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TOPOLOGY PROCEEDINGS

Volume 50, 2017

Pages 311–317

http://topology.nipissingu.ca/tp/

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Electronically published on March 6, 2017

Topology Proceedings

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E-mail: topolog@auburn.edu
ISSN: (Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

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Volume 50 (2017)
Pages 311-317

E-Published on March 6, 2017

PARTIAL ANSWERS TO SOME QUESTIONS ON MAPS TO ORDERED TOPOLOGICAL VECTOR SPACES

ER-GUANG YANG

Abstract. In this paper, we give partial answers to some questions posed by Kaori Yamazaki (*Monotone countable paracompactness and maps to ordered topological vector spaces*, Topology Appl. **169** (2014), 51–70).

1. Introduction and Preliminaries

A space always means a T_1 topological space and a function always means a real-valued function. The set of all positive integers is denoted by $\mathbb{N}$. A vector space always means a real vector space. The origin of a vector space is denoted by $\mathbf{0}$ For a space X and $A \subset X$, we use $intA$ and $\overline{A}$ to denote the interior and the closure of A in X, respectively. Also, we use χ_A to denote the characteristic function of A.

A vector space Y equipped with a partial order $\leq$ is called an *ordered vector space* if $\leq$ is compatible with its linear structure. A topological vector space Y is called an *ordered topological vector space* if Y is an ordered vector space and the positive cone $Y^+ = \{y \in Y : y \geq 0\}$ is closed in Y.

Let Y be an ordered topological vector space and $e \in Y^+$. Then e is called an *interior point* of Y^+ if $e \in int_Y(Y^+)$. If e is an interior point of Y^+ and $e > 0$, then e is called a *positive interior point.* e is called an *order unit* if for each $y \in Y$, there exists $\lambda > 0$ such that $y \leq \lambda e$. It is

2010 *Mathematics Subject Classification.* 54D20, 46A40.

Key words and phrases. lower (upper) semi-continuous maps, monotone cb-spaces, monotonically countably metacompact spaces, monotonically countably paracompact spaces, ordered topological vector spaces.

This work is supported by NSFC (11271178) and NSFC (11401262).

clear (see [6]) that if e is an interior point of Y^+, then for each $r > 0$, both $-re + Y^+$ and $re - Y^+$ are 0-neighborhoods.

Recall that a function f on a space X is called *lower* (*upper*) *semi-continuous* if for any real number r, the set $\{x \in X : f(x) > r\}$ (respectively, $\{x \in X : f(x) < r\}$) is open. In [1], the notion of semi-continuous functions was generalized to the semi-continuous maps with values into ordered topological vector spaces as follows.

Let X be a topological space and Y an ordered topological vector space. A map $f : X \to Y$ is called *lower semi-continuous* [1] if the set-valued mapping $\varphi : X \to 2^Y$, defined by letting $\varphi(x) = f(x) - Y^+$ for each $x \in X$, is lower semi-continuous. f is *upper semi-continuous* if $-f$ is lower semi-continuous. f is called *locally upper bounded* [6] if for each $x \in X$ and each 0-neighborhood V, there exist a neighborhood O_x of x and $n \in \mathbb{N}$ such that $f(O_x) \subset nV - Y^+$. A real-valued function f on a space X is called *locally bounded* (*locally upper bounded*) if for each $x \in X$, there exist a neighborhood O_x of x and $n \in \mathbb{N}$ such that $|f(x')| < n$ (respectively, $f(x') < n$) for each $x' \in O_x$.

Lemma 1.1 ([1], [6]). *Let X be a topological space and Y an ordered topological vector space. For a map $f : X \to Y$,* (1) *and* (2) *are equivalent and* (1) *implies* (3).

