with structures as objects and admissible maps as morphisms; namely, a composition of admissible maps is admissible and $\text{id}_X : (X, \delta) \rightarrow (X, \delta)$ is admissible.

Definition 1.2. Let \mathcal{C} be a category of sets with structure, let (Y_i, ϵ_i) be a family of structures, and let $f_i : X \rightarrow Y_i$ be a family of functions. We say $\delta \in \sigma_X$ is an *optimal structure* for the f_i and that $f_i : (X, \delta) \rightarrow (Y_i, \epsilon_i)$ is an *optimal family* if the following hold:

- Each $f_i : (X, \delta) \rightarrow (Y_i, \epsilon_i)$ is admissible.
- Given $g : (W, \gamma) \rightarrow X$ with each $gf_i : (W, \gamma) \rightarrow (Y_i, \epsilon_i)$ admissible, then $g : (W, \gamma) \rightarrow (X, \delta)$ is also admissible.

Similarly, given a family (X_i, δ_i) of structures and functions $g_i : X_i \rightarrow Y$, $g_i : (X_i, \delta_i) \rightarrow (Y, \epsilon)$ is *co-optimal* if all $g_i : (X_i, \delta_i) \rightarrow (Y, \epsilon)$ are admissible and if whenever $h : Y \rightarrow (Z, \beta)$ with each $hg_i : (X_i, \delta_i) \rightarrow (Z, \beta)$ admissible, then $h : (Y, \epsilon) \rightarrow (Z, \beta)$ is admissible.

Example 1.3. The category of topological spaces and continuous maps is a category of structured sets. σ_X is the set of topologies on X . The admissible maps are the continuous ones.

In this example, if $f_i : (X, \delta) \rightarrow (Y_i, \epsilon_i)$ is optimal, then δ is the weakest topology making the f_i continuous. If $g_i : (X_i, \delta_i) \rightarrow (Y, \epsilon)$ is co-optimal,

then ϵ is the strongest topology making all g_i continuous. Given a space (X, δ) , if $A \subset X$ with inclusion $i : A \rightarrow X$, then the topology making i optimal is the subspace topology. If $f : X \rightarrow Y$ is surjective, then the quotient topology on Y induced by f makes f co-optimal.

Definition 1.4. A *tight category* is a category \mathcal{C} of sets with structure which satisfies the following four axioms.

- (tc.1) Let $f, g : (X, \delta) \rightarrow (Y, \epsilon)$ be admissible. Let $A = \{x \in X : fx = gx\}$ with inclusion $i : A \rightarrow X$. Let $B = f(X) = \{fx : x \in X\}$ be the image of f with inclusion $j : B \rightarrow Y$. Then there exist $\delta_o \in \sigma_A$, and $\epsilon_o \in \sigma_B$ with $i : (A, \delta_o) \rightarrow (X, \delta)$ and $j : (B, \epsilon_o) \rightarrow (Y, \epsilon)$ admissible.
- (tc.2) Let $f : (X, \delta) \rightarrow (Y, \epsilon)$ be admissible. Then if f is surjective, f is co-optimal. If f is injective, f is optimal.
- (tc.3) If (X_i, δ_i) is a family of structures in \mathcal{C} and if $X = \prod X_i$ is a product in the category of sets with projections $\pi_i : X \rightarrow X_i$, then there exists unique $\delta \in \sigma_X$ such that each $\pi_i : (X, \delta) \rightarrow (X_i, \delta_i)$ is admissible. Moreover, this family is optimal (and so constitutes a product in \mathcal{C}).
- (tc.4) There exists a cardinal function ψ and, for each cardinal α , there exists a set \mathcal{S}_α of cardinality at most $\psi(\alpha)$ consisting of (S, γ, t) , with (S, γ) a structure in \mathcal{C} and $t : n \rightarrow S$ a function, all such that the following condition holds: Given any structure (A, ϵ) in \mathcal{C} and any function $f : n \rightarrow A$, there exists $(S, \gamma, t) \in \mathcal{S}_{|n|}$ and admissible $g : (S, \gamma) \rightarrow (A, \epsilon)$ such that $n \xrightarrow{t} S \xrightarrow{g} A = f$.

For the balance of the section, we work in a tight category \mathcal{C} .

Example 1.5. The category of groups and group homomorphisms is tight.

To see axiom (tc.4), observe that if $t : n \rightarrow F$ is the inclusion of the generators into the free group (F, δ) generated by n , then $\{(F, \delta, t)\}$ is a one-element \mathcal{S}_n , $\psi(\alpha) = 1$. A more relaxed construction is to choose one copy up to isomorphism of each group S generated by a set of cardinality at most n and all (S, γ, t) for such S . Now $\psi(\alpha) = (\alpha \times \omega)^\alpha$.

Example 1.6. The category of complete Boolean algebras and complete homomorphisms is not tight.

All axioms hold except (tc.4). It was proved independently by [8] and [13] that there is no bound on the cardinality of a complete Boolean algebra generated by three or more elements.

Example 1.7. The category **CT2** of compact Hausdorff spaces and continuous maps is tight.

It is well known that a bijective continuous map between compact Hausdorff spaces is a homeomorphism. Using standard facts about products, subspaces, and quotients in topology then makes the first three axioms routine. For (tc.4), let $\mathcal{S}_n = \{(\beta X, \delta, \eta)\}$ where $(\beta X, \eta)$ is the beta-compactification of X with the discrete topology with inclusion $\eta : X \rightarrow \beta X$. Then $\Psi(\alpha) = 1$. As in Example 1.5, one can choose a more relaxed example by using one copy up to homeomorphism of all spaces S of density at most n , so that $\psi(\alpha) = \text{Max}(2^{2^\alpha}, \omega)$.

Definition 1.8. Let (X, δ) in \mathcal{C} . An injective function $m : A \rightarrow X$ is a *substructure* of (X, δ) if there exists δ_o with $m : (A, \delta_o) \rightarrow (X, \delta)$ admissible. A surjective function $q : X \rightarrow Q$ is a *quotient structure* of (X, δ) if there exists ϵ with $q : (X, \delta) \rightarrow (Q, \epsilon)$ admissible.

Since an isomorphism is the same thing as the product of a one-element family, every bijective admissible map is an isomorphism. It follows from (tc.2) that such δ_o is unique if it exists. Similarly, ϵ is unique.

Since any limit is the equalizer of a pair of maps between products ([1, Theorem 12.4], [17, ch. V, sec. 2, Theorem 2]), \mathcal{C} is a complete category, so has pullbacks, inverse limits, etc.

Definition 1.9. Let \mathcal{A} be any class of objects of \mathcal{C} . Define

$$\begin{aligned} S(\mathcal{A}) &= \text{all substructures of structures in } \mathcal{A} \\ Q(\mathcal{A}) &= \text{all quotient structures of structures in } \mathcal{A} \\ P(\mathcal{A}) &= \text{all products of sets of structures in } \mathcal{A}. \end{aligned}$$

Using identity maps (which are unary products in particular), we see that $\mathcal{A} \subset S(\mathcal{A})$, $\mathcal{A} \subset Q(\mathcal{A})$, and $\mathcal{A} \subset P(\mathcal{A})$. Since the composition of two injective admissible maps is injective and admissible, and similarly for surjective, we see that $SS(\mathcal{A}) \subset S(\mathcal{A})$ and $QQ(\mathcal{A}) \subset Q(\mathcal{A})$. Though perhaps tedious to write down carefully and verify, $PP(\mathcal{A}) \subset P(\mathcal{A})$ holds in any category which has products. We leave it to the reader to show the following (use pullbacks for the last one):

$$\begin{aligned} PQ(\mathcal{A}) &\subset QP(\mathcal{A}) \\ PS(\mathcal{A}) &\subset SP(\mathcal{A}) \\ SQ(\mathcal{A}) &\subset QS(\mathcal{A}). \end{aligned}$$

Proposition 1.10. *The following hold for any class \mathcal{A} of objects of \mathcal{C} .*

- (1) $SP(\mathcal{A})$ is the smallest subclass containing \mathcal{A} and closed under products and substructures.
- (2) $QS(\mathcal{A})$ is the smallest subclass containing \mathcal{A} and closed under quotients and substructures.

- (3) $QP(\mathcal{A})$ is the smallest subclass containing \mathcal{A} and closed under quotients and products.
- (4) $QSP(\mathcal{A})$ is the smallest subclass containing \mathcal{A} and closed under quotients, substructures and products.

Proof. These are immediate from the previous remarks. For example, $QQSP = QSP$; $SQSP \subset QSSP = QSP$; $PQSP \subset QPSP \subset QSPP = QSP$. \square

Definition 1.11. A *quasivariety* in \mathcal{C} is a full subcategory of \mathcal{C} closed under substructures and products. A *variety* in \mathcal{C} is a quasivariety which is also closed under quotients.

Thus, \mathcal{A} is a quasivariety if and only if $\mathcal{A} = SP(\mathcal{A})$ and \mathcal{A} is a variety if and only if $\mathcal{A} = QSP(\mathcal{A})$. For any \mathcal{A} , $SP(\mathcal{A})$ is the smallest quasivariety containing \mathcal{A} and $QSP(\mathcal{A})$ is the smallest variety containing \mathcal{A} . We say $QSP(\mathcal{A})$ is the *variety generated by \mathcal{A}* .

It is not hard to see that any quasivariety in a tight category is again a tight category.

Definition 1.12. For any set n , the structure in \mathcal{C} *freely generated by n* or the *free structure on n generators*, if it exists, is a structure (Tn, τ_n) together with a function called the *inclusion of the generators* $\eta_n : n \rightarrow Tn$ with the universal mapping property, and for each structure (X, δ) and each function $f : n \rightarrow X$, there exists a unique admissible $\psi : (Tn, \tau_n) \rightarrow (X, \delta)$ such that $\psi\eta_n = f$.