(1) *f is lower (upper) semi-continuous.*
(2) *For each $x \in X$ and each 0-neighborhood V, there exists a neighborhood O_x of x such that $f(O_x) \subset f(x) + V + Y^+$ (respectively, $f(O_x) \subset f(x) + V - Y^+$).*
(3) *$f^{-1}(y - Y^+)$ (respectively, $f^{-1}(y + Y^+)$) is closed in X for each $y \in Y$.*

Lemma 1.2 ([6]). *Let X be a topological space and Y an ordered topological vector space with a positive interior point e of Y^+. Then a map $f : X \to Y$ is locally upper bounded if and only if for each $x \in X$, there exist a neighborhood O_x of x and $n \in \mathbb{N}$ such that $f(O_x) \subset ne - Y^+$.*

For two sequences $\langle A_j \rangle$ and $\langle B_j \rangle$ of subsets of a space X, we write $\langle A_j \rangle \preceq \langle B_j \rangle$ if $A_n \subset B_n$ for every $n \in \mathbb{N}$.

Definition 1.3 ([3]). A space X is called *monotonically countably metacompact* if there is an operator O assigning to each decreasing sequence $\langle F_j \rangle$ of closed subsets of X with empty intersection, a sequence of open sets $\{O(n, \langle F_j \rangle) : n \in \mathbb{N}\}$ such that

(1) $F_n \subset O(n, \langle F_j \rangle)$ for each $n \in \mathbb{N}$,
(2) if $\langle F_j \rangle \preceq \langle H_j \rangle$, then $O(n, \langle F_j \rangle) \subset O(n, \langle H_j \rangle)$ for all $n \in \mathbb{N}$,
(3) $\bigcap_{n \in \mathbb{N}} O(n, \langle F_j \rangle) = \emptyset$.

X is called *monotonically countably paracompact* if, in addition, $(3')$ $\bigcap_{n\in\mathbb{N}} \overline{O(n,\langle F_j\rangle)} = \emptyset$.

Definition 1.4 ([7])**.** A space X is called a *monotone cb-space* if there exists an operator Φ assigning a continuous function $\Phi(f)$ to each locally bounded function f on X, such that $|f| \le \Phi(f)$ and $\Phi(f) \le \Phi(f')$ whenever $|f| \le |f'|$.

In [6], Kaori Yamazaki generalized real-valued functions in some insertion theorems to maps with values into ordered topological vector spaces and posed the following several questions.

Question 1.5. Let X be a topological space and Y an ordered topological vector space with a positive interior point of Y^+. Are the following conditions equivalent?

(1) X is monotonically countably paracompact.

(2) There exist operators Φ and Ψ assigning to each lower semi-continuous map $f : X \to Y^+ \setminus \{0\}$, an upper semi-continuous map $\Phi(f) : X \to Y^+ \setminus \{0\}$, and a lower semi-continuous map $\Psi(f) : X \to Y^+ \setminus \{0\}$ with $\Psi(f) \le \Phi(f) \le f$ such that $\Phi(f) \le \Phi(f')$ and $\Psi(f) \le \Psi(f')$ whenever $f \le f'$.

Question 1.6. Let X be a topological space and Y an ordered topological vector space with a positive interior point of Y^+. Are the following conditions equivalent?

(1) X is monotonically countably metacompact.

(2) There exists an operator Φ assigning to each lower semi-continuous map $f : X \to Y^+ \setminus \{\mathbf{0}\}$, an upper semi-continuous map $\Phi(f) : X \to Y^+ \setminus \{\mathbf{0}\}$ with $\Phi(f) \le f$ such that $\Phi(f) \le \Phi(f')$ whenever $f \le f'$.

Question 1.7. Let X be a topological space and Y an ordered topological vector space with a positive interior point of Y^+. Are the following conditions equivalent?

(1) X is a monotone cb-space.

(2) There exists an operator Φ assigning to each locally upper bounded map $f : X \to Y$, a continuous map $\Phi(f) : X \to Y$ with $f \le \Phi(f)$ such that $\Phi(f) \le \Phi(f')$ whenever $f \le f'$.

The main purpose of this paper is to give some partial answers to the above questions.

2. Partial Answers

In this section, we shall give partial answers to questions 1.5–1.7.

It is known that if X is a topological space, Y is a topological vector space, and f is a continuous function on X, then for each $y \in Y$, the map

$g : X \to Y$ defined by letting $g(x) = f(x)y$ for each $x \in X$ is continuous. As for semi-continuity, we have the following.

Proposition 2.1. *Let X be a topological space, Y an ordered topological vector space, and f a function on X. If f is lower (upper) semi-continuous, then for each $y \in Y^+$, the map $g : X \to Y$ defined by letting $g(x) = f(x)y$ for each $x \in X$ is lower (upper) semi-continuous.*

Proof. Let $x_0 \in X$ and let V be a $\mathbf{0}$-neighborhood. Then there exists $\delta > 0$ such that if $|\lambda| \leq \delta$, then $\lambda y \in V$. Put $O_{x_0} = \{x \in X : f(x) > f(x_0) - \delta\}$. Since f is lower semi-continuous, O_{x_0} is an open neighborhood of x_0. For each $x \in O_{x_0}$, $g(x) = f(x)y \geq f(x_0)y - \delta y$ which implies that $g(x) \in g(x_0) - \delta y + Y^+ \subset g(x_0) + V + Y^+$. By Lemma 1.1, g is lower semi-continuous. □

The following theorem gives a partial answer to Question 1.5.

Theorem 2.2. *Let X be a topological space and Y an ordered topological vector space with a positive interior point e of Y^+. If there exist operators Φ and Ψ assigning to each lower semi-continuous map $f : X \to Y^+ \setminus \{\mathbf{0}\}$, an upper semi-continuous map $\Phi(f) : X \to Y^+ \setminus \{\mathbf{0}\}$ and a lower semi-continuous map $\Psi(f) : X \to Y^+ \setminus \{\mathbf{0}\}$ with $\Psi(f) \leq \Phi(f) \leq f$ such that $\Phi(f) \leq \Phi(f')$ and $\Psi(f) \leq \Psi(f')$ whenever $f \leq f'$, then X is monotonically countably paracompact.*

Proof. Assume the condition and let $\langle F_j \rangle$ be a decreasing sequence of closed subsets of X with empty intersection. For each $x \in X$, let

$$f_{\langle F_j \rangle}(x) = (1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{F_n}(x))e.$$

By Proposition 2.1, $f_{\langle F_j \rangle} : X \to Y^+ \setminus \{\mathbf{0}\}$ is lower semi-continuous. For each $n \in \mathbb{N}$, let

$$O(n, \langle F_j \rangle) = int(\Phi(f_{\langle F_j \rangle})^{-1}(\frac{1}{2^{n-1}} e - Y^+))$$

and

$$E(n, \langle F_j \rangle) = \Psi(f_{\langle F_j \rangle})^{-1}(\frac{1}{2^{n-1}} e - Y^+).$$

Then $\{O(n, \langle F_j \rangle) : n \in \mathbb{N}\}$ is a decreasing sequence of open subsets of X. Since $\Psi(f_{\langle F_j \rangle})$ is lower semi-continuous, by Lemma 1.1, $E(n, \langle F_j \rangle)$ is closed. From $\Psi(f_{\langle F_j \rangle}) \leq \Phi(f_{\langle F_j \rangle})$, it follows that $O(n, \langle F_j \rangle) \subset E(n, \langle F_j \rangle)$, and thus $\overline{O(n, \langle F_j \rangle)} \subset E(n, \langle F_j \rangle)$. Since $\Psi(f_{\langle F_j \rangle})(x) > \mathbf{0}$ for each $x \in X$, we have that $\bigcap_{n \in \mathbb{N}} E(n, \langle F_j \rangle) = \emptyset$. Indeed, if $x \in \bigcap_{n \in \mathbb{N}} E(n, \langle F_j \rangle)$ for some $x \in X$, then $\Psi(f_{\langle F_j \rangle})(x) \leq \frac{1}{2^{n-1}} e$ for each $n \in \mathbb{N}$. Since Y^+ is

Archimedean, it follows that $\Psi(f_{\langle F_j\rangle})(x) \leq \mathbf{0}$, a contradiction. Hence, $\bigcap_{n\in\mathbb{N}} \overline{O(n,\langle F_j\rangle)} = \emptyset$.