In the above, we said “the” free structure because any two such are isomorphic. This is immediate. If $(F_n, \tau_n; \eta_n)$ and $(G_n, \gamma_n; \alpha_n)$ are both free, then there exists unique $\psi : (F_n, \tau_n) \rightarrow (G_n, \gamma_n)$ with $\psi\eta_n = \alpha_n$ and unique $\phi : (G_n, \gamma_n) \rightarrow (F_n, \tau_n)$ with $\phi\alpha_n = \eta_n$. The unique admissible map $t : (F_n, \tau_n) \rightarrow (F_n, \tau_n)$ with $t\eta_n = \eta_n$ is then both the identity map of F_n and $\phi\psi$. Similarly, $\psi\phi = id_{G_n}$.

Theorem 1.13. *In any tight category, each set n generates a free structure (Tn, τ_n) .*

Proof. This is an immediate application of the Freyd adjoint functor theorem. For an expository proof of that theorem, see [1, Theorem 18.12] or [17, ch. V, sec. 6, Theorem 2]. \square

Notation 1.14. For each set X , choose a free \mathcal{C} -object (TX, τ_X) generated by X with inclusion of the generators $\eta_X : X \rightarrow TX$. Given a structure (Y, ϵ) and a function $f : X \rightarrow Y$, the unique admissible $\psi : (TX, \tau_X) \rightarrow (Y, \epsilon)$ with $\psi\eta_X = f$ will be denoted as $f^\# : (TX, \tau_X) \rightarrow (Y, \epsilon)$.

Notice that if any structure in \mathcal{C} has two or more elements, then there exist structures of arbitrarily large cardinality by taking products. Hence, if X is a set, there exists a structure (Y, ϵ) and an injective function $f : X \rightarrow Y$. Since $f^\# \eta_X = f$, η_X is injective. Thus, the terminology “inclusion of the generators” is legitimate so long as \mathcal{C} is not trivial.

Proposition 1.15. *The following three properties hold. They are known as the monad laws.*

- (m.1) For $f : X \rightarrow TY$, $f^\# \eta_X = f$.
- (m.2) $(\eta_X)^\# = \text{id}_{TX}$.
- (m.3) For $f : X \rightarrow TY$, $g : Y \rightarrow TZ$, $(g^\# f)^\# = g^\# f^\#$.

Proof. (m.1) holds by definition. As $\text{id}_{TX} : (TX, \tau_X) \rightarrow (TX, \tau_X)$ is admissible and $\text{id}_{TX} \eta_X = \eta_X$, $\text{id}_{TX} = (\eta_X)^\#$, and this is (m.2). As both sides of (m.3) are admissible and $g^\# f^\# \eta_X = g^\# f = (g^\# f)^\# \eta_X$, (m.3) holds. \square

Example 1.16. Let M be a discrete monoid with unit e . An M -action $M \times X \rightarrow X$, $(t, x) \mapsto tx$ satisfies $ex = x$ and $(tu)x = t(ux)$ and a function $f : X \rightarrow Y$ between two M -actions is *equivariant* if $f(tx) = t(fx)$. Let \mathcal{C} be the category of compact Hausdorff M -actions (which means $M \times X \rightarrow X$ is continuous) with continuous equivariant maps as admissible maps.

We may see that this is a tight category as follows. The free structure generated by any set exists by the adjoint functor theorem, where (tc.4) uses Hausdorff to guarantee that for any subset A of an structure, $|A| < \overline{|MA|} \leq 2^{2^{|M \times A|}}$. The other properties are routine. We now give a specific construction of the free structure on one generator. It is the space βM (the usual Stone space of ultrafilters on the set M with the hull-kernel topology) with M -action $t\mathcal{U} = (\beta\lambda_t)\mathcal{U}$ where $\lambda_t : M \rightarrow M$ is left translation by t . Each $\beta\lambda_t$ is continuous because the Stone extension is and this is an action since $\beta(gf) = (\beta g)(\beta f)$ and $\beta \text{id}_X = \text{id}_{\beta X}$, whereas $\lambda_e = \text{id}_M$ and $\lambda_{tu} = \lambda_t \lambda_u$. The free generator is the principal ultrafilter on the monoid unit, $\text{prin}(e)$. Given a compact M -action X and $x \in X$, there is at most one admissible map $\beta M \rightarrow X$ mapping $\text{prin}(e)$ to x because M is dense in βM . To construct such an admissible map, start with the equivariant map $\psi : M \rightarrow X$, $\psi t = tx$. We claim that the Stone extension $\psi^\# : \beta M \rightarrow X$ is the desired map. We have $\psi^\# \text{prin}(e) = \psi e = ex = x$, so it remains only to show that $\psi^\#$ is equivariant. To that end, the map $\pi_t : X \rightarrow X$, $\pi_t x = tx$ is continuous by the definition of a topological

M -action. Because of this continuity, $(\pi_t \psi)^\# = \pi_t \psi^\#$. We have

$$\begin{aligned} \psi^\#(t\mathcal{U}) &= \psi^\#(\text{prin}_M \lambda_t)^\# \mathcal{U} = (\psi^\# \text{prin}_M \lambda_t)^\# \mathcal{U} \\ &= (\psi \lambda_t)^\# \mathcal{U} = (\pi_t \psi)^\# \mathcal{U} = \pi_t \psi^\# \mathcal{U}, \end{aligned}$$

as desired.

2. THE ENVELOPING SEMIGROUP

In this section, the enveloping semigroup of [7] is introduced in the setting of an arbitrary tight category.

We work in a fixed tight category \mathcal{C} . X is a structure in \mathcal{C} .

Proposition 2.1. *Let (F, η_0) be free on one generator in a tight category. Then F is a monoid with composition $\sigma \circ \tau = \tau^\# \sigma$ and unit η_0 , and this monoid acts on the left on each structure by $\sigma x = x^\# \sigma$.*

Proof. Let $\tau, \sigma, \rho \in F$. Then $\tau^\# \sigma^\# \eta_0 = \tau^\# \sigma = (\tau^\# \sigma)^\# \eta_0$. As two admissible maps agreeing on η_0 are equal, we have

$$(2.1) \quad \tau^\# \sigma^\# = (\tau^\# \sigma)^\#.$$

Associativity is then immediate since $(\tau \circ \sigma) \circ \rho = (\tau^\# \sigma) \circ \rho = (\tau^\# \sigma)^\# \rho = \tau^\# (\sigma^\# \rho) = \tau \circ (\sigma \circ \rho)$. As $\text{id}_F : F \rightarrow F$ is an admissible map, $(\eta_0)^\#$ is the identity function, so $\sigma \circ \eta_0 = (\eta_0)^\# \sigma = \text{id}_F \sigma = \sigma$ and $\eta_0 \circ \sigma = \sigma^\# \eta_0 = \sigma$, showing that η_0 is the monoid unit. The same type of proof establishes the action axioms. \square

It is routine to check that all admissible maps are equivariant. Lemma 2.13 below gives a partial converse.

Observation 2.2. For the monoid F above, right translations are admissible maps since the right translation $\sigma \mapsto \sigma \circ \tau$ is the admissible map $\tau^\#$. These are clearly the only admissible maps.

In Example 1.16, βM is free on one generator. We now compute the monoid structure. For $\mathcal{V} \in \beta M$, let $f_{\mathcal{V}} : M \rightarrow \beta M$ be the equivariant map $t \mapsto (\text{prin}_M \lambda_t)^\# \mathcal{V}$ so that $(f_{\mathcal{V}})^\#$ is the unique continuous equivariant map sending $\text{prin}(e)$ to \mathcal{V} . Then the monoid multiplication is $\mathcal{U} \circ \mathcal{V} = (f_{\mathcal{V}})^\# \mathcal{U}$. Therefore,

$$\begin{aligned} R \in \mathcal{U} \circ \mathcal{V} &\Leftrightarrow \{a \in M : R \in f_{\mathcal{V}} a\} \in \mathcal{U} \\ &\Leftrightarrow \{a \in M : R \in (\text{prin}_M \lambda_a)^\# \mathcal{V}\} \in \mathcal{U} \\ &\Leftrightarrow \{a \in M : \{c \in M : R \in \text{prin}_M(ac)\} \in \mathcal{V}\} \in \mathcal{U} \\ &\Leftrightarrow \{a \in M : \{c : ac \in R\} \in \mathcal{V}\} \in \mathcal{U}. \end{aligned}$$

Neil Hindman and Dona Strauss [14] study a semigroup βS induced by a semigroup S . In the case that S is a monoid, their formula for multiplication is exactly that given above. It is easy to check that if S is any subsemigroup of M , then βS is a subsemigroup of βM . Hence, the semigroup case can be recovered from the monoid case by regarding S as a subsemigroup of the monoid $S^1 = S + 1$ with $x1 = xs = 1x$.

We are now ready for the main definition of this section.

Definition 2.3. Let \mathcal{V} be any class of structures in \mathcal{C} . The *enveloping semigroup* $E(\mathcal{V})$ of \mathcal{V} is the free structure on one generator in $QSP(\mathcal{V})$.

Although $E(\mathcal{V})$ is both a structure and a monoid (the latter as in Proposition 2.1), the term “enveloping semigroup” coined by Ellis [7] has become standard. For X a structure, we write $E(X)$ for $E(\{X\})$.

Proposition 2.4. $E(E(X)) \cong E(X)$.

Proof. $E(X) \in SP(X)$ so that $QSP(E(X)) \subset QSP(X)$. Thus, $E(X)$ acts as the free structure on one generator in $QSP(E(X))$ and therefore is isomorphic to $E(E(X))$. \square

In order to recapture the original construction of $E(X)$ given by Ellis, we must pause to consider how substructures are generated.

Proposition 2.5. For $f : X \rightarrow Y$ a function, define $Tf : (TX, \tau_X) \rightarrow (TY, \tau_Y)$ as the admissible map

$$Tf = (\eta_Y f)^\#.$$

Then such T is a functor $\mathbf{Set} \rightarrow \mathbf{Set}$ and $\eta : id \rightarrow T$ is a natural transformation.

Proof. $T(id_X) = (\eta_X)^\# = id_{TX}$. Further, if $g : Y \rightarrow Z$, we have

$$\begin{aligned} (Tg)(Tf) &= (\eta_Z g)^\# (\eta_Y f)^\# = ((\eta_Z g)^\# \eta_Y f)^\# \\ &= (\eta_Z gf)^\# = T(gf). \end{aligned}$$

To see that η is a natural transformation from the identity functor to T , we must verify

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \eta_X \downarrow & & \downarrow \eta_Y \\ TX & \xrightarrow{Tf} & TY \end{array} \quad .$$

Indeed, $(Tf)\eta_X = (\eta_Y f)^\# \eta_X = \eta_Y f$. \square

Definition 2.6. For (X, δ) , a structure in \mathcal{C} , the *structure map* of (X, δ) is the admissible map $\xi_\delta : (TX, \tau_X) \rightarrow (X, \delta)$ defined by $\xi_\delta = (\text{id}_X)^\#$.

Theorem 2.7. For (X, δ) and (Y, ϵ) , structures in \mathcal{C} , and $f : X \rightarrow Y$, a function, the following hold:

- (1) $f^\# = TX \xrightarrow{Tf} TY \xrightarrow{\xi_\epsilon} Y$.
- (2) $f : (X, \delta) \rightarrow (Y, \epsilon)$ is admissible if and only if the following square commutes:

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ \xi_\delta \downarrow & & \downarrow \xi_\epsilon \\ X & \xrightarrow{f} & Y \end{array} .$$

Proof. For the first statement,

$$\begin{aligned} \xi_\epsilon(Tf) &= (\text{id}_Y)^\# (\eta_Y f)^\# = ((\text{id}_Y)^\# \eta_Y f)^\# \\ &= (\text{id}_Y f)^\# = f^\#. \end{aligned}$$

For the second statement, as $\xi_\delta \eta_X = \text{id}_X$, ξ_δ is surjective. If the square commutes, f is admissible because ξ_δ is co-optimal and $f^\#$ is admissible. Conversely, let f be admissible. Then, as $(f \xi_\delta) \eta_X = f(\text{id}_X) = f$, $f \xi_\delta = f^\# = \xi_\epsilon(Tf)$. \square

We have yet to apply the part of axiom (tc.1) that asserts that the image of an admissible map is a substructure. This is needed for the next results.

Corollary 2.8. Let (X, δ) be a structure in \mathcal{C} and let $m : A \rightarrow X$ be injective. Then m is a substructure if and only if there exists a factorization ξ_o

$$\begin{array}{ccc} TA & \xrightarrow{Tm} & TX \\ \xi_o \downarrow & & \downarrow \xi_\delta \\ A & \xrightarrow{m} & X \end{array} ,$$

and then ξ_o is the structure map of A .

Proof. If A is a substructure, let ξ_o be its structure map and apply the previous theorem. Conversely, if ξ_o exists, then, using Theorem 2.7(1), A is the image of the admissible map $m^\#$ and therefore is a substructure. The structure map of that substructure also makes the square commute and so coincides with ξ_o since m is injective. \square

Definition 2.9. For (X, δ) , a structure in \mathcal{C} , $A \subset X$ with inclusion $i : A \rightarrow X$, denote by $\langle A \rangle$ the image of $i^\# : TA \rightarrow X$.

Proposition 2.10. $\langle A \rangle$ is the substructure of (X, δ) generated by A .

Proof. As $i^\# \eta_A = i$, $A \subset \langle A \rangle$. $\langle A \rangle$ is a substructure by (tc.1). Now let $B \subset X$ be a substructure of (X, δ) with inclusion $j : B \rightarrow X$ and suppose that $A \subset B$ so that there is an inclusion k with $i = A \xrightarrow{k} B \xrightarrow{j} X$. By Theorem 2.7, there is a commutative square

$$\begin{array}{ccc} TB & \xrightarrow{Tj} & TX \\ \xi_{\delta_o} \downarrow & & \downarrow \xi_\delta \\ B & \xrightarrow{j} & X \end{array},$$

where δ_o is the unique structure making j admissible. By the same theorem,

$$\begin{aligned} i^\# &= \xi_\delta(Ti) = \xi_\delta(Tj)(Tk) \\ &= j \xi_{\delta_o}(Tk), \end{aligned}$$

and this shows that $\langle A \rangle \subset B$. □

Proposition 2.11. If $f : (X, \delta) \rightarrow (Y, \epsilon)$ is admissible and $A \subset X$, then $\langle fA \rangle = f \langle A \rangle$.

Proof. Let $i : A \rightarrow X$, $j : fA \rightarrow Y$ be the inclusion maps. We have a factorization in the square on the left with p surjective, and the square on the right

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ p \downarrow & & \downarrow f \\ fA & \xrightarrow{j} & Y \end{array} \quad \begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ \xi_\delta \downarrow & & \downarrow \xi_\epsilon \\ X & \xrightarrow{f} & Y \end{array}$$

from Theorem 2.7. By Proposition 2.5, Tp is surjective (that is, if $pi = \text{id}$, then $(Tp)(Ti) = \text{id}$). We then have that

$$\begin{aligned} \langle fA \rangle &= \text{Im}(\xi_\epsilon(Tj)) = \text{Im}(\xi_\epsilon(Tj)(Tp)) \\ &= \text{Im}(\xi_\epsilon(Tf)(Ti)) = \text{Im}(f \xi_\delta(Ti)) \\ &= \text{Im}(fi^\#) = f(\text{Im}(i^\#)) = f \langle A \rangle. \end{aligned} \quad \square$$

We are now ready to establish Ellis's original definition.

Theorem 2.12. *$E(X)$ may be constructed as the substructure of the product structure X^X generated by id_X . The associated monoid structure is function composition and the action of $E(X)$ on X is function evaluation.*

Proof. Define $E = \langle \text{id}_X \rangle \subset X^X$ so that $E \in SP(X) \subset QSP(X)$. Let \mathcal{W} be the class of all structures W such that, for each $w \in W$, there exists an admissible map $\psi : E \rightarrow W$ with $\psi(\text{id}_X) = w$. Such ψ is necessarily unique. Moreover, $X \in \mathcal{W}$ because for any $x \in X$, $pr_x : E \rightarrow X$ is admissible and maps id_X to x . Thus, E is free on one generator in $QSP(X)$, providing \mathcal{W} is shown to be closed under products, substructures, and quotients. Products and substructures are obvious. For quotients, let $h : W \rightarrow Z$ be an admissible surjection with $W \in \mathcal{W}$. By the axiom of choice, there exists a function $f : Z \rightarrow W$ with $hf = \text{id}_Z$. Let $z \in Z$. By hypothesis, there exists an admissible map $\psi : E \rightarrow W$ mapping id_X to hz . Then $h\psi : E \rightarrow Z$ is admissible and maps id_X to z and indeed Z is in \mathcal{W} . We next verify that the monoid multiplication induced by freeness coincides with function composition. For any function $f : X \rightarrow X$, composing with f is an admissible map $-\circ f : X^X \rightarrow X^X$ because $pr_x(-\circ f) = pr_{fx}$. By Proposition 2.11, $-\circ f$ maps E into itself so long as $f \in E$. But then $-\circ f$ is the unique admissible map $f^\#$ mapping id_X to f . Since such $f^\#$ is the right translation of the monoid structure by Observation 2.2, the two monoids coincide. Finally, for $x \in X$, $pr_x : E \rightarrow X$ is the unique admissible map $x^\#$ mapping id_X to x so the action of E on X is function evaluation. \square

Notice that the action of E on X is effective. We conclude the section with a few further facts about the enveloping semigroup.

Lemma 2.13. *Let F be the free structure on the generator η_0 and let X and Y be structures with $f : X \rightarrow Y$, an F -equivariant map. Then if X is singly-generated, f is admissible.*

Proof. For $x \in X$ we have the commutative triangle

$$\begin{array}{ccc} F & & \\ x^\# \downarrow & \searrow (fx)^\# & \\ X & \xrightarrow{f} & Y \end{array}$$

because, by equivariance, $(fx^\#)t = f(tx) = t(fx) = (fx)^\#t$. Since X is singly-generated, $x \in X$ exists with $x^\# : F \rightarrow X$ surjective. Now use (tc.2). \square

Proposition 2.14. *Let \mathcal{W} and \mathcal{V} be subclasses of a tight category with $\mathcal{W} \subset QSP(\mathcal{V})$, and let $E(\mathcal{V})$ and $E(\mathcal{W})$ have free generators η_0 and ξ_0 . Then the unique admissible map $\psi : E(\mathcal{V}) \rightarrow E(\mathcal{W})$ mapping η_0 to ξ_0 is surjective and is a monoid homomorphism.*

Proof. ψ is surjective because its image is a substructure containing the generator, and ψ maps the monoid unit to the monoid unit. The right translation by τ in $E(\mathcal{V})$ is $\tau^\#$. Proving that ψ is a semigroup homomorphism, is equivalent to showing that it commutes with right translations, that is, that the following square commutes.

$$\begin{array}{ccc} E(\mathcal{V}) & \xrightarrow{\tau^\#} & E(\mathcal{V}) \\ \psi \downarrow & & \downarrow \psi \\ E(\mathcal{W}) & \xrightarrow{(\psi\tau)^\#} & E(\mathcal{W}). \end{array}$$

As all four maps are admissible, we need only check that both paths map η_0 to the same element. Indeed, $(\psi\tau)^\# \psi(\eta_0) = (\psi\tau)^\#(\xi_0) = \psi(\tau) = \psi\tau^\#(\eta_0)$. \square

Lemma 2.15. *Let X be a structure, $E = E(X)$ and $x \in X$. Then $\langle x \rangle = Ex$.*

Proof. $Ex = pr_x(\langle x \rangle) = \langle pr_x(\text{id}_X) \rangle = \langle x \rangle$. \square

3. LG-SEMIGROUPS

The dynamical information of a structure in the tight categories we wish to study is stored in the core of an lg-semigroup. Some of the basic theory is available in texts such as [6] and [14], so we omit proofs in these cases. Note, however, that our approach puts special emphasis on Green's relations [12]. We present here those aspects which we believe to be new. For semigroups per se, we follow standard notations and definitions such as may be found in [15].