It is clear that if $\langle F_j\rangle \preceq \langle H_j\rangle$, then $O(n,\langle F_j\rangle) \subset O(n,\langle H_j\rangle)$ for each $n \in \mathbb{N}$.

For a fixed $n \in \mathbb{N}$, let $x \in F_n$ and $k = \max\{n \in \mathbb{N} : x \in F_n\}$. Then $n \leq k$ and so

$$\Phi(f_{\langle F_j\rangle})(x) \leq f_{\langle F_j\rangle}(x) = (1 - \sum_{n=1}^{k} \frac{1}{2^n})e = \frac{1}{2^k}e \leq \frac{1}{2^n}e.$$

Since $\Phi(f_{\langle F_j\rangle})$ is upper semi-continuous and $\frac{1}{2^n}e - Y^+$ is a $\mathbf{0}$-neighborhood, there exists an open neighborhood O_x of x such that

$$\Phi(f_{\langle F_j\rangle})(x') \in \Phi(f_{\langle F_j\rangle})(x) + \frac{1}{2^n}e - Y^+ - Y^+ \subset \frac{1}{2^{n-1}}e - Y^+$$

for all $x' \in O_x$. This implies that $x \in int(\Phi(f)^{-1}(\frac{1}{2^{n-1}}e - Y^+))$ and so $F_n \subset O(n,\langle F_j\rangle)$. Consequently, X is monotonically countably paracompact. $\square$

We don't know whether the condition in the above theorem is necessary. However, if we require additionally that each point of $Y^+ \setminus \{\mathbf{0}\}$ be an order unit, then the converse of the above theorem is also true. We sketch the proof as follows.

Proof. Let $f : X \to Y^+ \setminus \{\mathbf{0}\}$ be a lower semi-continuous map. For each $j \in \mathbb{N}$, let $F_j(f) = X \setminus int(f^{-1}(\frac{1}{2^{j-1}}e + Y^+))$. Then $\langle F_j(f)\rangle$ is a decreasing sequence of closed subsets of X. Let $x \in X$, since $f(x)$ is an order unit, there exists $n > 2$ such that $f(x) \geq \frac{1}{2^{n-2}}e$. Since f is lower semi-continuous and $-\frac{1}{2^{n-1}}e + Y^+$ is a $\mathbf{0}$-neighborhood, there exists an open neighborhood O_x of x such that $f(x') \in f(x) - \frac{1}{2^{n-1}}e + Y^+ + Y^+ \subset \frac{1}{2^{n-1}}e + Y^+$ for all $x' \in O_x$. Thus, $x \in int(f^{-1}(\frac{1}{2^{n-1}}e + Y^+))$ which implies that $x \notin F_n(f)$ and so $\bigcap_{n\in\mathbb{N}} F_n(f) = \emptyset$.

Let O be the operator in Definition 1.3. For each $x \in X$, let

$$\Phi(f)(x) = (1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\chi_{O(n,\langle F_j(f)\rangle)}(x))e$$

and

$$\Psi(f)(x) = (1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\chi_{\overline{O(n,\langle F_j(f)\rangle)}}(x))e.$$

Then $\Phi(f) : X \to Y^+ \setminus \{\mathbf{0}\}$ is upper semi-continuous and $\Psi(f) : X \to Y^+ \setminus \{\mathbf{0}\}$ is lower semi-continuous and $\Psi(f) \leq \Phi(f)$.