Let S be a semigroup. The equivalence class of x under Green's equivalence relations \mathcal{L} , \mathcal{R} , \mathcal{H} , and \mathcal{D} are denoted L_x , R_x , H_x , and D_x , respectively. Here, $x\mathcal{L}y \Leftrightarrow Sx \cup \{x\} = Sy \cup \{y\}$, $x\mathcal{R}y \Leftrightarrow xS \cup \{x\} = yS \cup \{y\}$, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, and $\mathcal{LR} = \mathcal{D} = \mathcal{RL}$ (relation composition) is the smallest equivalence relation containing \mathcal{L} and \mathcal{R} . $H_x = L_x \cap R_x$ is a group if and only if it has an idempotent. The maximal subgroups of the semigroup are precisely those \mathcal{H} -classes which have idempotents and every group in the semigroup is contained in a unique \mathcal{H} -class. If $u^2 = u$, $v^2 = v$, and

uDv , then H_u and H_v are isomorphic groups. An element x in a semigroup is *regular* if there exists a with $axa = x$. In a *regular semigroup*, all elements are regular. If x is regular, all elements of D_x are regular and D_x is called a *regular \mathcal{D} -class*. In a regular \mathcal{D} -class, each \mathcal{L} -class and each \mathcal{R} -class has at least one idempotent.

Definition 3.1 ([5]). A semigroup I is a *left group* if, for all $x, y \in I$, there exists unique $z \in I$ with $zx = y$. The dual concept, $xz = y$, is a *right group*.

The relationship of left groups to semigroups is in many ways like the relationship between groups and semigroups. A subsemigroup of a group need not be a group and a subsemigroup of a left group need not be a left group, but both groups and left groups are closed under products and quotients in semigroups. Both groups and left groups are equationally definable if a new operation is added. In the case of groups, add the unary operation of inverse. In the case of left groups, add a new binary operation $\frac{y}{x}$ and impose the equations

$$\begin{aligned} \frac{y}{x} x &= y \\ \frac{yx}{x} &= y. \end{aligned}$$

A semigroup homomorphism between two groups always preserves inverse. A semigroup homomorphism between two left groups always preserves $\frac{y}{x}$.

It is immediate from the definition that a left group comprises a single \mathcal{L} -class and is a right cancellative semigroup. The following is a useful characterization of left groups.

Lemma 3.2. *A semigroup I is a left group if and only if all of its elements are regular and each of its idempotents is a right unit.*

Proof. \Rightarrow For $x \in I$, let $v \in I$ with $vx = x$. As $v^2x = v(vx) = vx$, we can cancel x to get $v^2 = v$. As v is regular and $I = L_v = D_v$, all elements of I are regular. If $u^2 = u \in I$ and $x \in I$, write $zu = x$. Then $xu = (zu)u = zu = x$.

\Leftarrow Let $x, y \in I$ and write $axa = x$. As ax is idempotent, $(ya)x = y(ax) = y$. As xa is also idempotent, if $zx = y$, then $z = z(xa) = (zx)a = ya$. \square

Lemma 3.3. *Let I be a left group. Then every \mathcal{H} -class is a group. If $u^2 = u \in I$, $H_u = uI$.*

Proof. In a left group, $\mathcal{R} = \mathcal{H}$ and all elements are regular, so every \mathcal{H} -class has an idempotent and therefore is a group. $H_u \subset uI$ since H_u is a group with unit u . Conversely, let $x \in I$ and show $ux \in H_u$. There exist

unique y and z with $yx = u$ and $zxy = u$. Then $ux = (zxy)x = zxu = zx$. Cancelling x , $z = u$. We have $(ux)y = zxy = u$, whereas $ux = ux$ so that $ux\mathcal{R}u$. As I is a single \mathcal{L} -class, $ux\mathcal{H}u$. \square

Definition 3.4. If S is a set, S is a semigroup if $xy = x$. Such is called a *left zero semigroup*.

The following result gives rather complete information on how to construct left groups. The proof is standard, and we omit it.

Theorem 3.5. Let S be a semigroup. Then S is a left group $\Leftrightarrow S$ is isomorphic to the product of a left zero semigroup with a group. In that case, the left zero semigroup may be taken to be the set U of idempotents of S and the group may be taken to be any \mathcal{H} -class G of S . The map $U \times G \rightarrow S$, $(u, g) \mapsto ug$ is a semigroup isomorphism. \square

Definition 3.6. Let S be a semigroup, $I \subset S$. I is a *left ideal* of S if $I \neq \emptyset$ and $SI \subset I$. I is a *right ideal* of S if $I \neq \emptyset$ and $IS \subset I$. I is an *ideal* of S if it is both a left and a right ideal. I is an *lg-ideal* if it is a left ideal which, as a semigroup, is a left group. I is an *rg-ideal* if it is a right ideal which, as a semigroup, is a right group. Finally, S is an *lg-semigroup* if it has an lg-ideal.

The following alternate characterization of lg-ideals is the one usually found in the literature.

Lemma 3.7. Let I be a left ideal of a semigroup S . Then I is an lg-ideal \Leftrightarrow it is a minimal left ideal with an idempotent.

Proof. \Rightarrow If $x \in I$, then $Ix = I$. As I is a left ideal, $Sx = I$. This shows that I properly contains no left ideal. If $x \in I$, x is regular in I so there exists $a \in I$ with $xax = x$. Thus, ax is an idempotent in I .

\Leftarrow Given $x, y \in I$, $Sx = I = Sy$. It follows that $x\mathcal{L}y$ in S so all elements of I are regular since I has an idempotent. By Lemma 3.2, it suffices to show for $x \in I$ and $u^2 = u \in I$ that $xu = x$. Let $zu = x$. Then $xu = (zu)u = zu = x$. \square

Example 3.8. Let S be a semigroup admitting a compact Hausdorff topology for which all right translations are continuous. Zorn's lemma gives a minimal closed left ideal I . Because right translations are closed, I is a minimal left ideal. It is well known that every closed subsemigroup contains an idempotent (see [7], [14], [21]), so S is an *lg* semigroup.

Note that all finite semigroups are lg-semigroups. The preceding example is a primary source of infinite lg-semigroups.

Because of Lemma 3.7, lg-semigroups have been studied in the literature. We omit proofs that can be referred to [7] or [14, sections 1.6 and 1.7].

Lemma 3.9. *The following hold for an lg-semigroup.*

- (1) *Each lg-ideal is an \mathcal{L} -class.*
- (2) *Let I be an lg-ideal, $x \in I$. Then the \mathcal{H} -class H_x of x in S is a subset of I and is a group.*
- (3) *If I is an lg-ideal and $u^2 = u \in I$, then $H_u = uSu = uIu$ and Iu is an rg-ideal.* \square

It follows that the dual of an lg-semigroup is again an lg-semigroup.

Lemma 3.10. *Let S be an lg-semigroup and let I and J be lg-ideals of S . The following hold.*

- (1) ([7, Lemma 2]) *If $x, y \in I$ and $a \in S$ with $xa = ya$, then $x = y$.*
- (2) ([7, Lemma 3]) *Let $u^2 = u \in I$. Then there exists a unique $v \in J$ with $vu = u$. Further, $v^2 = v$ and $uv = v$.*
- (3) *Every minimal left ideal of S is an lg-ideal.*
- (4) *For any $x \in S$, Ix is an lg-ideal.*
- (5) *Every left ideal contains an lg-ideal, an idempotent in particular.* \square

Lemma 3.11. *Let S be an lg-semigroup and let I and J be lg-ideals of S . Suppose that $\psi : I \rightarrow J$ is S -equivariant; that is, $\psi(tx) = t\psi(x)$ for $t \in S$ and $x \in I$. Then there exists $p \in J$ with $\psi(x) = xp$ and ψ is bijective.*

Proof. Let $u^2 = u \in I$ so that u is a right unit for I . Define $p = \psi u$. For $x \in I$, $xp = x(\psi u) = \psi(xu) = \psi(x)$. By Lemma 3.10(1), ψ is injective. As Ip is a left ideal and $Ip \subset J$, $Ip = J$, so ψ is surjective. \square

If S is any semigroup and if I and J are ideals in S , then IJ is an ideal and $IJ \subset I \cap J$. Thus, if the intersection of all ideals is non-empty, it is the minimum ideal.

Definition 3.12. Let S be an lg-semigroup. The *core* of S is the union of all the lg-ideals. We denote it $D(S)$.

Theorem 3.13. *Let S be an lg-semigroup with core D . Then D is the minimum ideal of S and comprises a single \mathcal{D} -class.*

Proof. The first statement is standard, so we omit the proof. Let $x, y \in D$. We must first show $x\mathcal{D}y$. There exist lg-ideals I and J with $x \in I$ and $y \in J$. Let $u^2 = u \in I$. By Lemma 3.10(2), there exists $v^2 = v \in J$ with $uv = v$ and $vu = u$. This shows $u\mathcal{R}v$. As I and J are \mathcal{L} -classes, it follows that $x\mathcal{D}y$. So far, D is contained in a single \mathcal{D} -class. To complete the proof, let $x \in D$ and $y \in S$ with $x\mathcal{D}y$, and show that $y \in D$. There exists $z \in S$ with $x\mathcal{L}z$ and $z\mathcal{R}y$. $z \in D$ as the lg-ideal containing x is an \mathcal{L} -class. If $y \neq z$, there exists $w \in S$ with $y = zw$. But then $y \in D$ since D is a right ideal. \square

Every \mathcal{D} -class in a semigroup has a Clifford–Preston egg-box picture in which the columns are the \mathcal{L} -classes, the rows are the \mathcal{R} -classes, and the cells are the \mathcal{H} -classes. In general, a \mathcal{D} -class need not be a subsemigroup, let alone an ideal. If D_x has an idempotent u , one can show that $L_u R_u = D_u$, but D_u may be a different \mathcal{D} -class than D_x ; in this case, $L_u R_u$ does not intersect D_x since different equivalence classes must be disjoint. For S an lg-semigroup, the \mathcal{D} -class $D(S)$ is the minimum ideal and the columns of the egg-box are the lg-ideals and the rows are the rg-ideals. Every \mathcal{H} -class in $D(S)$ has an idempotent and therefore is a group. For $x, y \in D(S)$, L_x is a right cancellative semigroup and L_x and L_y are isomorphic semigroups. R_x is a left cancellative semigroup and R_x and R_y are isomorphic semigroups. Finally, H_x and H_y are isomorphic groups.

Proposition 3.14. *The following hold.*

- (1) *Any product X of lg-semigroups X_i is an lg-semigroup and $D(X) = \prod D(X_i)$.*
- (2) *If $\psi : S \rightarrow W$ is a surjective semigroup homomorphism and S is an lg-semigroup, then W is an lg-semigroup and ψ maps $D(S)$ onto $D(W)$.*
- (3) *Let S be an lg-semigroup and let $e^2 = e \in D(S)$. Let W be a subsemigroup of S with $e \in W$. Then W is an lg-semigroup and $D(W) = W \cap D(S)$.*

Proof. (1) This is obvious since a Cartesian product of left ideals is a left ideal and a Cartesian product of left groups is a left group.

(2) Let I be an lg-ideal of S . It is obvious that $\psi(I)$ is a left ideal of $\psi(S)$ and that $\psi(I)$ is a left group if I is; therefore, W is an lg-semigroup and ψ maps $D(S)$ into $D(W)$. Let u be an idempotent of $D(W)$. As all elements of $D(S)$ are regular, it follows from Lallement's lemma applied to the homomorphism $D(S) \rightarrow D(W)$ that there exists an idempotent $v \in D(S)$ with $\psi v = u$. Since any semigroup homomorphism preserves Green's relations, $\psi(L_v) = L_u$, so ψ maps $D(S)$ onto $D(W)$.

(3) Let I be the lg-ideal of S with $e \in I$. Then $We \subset W \cap I$. If $x \in W \cap I$, then $x = xe$, so $We = W \cap I$. Let $x \in We$. Let $A = \{u \in W \cap I : u^2 = u\}$. By Theorem 3.5, we can write $x = ug$ with $u^2 = u \in I$ and $g \in H_e$, where H_e means the \mathcal{H} -class of e in I . Let H be the \mathcal{H} -class of e in W . As $x \in W$, $ug \in W$, so $g = eg = (eu)g = e(ug) = ex \in W$. Thus, $g \in H$. As also $g^{-1} \in H$, $u = u(gg^{-1}) = (ug)g^{-1} = xg^{-1} \in W$. Thus, $u \in A$ and $x \in AH$. We have $We \subset AH \subset W \cap I = We$; therefore, by Theorem 3.5, We is a left group. Thus, We is an lg-ideal of W and W is an lg-semigroup. Since any semigroup homomorphism preserves Green's relations, an element \mathcal{D} -related to e in W is \mathcal{D} -related to e in S as well. Thus, $D(W) \subset D(S) \cap W$. Let $x \in D(S) \cap W$. Let I be the

lg-ideal of S with $x \in I$. $I \cap W$ is a left ideal of W and so contains an lg-ideal J of W by Lemma 3.10(5). Let $u^2 = u \in J \subset I \cap W$ so that, by minimality, $Wu = J$. As $u \in I \cap W$, $Iu = I = Ix$. Thus, $x \in Iu$, so $x = xu \in Wu = J \subset D(W)$. \square

Proposition 3.14(3) raises the question if any subsemigroup of an lg-semigroup is lg. In fact, the opposite is true. Any semigroup is a subsemigroup of an lg-semigroup. To see this, first embed the semigroup as a subsemigroup of the endomorphism monoid X^X of a set X using Cayley's theorem. Then observe that X^X has a unique lg-ideal, namely the constant functions.

4. DYNAMIC STRUCTURES

Definition 4.1. A structure X in a tight category \mathcal{C} is a *dynamic structure* if its enveloping semigroup $E(X)$ is an lg-semigroup.

Evidently, every finite structure is a dynamic structure since its enveloping semigroup is also finite (notice that every finite semigroup has an idempotent).

Example 4.2. A product of dynamic structures need not be dynamic. For example, in the tight category of semigroups, \mathbf{Z}_n is a dynamic structure (being finite), but $X = \prod_{n \geq 0} \mathbf{Z}_n$ is not dynamic.

To see this, let $N = \{1, 2, 3, \dots\}$ and note that $E(X) = (N, +)$ as a semigroup and (N, \cdot) as a monoid. Clearly, the latter has no minimal ideal.

Powers behave better, however. If X is a structure and K is a set, then X^K is either finite (which includes the case that K is empty) or else $X^K \in P(X)$, $X \in Q(X^K)$. Thus, $E(X) = E(X^K)$, so X^K is dynamic if X is.

We also have the following.

Lemma 4.3. *Every substructure or quotient structure of a dynamic structure is dynamic.*

Proof. Let X be a dynamic structure and let either $Y \in S(X)$ or $Y \in Q(X)$. Either way, $Y \in QSP(X)$. By Proposition 2.14, $E(Y)$ is a monoid quotient of $E(X)$, so $E(Y)$ is an lg-semigroup by Proposition 3.14. \square

Thus, if \mathcal{V} is a class of dynamic structures closed under products, $QSP(\mathcal{V}) = QS(\mathcal{V})$ is a dynamic category.

For the time being, fix a dynamic structure X with enveloping semigroup $E = E(X)$ with core $D = D(E(X))$. (In general, $D(X)$ will denote $D(E(X))$.) Recall that E acts on X by evaluation, $tx = t(x)$.

We now introduce dynamic notions for dynamic structures.

Definition 4.4. Say that $x \in X$ is an *almost periodic* point, abbreviated x ap, if there exists $t \in D$ with $tx = x$. Say that $x, y \in X$ are *proximal*, written xPy , if there exists $t \in E$ with $tx = ty$. Finally, say that x and y are *distal*, written $x \perp y$, if $x \neq y$ and x and y are not proximal.

Lemma 4.5. *The following hold.*

- (1) x is almost periodic if and only if every lg-ideal has an idempotent u with $ux = x$.
- (2) xPy if and only if there exists an lg-ideal I with $tx = ty$ for all $t \in I$.

Proof. (1) Let $t \in D$ with $tx = x$. Let I be an lg-ideal and let H be the \mathcal{H} -class $I \cap R_t$, so that H is a group with unit u . As u is a left-unit in its \mathcal{R} -class, $ux = u(tx) = (ut)x = tx = x$.

(2) The set $\{p \in E : px = py\}$ is non-empty, therefore, clearly a left ideal. But every left ideal contains an lg-ideal by Lemma 3.10(5). \square

Theorem 4.6. *The following hold.*

- (1) Every element of X is proximal to an almost periodic point.
- (2) If xPy with y ap, then there exists $u^2 = u \in D$ with $ux = y$.
- (3) If $u^2 = u \in D$ and $x, y \in X$, then either $ux = uy$ or $ux \perp uy$.

Proof. (1) Let $u^2 = u \in D$. Then the equation $u(ux) = ux$ gives that ux is an almost periodic point proximal to x .

(2) Let I be an lg-ideal with $tx = ty$ for all $t \in I$. There exists $u^2 = u \in I$ with $uy = y$. Thus, $ux = uy = y$.

(3) Suppose $ux \not P uy$ so that there exists an lg-ideal I with $tux = tuy$ for all $t \in I$. By Lemma 3.10(2), there exists $v^2 = v \in I$ with $uv = v$ and $vu = u$. Thus, $ux = (vu)x = v(ux) = v(uy) = (vu)y = uy$. \square

Definition 4.7. Say that X is *distal* if $x \neq y \Rightarrow x \perp y$.

Thus, Theorem 4.6(1) gives that all points in a distal structure are almost periodic.

Lemma 4.8. *E has minimal substructures. They are precisely the lg-ideals.*

Proof. For $f \in E$, $\langle f \rangle = Ef$. It is then clear that the minimal substructures coincide with the minimal left ideals. These are the lg-ideals by Lemma 3.10(3) \square

Theorem 4.9. *In a dynamic structure, x is an almost periodic point if and only if it generates a minimal substructure.*

Proof. If x ap, there exists $u^2 = u \in D$ with $ux = x$. The lg-ideal Eu is a minimal substructure of E . Thus, $pr_x(Eu) = pr_x \langle u \rangle = \langle pr_x(u) \rangle = \langle ux \rangle = \langle x \rangle$. Since the inverse image of a substructure is a substructure (use a pullback), it is clear that the image of a minimal substructure under an admissible map is minimal. Conversely, let x generate a minimal substructure and let I be any lg-ideal of E . As $Ix = pr_x(I)$ is a substructure, $Ix = \langle x \rangle$, so $t \in I$ exists with $tx = x$. \square

When the tight category is compact M -actions as in Example 1.16, the above theorem establishes that the definition of “almost periodic” for a dynamic structure coincides with the original topological one. It is not hard to show that the proximal relation for compact M -actions also coincides with the lg-definition, but we will not do that in this brief paper. See, e.g., [6].

Theorem 4.10. *For $x \in X$, the set of almost periodic points of $\langle x \rangle$ is precisely Dx .*

Proof. Let $t \in D$. Let u be the unit of the group H_t . Then $u(tx) = (ut)x = tx$, so tx ap. Conversely, let $y \in \langle x \rangle$ be ap so that $\langle y \rangle$ is minimal. Let I be an lg-ideal of E . Then $\langle y \rangle = Iy$, so there exists $p \in I$ with $py = y$. As $y \in \langle x \rangle = Ex$ and D is a right ideal of E , $y \in Dy \subset DEx \subset Dx$. \square

Corollary 4.11. *The set of almost periodic points of E is precisely D .*

Proof. $E = \langle \text{id}_X \rangle$, so the set of almost periodic points is $D \text{id}_X = D$. \square

Theorem 4.12. *Let \mathcal{V} be a class of dynamic structures closed under products. Then there exists $U \in QSP(\mathcal{V})$ such that U is a minimal structure and for every minimal structure M in $QSP(\mathcal{V})$ there exists a (necessarily surjective) admissible map $U \rightarrow M$. Further, such U is unique up to isomorphism.*

Proof. $QSP(\mathcal{V})$ is a dynamic category. Let $F = E(\mathcal{V})$. As F is free on one generator in $QSP(\mathcal{V})$, F admits a surjective admissible map onto every singly-generated structure. Let $U \subset F$ be any lg-ideal. Then U admits an admissible map to every singly-generated structure, and so admits a surjective admissible map to every minimal structure. Additionally, U is a minimal structure by Lemma 4.8. Suppose that V is also minimal and admits an admissible map to every other minimal. Thus, there exist admissible $U \xrightarrow{\psi} V \xrightarrow{\phi} U$. Recalling that admissible maps $E \rightarrow E$ are right translations, it follows from Lemma 3.11 that $\phi\psi$ is bijective. Hence, the surjective map ψ is also injective and so is an isomorphism since bijective admissible maps are isomorphisms in a tight category. \square

A structure U as in the previous theorem is usually called the *universal minimal set* in topological dynamics.