Let $x \in X$. Since $\bigcap_{n\in\mathbb{N}} O(n, \langle F_j(f)\rangle) = \emptyset$, then $x \notin O\ (n, \langle F_j(f)\rangle)$ for some $n \in \mathbb{N}$. Let $k = \min\{n \in \mathbb{N} : x \notin O(n, \langle F_j(f)\rangle)\}$; then $x \notin O(k, \langle F_j(f)\rangle) \supset F_k(f)$, from which it follows that $f(x) \geq \frac{1}{2^{k-1}}e$. Thus,

$$\Phi(f)(x) = (1 - \sum_{n=1}^{k-1} \frac{1}{2^n})e = \frac{1}{2^{k-1}}e \leq f(x). \qquad \square$$

Remark 2.3. The referee pointed out that the condition that each point of $Y^+ \setminus \{\mathbf{0}\}$ be an order unit may be too strong and asked whether there exists an ordered topological vector space with positive interior points, each point of which is an order unit, except spaces which are isomorphic to $\mathbb{R}$ (as ordered topological vector spaces). We have no idea on it.

As for monotonically countably metacompact spaces, the following theorem gives a partial answer to Question 1.6.

Theorem 2.4. *Let X be a topological space and Y an ordered topological vector space with a positive interior point e of Y^+. Then condition* (1) *implies condition* (2).

(1) *There exists an operator Φ assigning to each lower semi-continuous map $f : X \to Y^+ \setminus \{\mathbf{0}\}$, an upper semi-continuous map $\Phi(f) : X \to Y^+ \setminus \{\mathbf{0}\}$ with $\Phi(f) \leq f$ and $\Phi(f) \leq \Phi(f')$ whenever $f \leq f'$.*

(2) *X is monotonically countably metacompact.*

If, in addition, each point of $Y^+ \setminus \{\mathbf{0}\}$ is an order unit, then (2) *implies* (1).

The following theorem partially answers Question 1.7. The proof of the following theorem was suggested by the referee who thought the original proof was too complicated.

Theorem 2.5. *Let X be a topological space and Y an ordered topological vector space with a positive interior point of Y^+. If X is a monotone cb-space, then there exists an operator Ψ assigning to each locally upper bounded map $f : X \to Y$, a continuous map $\Psi(f) : X \to Y$ with $f \leq \Psi(f)$ such that if $f \leq f'$, then $\Psi(f) \leq \Psi(f')$.*

Proof. For each locally upper bounded map $f : X \to Y$ and $x \in X$, let $n(f)(x) = \min\{n \in \mathbb{N} : x \in int(f^{-1}(ne - Y^+))$. By Lemma 1.2, for each $x \in X$, there exist an open neighborhood O_x of x and $n \in \mathbb{N}$ such that $f(O_x) \subset ne - Y^+$. Thus, for each $x' \in O_x$, we have $x' \in int(f^{-1}(ne - Y^+))$ from which it follows that $n(f)(x') \leq n$. Thus, the real valued function $n(f) : X \to \mathbb{R}^+$ is locally upper bounded. If $f \leq f'$, then for each $x \in X$, $x \in int((f')^{-1}(n(f')(x)e - Y^+)) \subset int(f^{-1}(n(f')(x)e - Y^+))$ from which it follows that $n(f)(x) \leq n(f')(x)$. Thus, $n(f) \leq n(f')$. Since $n(f) > 0$, it follows that $n(f)$ is locally bounded and $n(f) = |n(f)|$.

Let Φ be the operator in Definition 1.4 and let $\Psi(f)(x) = \Phi(n(f))(x) \cdot e$ for each $x \in X$. Then the map $\Psi(f) : X \to Y$ is continuous. It is easy to see that $\Psi(f) \leq \Psi(f')$ whenever $f \leq f'$. For each $x \in X$, since $x \in int(f^{-1}(n(f)(x)e - Y^+))$, we have that $f(x) \leq n(f)(x)e \leq \Phi(n(f))(x)e = \Psi(f)(x)$. Thus, $f \leq \Psi(f)$. □

We don't know whether the condition in the above theorem is sufficient.

Acknowledgment. The author would like to thank the referee for the valuable suggestions which have greatly improved the original manuscript.

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