Let \mathcal{C} be an equational class for which there exists a cardinal κ for which each \mathcal{C} -operation has arity $< \kappa$. The category \mathcal{KC} of *compact \mathcal{C} -structures* has as objects all \mathcal{C} -structures X equipped with a compact Hausdorff topology in such a way that each \mathcal{C} -operation $X^n \rightarrow X$ is continuous, and with morphisms the continuous \mathcal{C} -homomorphisms. It is not hard to see that both \mathcal{C} and \mathcal{KC} are tight categories (see [18], [19]).

Proposition 4.13. *Every structure in \mathcal{KC} is dynamic.*

Proof. For X a compact \mathcal{C} -structure, each right translation $- \circ f : X^X \rightarrow X^X$ is continuous, so that X^X is a compact semigroup with continuous right translations. $E(X)$ is a closed subsemigroup of X^X . It follows from Example 3.8 that $E(X)$ is an lg-semigroup. \square

Example 4.14. Let \mathcal{JT} be the equational class of Jónsson–Tarski algebras [16, Theorem 5]. These are defined with a binary operation m and two unary operations u and v provided with the equations that express that m is bijective with u and v giving the inverse, specifically,

$$\begin{aligned} m(ux, vx) &= x \\ u m(x, y) &= x \\ v m(x, y) &= y. \end{aligned}$$

These algebras were invented to give an example where all finitely-generated free algebras are isomorphic. The free algebra on one generator x (and also on two generators ux and vx) is constructed by building one-variable terms in m , u , and v and reducing modulo the equations. Since such a free algebra is countably infinite, if minimal algebras exist they must be countable. The standard Cantor pairing function $\omega \times \omega \rightarrow \omega$, namely

$$m(x, y) = \frac{(x + y)(x + y + 1)}{2} + y$$

is not minimal since $\{0\}$ is a subalgebra. If, however, m is modified replacing $m(0, 0)$ with 4 and replacing $m(1, 1)$ with 0, the new bijection is minimal, as is not hard to show. D. M. Smirnov [22] shows that there are in fact c non-isomorphic minimal Jónsson–Tarski algebras.

We can argue that the free Jónsson–Tarski algebra on one generator is not an lg-semigroup because it has no minimal left ideals as follows. There are complex terms beginning with m such as $m(m(ux, m(vux, x)), vux)$. By applying u sufficiently many times to any term τ , the result is a term of form $\sigma = w_1 \cdots w_k x$ ($k \geq 0$) with each w_i , either u or v . Such σ belongs to any left ideal containing τ . But it seems clear that no term ν

exists with $\nu u \sigma = \sigma$, so for arbitrary τ , no left ideal containing τ can be minimal.

Example 4.15. The category \mathcal{KJT} of compact Jónsson–Tarski algebras has all structures dynamic.

The objects here, of course, are compact Hausdorff spaces which are homeomorphic to their square, but the morphisms are more than continuous; they must also preserve m , u , and v . The Cantor set 2^ω is a minimal structure in this category [18]. Because of Lemma 2.13, it is tempting to conjecture that the free structure on one generator in this category is βM if the monoid M is the free structure on one generator in \mathcal{JT} . The conjecture is false, however, since it is known [23, 8.12] that $\beta\omega$ is not homeomorphic to $\beta\omega \times \beta\omega$.

5. COMFORT TYPES ARE MONADS

In this section, we introduce subfunctors and submonads of the ultrafilter monad β which play a role in the rest of this paper. In this section we show that a concrete model of the Comfort type of $r \in \omega^*$ is the submonad of β that r generates. We begin with the definition of a monad in the category of sets.

Definition 5.1. A *monad* in the category of sets is given by the following data and axioms. Data: A function T which, for each set X , produces another set TX ; a function $\eta_X : X \rightarrow TX$ for each set X ; for each function of form $f : X \rightarrow TY$ is given a function $f^\# : TX \rightarrow TY$. The axioms are the monad laws (m.1, m.2, m.3) of Proposition 1.15.

Thus, any tight category gives rise to a monad which is explicitly known once it is understood how to construct the free structures.

Example 5.2. The *ultrafilter monad* $(\beta, \text{prin}, (\cdot)^\#)$ arises from the tight category **CT2** of Example 1.7. We know that free compact spaces are constructed as the β -compactification of discrete spaces. In detail, let βX be the set of ultrafilters on the set X and let $\text{prin}_X : X \rightarrow \beta X$ map an element x to its principal ultrafilter $\text{prin}(x)$. Let $f^\# : \beta X \rightarrow \beta Y$ be the Stone extension of f .

For the ultrafilter monad, it is easily checked that the Stone extension $f^\# : \beta X \rightarrow \beta Y$ is given by

$$(5.1) \quad f^\# \mathcal{U} = \{B \subset Y : \{x : B \in fx\} \in \mathcal{U}\}.$$

The usual “image of an ultrafilter” $f\mathcal{U}$ is precisely

$$(5.2) \quad (\beta f)\mathcal{U} = \{B \subset Y : \{x : B \in fx\} \in \mathcal{U}\}.$$

We now turn to exploring subfunctors and submonads of β . We use G as the generic symbol for a subfunctor of β , where we exclude the trivial case $GX = \emptyset$ for all X . *Subfunctor* means that for all $f : X \rightarrow Y$, βf maps GX into GY . In that case, the resulting function $GX \rightarrow GY$ is denoted Gf , as is customary for functors.

Notice that every principal ultrafilter on X is in GX (consider the effect of constant functions).

We use T as the generic symbol for a submonad of β . By a *submonad*, we mean a subfunctor $T \subset \beta$ with some $TX \neq \emptyset$ (so that all principal ultrafilters on X belong to TX) with the additional property of being closed under the Stone extension. This means that if $f : X \rightarrow TY$, then $(X \xrightarrow{f} TY \subset \beta Y)^\#$ maps TX into TY . In that case we denote the resulting map also by $f^\# : TX \rightarrow TY$.

Example 5.3. $\beta_\omega X = \{\mathcal{U} \in \beta X : \mathcal{U} \text{ has a countable member}\}$ is a submonad of β [20, Lemma 3.6].

Any (pointwise) intersection or union of subfunctors is again a subfunctor. In particular, every set of ultrafilters generates a subfunctor. “Small” subfunctors are generated by a single ultrafilter. These are easily described.

Example 5.4. Let $r \in \beta n$ for some set n . Then the subfunctor G_r generated by r is given by

$$G_r X = \{fr : n \xrightarrow{f} X\}$$

as is obvious. Here we use the customary notation fr for $(\beta f)r$ when no confusion would result.

For $r, s \in \beta\omega$, the usual Rudin–Keisler order $r \leq_{RK} s$ evidently coincides with $G_r \subset G_s$. Hence, the functor G_r is a concrete model for the Rudin–Keisler type of r .

Definition 5.5. Let $G \subset \beta$ be a subfunctor. A topological space X is *G-compact* if every ultrafilter in GX converges in X . For $r \in \omega^*$, X is *r-compact* if it is G_r -compact. The *Comfort preorder* (see [9], [10]) on ω^* is defined by $r \leq_c s \Leftrightarrow$ every s -compact Tychonoff space is r -compact.

To learn more about the Comfort order we shall use the following result.

Theorem 5.6 (Börger [4]). *Let H be a functor from the category of sets to itself which preserves binary coproducts. Then there exists a unique natural transformation $H \rightarrow \beta$.*

Any composition of subfunctors of β preserves binary coproducts. This is clear if we ensure that a subfunctor G of β preserves binary coproducts.

Let $\mathcal{U} \in GX$, $A \in \mathcal{U}$. Then $\mathcal{U} \in \beta A$ by tracing with A . This trace is $s\mathcal{U}$ if $s : X \rightarrow A$ extends the inclusion of A . For that reason, $A \in \mathcal{U} \Rightarrow \mathcal{U} \in GA$. Preserving binary coproducts then amounts to the fact that if a disjoint union of two subsets belongs to \mathcal{U} , exactly one of them does.

Lemma 5.7. *Let X be compact Hausdorff and let $\xi : \beta X \rightarrow X$ be ultrafilter convergence. Then ξ is continuous.*

Proof. Let \mathcal{U}_α be a net in βX which converges to \mathcal{U} , and let U be open with $\xi\mathcal{U} \in U$. As X is regular, there exists open V with $\xi\mathcal{U} \in V \subset \overline{V} \subset U$. Then $\xi\mathcal{U} \in V \Rightarrow V \in \mathcal{U} \Rightarrow$ eventually $V \in \mathcal{U}_\alpha$ so that eventually $\xi\mathcal{U}_\alpha \in \overline{V} \subset U$. This shows that $\xi\mathcal{U}_\alpha$ converges to $\xi\mathcal{U}$ in X . \square

Lemma 5.8. *Let $\mu_X : \beta\beta X \rightarrow \beta X$ be the ultrafilter convergence map of βX . Then $\mu : \beta\beta \rightarrow \beta$ is a natural transformation.*

Proof. By the previous lemma, $\mu_X = (\text{id}_{TX})^\#$. Naturality is then a formal consequence of the monad laws. For details, see [19, Proposition 2.14]. \square

We now turn to characterizing the Comfort preorder.

Theorem 5.9. *Let S be a subfunctor of β and let T be a submonad of β . The following are equivalent.*

- (1) *Every T -compact space is S -compact.*
- (2) *Every T -compact Tychonoff space is S -compact.*
- (3) *S is a subfunctor of T .*

Proof. That (1) \Rightarrow (2) and (3) \Rightarrow (1) are obvious. We show (2) \Rightarrow (3). Let $i : S \rightarrow \beta$ and $j : T \rightarrow \beta$ be the inclusion natural transformations. Regard TX as a subspace of βX so that TX is a Tychonoff space. Because T is a submonad, we have a commutative square

$$\begin{array}{ccc} TTX & \xrightarrow{(ii)_X} & \beta\beta X \\ (\text{id}_{TX})^\# \downarrow & & \downarrow \mu_X \\ TX & \xrightarrow{i_X} & \beta X \end{array}$$

where ii is the horizontal composition of natural transformations; here

$$TTX \xrightarrow{Ti_X} T\beta X \xrightarrow{i_{\beta X}} \beta\beta X = (ii)_X = TTX \xrightarrow{i_{TX}} \beta TX \xrightarrow{\beta i_X} \beta\beta X.$$

(For the reader familiar with monads, we are only asserting that the inclusion of a submonad is a monad map.) The diagram then asserts that the ultrafilters in $T\beta X$ converge in TX so that TX is T -compact. By

hypothesis, TX is S -compact. It follows that γ_X exists, rendering the following diagram commutative because the point to which an ultrafilter in STX converges in βX lies in TX .

$$\begin{array}{ccccc}
 STX & \xrightarrow{i_{TX}} & \beta TX & \xrightarrow{\beta j_X} & \beta\beta X \\
 \downarrow \gamma_X & & & & \downarrow \mu_X \\
 TX & \xrightarrow{j_X} & & & \beta X
 \end{array}$$

This results in a natural transformation $ST \xrightarrow{i^T} \beta T \xrightarrow{\beta j} \beta\beta \xrightarrow{\mu} \beta$. As j is a pointwise monic natural transformation with $j\gamma$ natural, it follows that γ is a natural transformation as well. Thus, $\Gamma = S \xrightarrow{s\text{prin}} ST \xrightarrow{\gamma} T$ is a natural transformation. But then the triangle

$$\begin{array}{ccc}
 S & \xrightarrow{\Gamma} & T \\
 & \searrow i & \downarrow j \\
 & & \beta
 \end{array}$$

must commute by Theorem 5.6 so that $S \subset T$. \square

Definition 5.10. For $r \in \omega^*$, let T_r be the submonad of β generated by r (in effect, the intersection of all submonads containing G_r).

Evidently, T_r is a submonad of the monad β_ω of Example 5.3.

It is shown in [20, Lemma 11.8] that every r -compact space is T_r -compact. Thus, the terms “ r -compact,” “ G_r -compact,” and “ T_r -compact” are synonymous. The following is then immediate from the previous theorem.

Theorem 5.11. For $r, s \in \omega^*$, $r \leq_c s \Leftrightarrow T_r \subset T_s$. \square

It follows that a concrete model of a Comfort type is a submonad T_r . A comfort type can also be characterized as a single topological space. See Theorem 6.12 below.

6. T -SPACES

Definition 6.1. Let G be a subfunctor of β and let X be a topological space. $A \subset X$ is G -open if whenever $\mathcal{U} \in GX$ and $\mathcal{U} \rightarrow x \in A$, then $A \in \mathcal{U}$. A is G -closed if whenever $A \in \mathcal{U} \rightarrow x$, then $x \in A$. X is a G -space if every G -open set is open.

Here, “ \rightarrow ” indicates ultrafilter convergence. Since every neighborhood filter is an intersection of ultrafilters, a set is β -open if and only if it is a neighborhood of each of its points, that is, is open. In general, every open set is G -open so there are more G -open sets than open sets. It is trivial to prove that a subset is G -closed if and only if its complement is G -open, so a G -space is a space in which every G -closed set is closed.

Lemma 6.2. *Let (X, \mathcal{T}) be a G -space, let (Y, \mathcal{S}) be any space, and let $f : X \rightarrow Y$ be a function. Then f is continuous if and only if for every ultrafilter $\mathcal{U} \in GX$, if $\mathcal{U} \rightarrow x$, then $f\mathcal{U} \rightarrow fx$.*

Proof. Let U be open in (Y, \mathcal{S}) and let $\mathcal{U} \rightarrow x \in f^{-1}U$ with $\mathcal{U} \in GX$. By hypothesis, $f\mathcal{U} \rightarrow fx \in U$, so $U \in f\mathcal{U}$ and $f^{-1}U \in \mathcal{U}$. This shows that $f^{-1}U$ is G -open. As (X, \mathcal{T}) is a G -space, f is continuous. \square

Proposition 6.3. *For a subfunctor $G \subset \beta$, let (X, \mathcal{T}) be a topological space and let \mathcal{T}_G be the G -open subsets of (X, \mathcal{T}) . Then (X, \mathcal{T}_G) is the G -space coreflection of (X, \mathcal{T}) .*

Proof. That \mathcal{T}_G is a topology is trivial. Let A be G -open in (X, \mathcal{T}_G) and let \mathcal{U} converge to x in (X, \mathcal{T}_G) with $x \in A$. As also \mathcal{U} converges to x in (X, \mathcal{T}) , $A \in \mathcal{U}$. But then A is also G -open in (X, \mathcal{T}) ; that is, A is open in (X, \mathcal{T}_G) . This shows that (X, \mathcal{T}_G) is a G -space. For the coreflective property, let $f : (Z, \mathcal{S}) \rightarrow (X, \mathcal{T})$ be continuous with (Z, \mathcal{S}) a G -space. We must show that $f : (Z, \mathcal{S}) \rightarrow (X, \mathcal{T}_G)$ is continuous. To that end, let $U \in \mathcal{T}_G$. Let $\mathcal{U} \in GX$, $\mathcal{U} \rightarrow z \in f^{-1}U$. By hypothesis, $f\mathcal{U} \rightarrow fz \in U$. As U is G -open, $f^{-1}U \in \mathcal{U}$. This shows that $f^{-1}U$ is G -open, and hence open. \square

Lemma 6.4. *Let $R \subset GX \times X$ be any relation and define $\mathcal{T}_R = \{U \subset X : \mathcal{U} R x, x \in U \Rightarrow U \in \mathcal{U}\}$. Then (X, \mathcal{T}_R) is a G -space whose G -restricted ultrafilter convergence relation contains R .*

Proof. That \mathcal{T}_R is a topology is obvious. Let $\xi \subset GX \times X$ be the G -restricted ultrafilter convergence relation of (X, \mathcal{T}_R) . If $U \in \mathcal{T}_R$ and $\mathcal{U} R x$, $x \in U$, then $U \in \mathcal{U}$, so $R \subset \xi$. Let U be G -open. Let $\mathcal{U} R x$ with $x \in U$. Since $\mathcal{U} \in GX$ and U is G -open, $U \in \mathcal{U}$. This shows that $U \in \mathcal{T}_R$ so that (X, \mathcal{T}_R) is a G -space. \square

If X is a space and R is its G -restricted ultrafilter convergence relation, then \mathcal{T}_R of the previous lemma is just the G -open sets, so (X, \mathcal{T}_R) is the G -space coreflection of X .

Definition 6.5. Let \mathbf{Top}_G denote the category of G -spaces and continuous maps.

The category \mathbf{Top}_G has products, namely the G -space coreflection of the Tychonoff product. This is general category theory: right adjoints preserve products.

Lemma 6.6. *Let X be the G -space product of the G -spaces X_i . Then each projection $pr_i : X \rightarrow X_i$ is open. Moreover, convergence is pointwise in that for $\mathcal{U} \in GX$, $\mathcal{U} \rightarrow (x_i) \Leftrightarrow$ for each i , $pr_i \mathcal{U} \rightarrow x_i$.*

Proof. See [20, Theorem 5.4 and Theorem 5.5]. \square

Corollary 6.7. *In \mathbf{Top}_G , a product of G -compact spaces is G -compact.*

Proposition 6.8. *The following hold.*

- (1) *Every G -compact subspace of a Hausdorff G -space is closed.*
- (2) *If $f : X \rightarrow Y$ is continuous and surjective with X G -compact, then Y is also G -compact.*

Proof. (1) Let $A \in \mathcal{U} \rightarrow x$, with $A \subset X$ G -compact and $\mathcal{U} \in GX$. There exists $a \in A$ with $\mathcal{U} \rightarrow a$. As X is Hausdorff, $x \in A$. Thus, A is G -closed, and hence closed.

(2) Let $\mathcal{V} \in GY$. Let $s : Y \rightarrow X$ with $fs = \text{id}_Y$ and let $\mathcal{U} = s\mathcal{V} \in GX$. By hypothesis, x exists with $\mathcal{U} \rightarrow x$. By continuity, $\mathcal{V} = fs\mathcal{V} = f\mathcal{U} \rightarrow fx$. \square

From this point forward, we fix a submonad T of β .

From the diagram for $i_X : TX \rightarrow \beta X$ in the proof of Theorem 5.8, we see that $\mu_X : \beta\beta X \rightarrow \beta X$ restricts to a map $TTX \rightarrow TX$. We use the same symbol μ to denote this map, $\mu_X : TTX \rightarrow TX$. From the formula (5.1) for the Stone extension, we see that

$$\mu_X(\mathcal{H}) = \{B \subset X : TB \in \mathcal{H}\},$$

where we identify TB with $\{\mathcal{U} \in TX : B \in \mathcal{U}\}$.

Definition 6.9. The T -space $(TX, \mathcal{T}_{\mu_X})$ of Lemma 6.4 will be denoted TX .

As the T -restricted ultrafilter convergence relation of TX contains μ_X , TX is T -compact.

Lemma 6.10. *For $A \subset X$, TA is clopen in TX .*

Proof. Let $\mu_X(\mathcal{H}) = \mathcal{U}$ and let $\mathcal{U} \in TA$. As $A \in \mathcal{U}$, $TA \in \mathcal{H}$. This shows that TA is open in TX . As $(TA)' = TA'$, TA is clopen. \square

The converse of the previous lemma is also true, but we don't need that in this paper.

Proposition 6.11. *TX is a Urysohn space.*

Proof. It suffices to show that the inclusion $i : TX \rightarrow 2^{2^X}$ is continuous since every compact Hausdorff space is Urysohn. For $A \in 2^X$, the composition $TX \xrightarrow{i} 2^{2^X} \xrightarrow{pr_A} 2$ is the characteristic function of TA , and this is continuous by the previous lemma. \square

It follows that TX is a T -compact, Hausdorff T -space whose T -restricted ultrafilter convergence function is precisely $\mu_X : TTX \rightarrow TX$.

Theorem 6.12. *Let S and T be submonads of the monad β_ω of Example 5.3. Then if the $S\omega$ and $T\omega$ are homeomorphic, $S = T$. In particular, the Comfort type T_r is completely characterized by the topological space $T_r\omega$.*

Proof. See [20, Theorem 3.9]. \square

Definition 6.13. For T a submonad of β , denote by \mathcal{C}_T the category of all T -compact, Hausdorff T -spaces, and continuous maps.

Theorem 6.14. *Let T be a submonad of β . The following properties are established in [20].*

- (1) *Let $f : X \rightarrow Y$ be a function with Y in \mathcal{C}_T . Then $f^\# = TX \xrightarrow{Tf} TY \xrightarrow{\theta} Y$, where θ is the T -restricted ultrafilter convergence function of Y , is the unique continuous map ψ with $\psi \text{prin}_X = f$.*
- (2) *For $A \subset X \in \mathcal{C}_T$ with inclusion $i : A \rightarrow X$, \bar{A} is the image of $i^\# : TA \rightarrow X$.*
- (3) *If X is a T -space and $A \subset X$, then $x \in \bar{A} \Leftrightarrow$ there exists $\mathcal{U} \in TX$ with $A \in \mathcal{U} \rightarrow x$.*

Proposition 6.15. *The following hold.*

- (1) *A space is a T -space if and only if each of its neighborhood filters is an intersection of ultrafilters in TX .*
- (2) *Every subspace of a T -space is a T -space.*

Proof. (1) Suppose $A \notin \mathcal{N}_x$. Then $x \in (A^\circ)' = \bar{A}'$, so there exists $\mathcal{U} \in TX$ with $A' \in \mathcal{U}, \mathcal{U} \rightarrow x$. As $A \notin \mathcal{U}$, we are done.

(2) Let X be a T -space, let $A \subset X$ be a subspace, and let $D \subset A$ with D T -closed in A . Let \bar{D} be the closure of D in X . To show $\bar{D} \cap A \subset D$, let $x \in \bar{D} \cap A$ so that there exists $\mathcal{U} \in TX$ with $D \in \mathcal{U}, \mathcal{U} \rightarrow x$. $D \in \mathcal{U} \Rightarrow A \in \mathcal{U}$, so we may consider $\mathcal{U} \in TA$. As $x \in A$ and A is a subspace, $\mathcal{U} \rightarrow x$ in A . As D is T -closed in A , $x \in D$. \square

From the facts in this section, we obtain the following result.

Theorem 6.16. *For T a submonad of β , the category \mathcal{C}_T of all T -compact, Hausdorff T -spaces and continuous maps is a tight category. Substructures coincide with closed subsets.*

7. COUNTABLY TIGHT DYNAMICS

For M a monoid, let \mathcal{C}_M be the category of M -sets. Mainstream topological dynamics studies the category \mathcal{KC}_G of compact Hausdorff G -sets, where G is a group. All objects here are dynamic structures and the notions of almost periodicity, proximal as well as other properties not discussed herein, as developed for dynamic structures, generally coincide with their original topological definitions. The classical examples of topological dynamics are based on compact metric spaces. In \mathcal{KC}_G , however, spaces need not be countably tight as is even the case for the free structure on one generator βG .

In this final section, we consider submonads T of β for which every compact metrizable space is in \mathcal{C}_T , whereas every space in \mathcal{C}_T is countably tight. There are 2^c choices of such T which are “small” in that the closure of a countable set has cardinality at most c . For any submonad T and group G , the category of G -actions $\rho : G \times X \rightarrow X$ with $X \in \mathcal{C}_T$, G discrete and ρ continuous is a tight category. We are interested in finding conditions guaranteeing that all the structures in such a category are dynamic.

Theorem 7.1. *Let β_ω be the submonad of β of Example 5.3. Then the β_ω -spaces are precisely the countably tight spaces.*

Proof. See [20, Theorem 4.9]. □

It is evident from the definition that if $S \subset T$, then every S -space is a T -space. Thus, if $T \subset \beta_\omega$, every T -space is countably tight. Notice that $T_r \subset \beta_\omega$ if $r \in \omega^*$.

Lemma 7.2. *For any $r \in \omega^*$, every sequential space is a G_r -space.*

Proof. Let X be sequential and let $A \subset X$ be G_r -closed. Let $f : \omega \rightarrow X$ be a sequence which converges to $x \in X$. If U is open with $x \in U$, $f^{-1}U$ is cofinite, hence in r , so $fr \rightarrow x$. As A is G_r -closed, $x \in A$. Since then A is sequentially closed, A is closed because X is sequential. □

We now see that there are a wealth of categories of countably tight Hausdorff spaces which contain all compact metrizable spaces.

Corollary 7.3. *For T a submonad of β_ω such that $T\omega$ has a nonprincipal ultrafilter, every compact metrizable space is in \mathcal{C}_T .*

Proof. There exists $r \in \omega^*$ with $G_r \subset T$. A metrizable space is sequential, hence a G_r -space, and hence a T -space. A compact metrizable space is T -compact and Hausdorff. □

Theorem 7.4. *For $r \in \omega^*$, $|T_r\omega| = |T_r2^\omega| = c$.*

Proof. By [20, Corollary 11.6], $|T_r\omega| \leq |T_r2^\omega| \leq c$. The unit interval belongs to \mathcal{C}_{T_r} so admits a continuous surjection from $T_r\omega$ so that $c \leq |T_r\omega|$. \square

We conclude the paper by delivering on our main thesis that there is a universe for topological dynamics in which the acting group G is countable and all spaces are countably tight. Start with compact group actions, so that the free structure on one generator has lg-monoid structure βG as in Example 1.16. Let e be any idempotent in the core of βG . Applying Proposition 3.14(3), G -actions with spaces in \mathcal{C}_{T_e} have $T_e G$ as the free structure on one generator with lg-monoid structure. All enveloping semigroups are monoid quotients of $T_e G$ and so are lg-semigroups rendering all structures dynamic. T_e -spaces are countably tight because, since G is countable, T_e is a submonad of β_ω .

The method of the paragraph above will not work for uncountable groups because if G is uncountable, the uniform ultrafilters in βG form a two-sided ideal and so all elements of the core are uniform [3, Proposition 2.4]. What goes wrong is that if $e^2 = e$ is in the core of βG , T_e is a submonad of β but is not a submonad of β_ω . One will get dynamic structures based on T_e -spaces, but these need not be countably tight and the cardinality of the closure of a countable set in such spaces may exceed c . Other approaches will be discussed elsewhere.

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REFERENCES

- [1] Jiří Adámek, Horst Herrlich, and George E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. New York: John Wiley & Sons, Inc., 1990.
- [2] George D. Birkhoff, *Dynamical Systems*. American Mathematical Society Colloquium Publications, Vol. IX. Providence, RI: AMS, 1927.
- [3] William Bryan, *Ideals and idempotents in the uniform ultrafilters*. Preprint. Available at arXiv:1505.02102v1 [math.RA].
- [4] Reinhard Börger, *Coproducts and ultrafilters*, J. Pure Appl. Algebra **46** (1987), no. 1, 35–47.
- [5] A. H. Clifford, *A system arising from a weakened set of group postulates*, Ann. of Math. (2) **34** (1933), no. 4, 865–871.
- [6] J. de Vries, *Elements of Topological Dynamics*. Mathematics and its Applications, 257. Dordrecht: Kluwer Academic Publishers Group, 1993.
- [7] Robert Ellis, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc. **94** (1960), 272–281.
- [8] H. Gaifman, *Infinite Boolean polynomials. I*, Fund. Math. **54** (1964), 229–250.

- [9] Salvador García-Ferreira, *Comfort types of ultrafilters*, Proc. Amer. Math. Soc. **120** (1994), no. 4, 1251–1260.
- [10] Salvador García-Ferreira, Neil Hindman, and Dona Strauss, *Orderings of the Stone-Čech remainder of a discrete semigroup*, Special issue in honor of W. W. Comfort (Curacao, 1996). Topology Appl. **97** (1999), no. 1-2, 127–148.
- [11] Walter Helbig Gottschalk and Gustav Arnold Hedlund, *Topological Dynamics*. American Mathematical Society Colloquium Publications, Vol. 36. Providence, R. I.: American Mathematical Society, 1955.
- [12] J. A. Green, *On the structure of semigroups*, Ann. of Math. (2) **54**, (1951), 163–172.
- [13] A. W. Hales, *On the non-existence of free complete Boolean algebras*, Fund. Math. **54** (1964), 45–66.
- [14] Neil Hindman and Dona Strauss, *Algebra in the Stone-Čech Compactification. Theory and Applications*. De Gruyter Expositions in Mathematics, 27. Berlin: Walter de Gruyter & Co., 1998.
- [15] J. M. Howie, *An Introduction to Semigroup Theory*. L.M.S. Monographs, No. 7. London-New York: Academic Press, 1976.
- [16] Bjarni Jónsson and Alfred Tarski, *On two properties of free algebras*, Math. Scand. **9** (1961), 95–101.
- [17] Saunders MacLane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Vol. 5. New York-Berlin: Springer-Verlag, 1971.
- [18] Ernest Manes, *Minimal subalgebras for dynamic triples* in Category Theory, Homology Theory and Their Applications III. Ed. Peter J. Hilton. Lecture Notes in Mathematics, 99. Berlin-Heidelberg-New York: Springer-Verlag, 1969. 419–447.
- [19] Ernie Manes, *Monads of sets* in Handbook of Algebra, Vol. 3. Ed. M. Hazewinkel. Amsterdam: Elsevier/North-Holland, 2003. 67–153.
- [20] Ernie Manes, *Monads in topology*, Topology Appl. **157** (2010), no. 5, 961–989.
- [21] Wolfgang Ruppert, *Compact Semitopological Semigroups: An Intrinsic Theory*. Lecture Notes in Mathematics, 1079. Berlin: Springer-Verlag, 1984.
- [22] D. M. Smirnov, *Cantor algebras with single generator, I*, Algebra and Logic (English translation) **10** (1971), 40–49.
- [23] Russell C. Walker, *The Stone-Čech Compactification*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 83. New York-Berlin: Springer-Verlag, 1974.

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