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by

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CONTINUOUS INJECTIONS BETWEEN THE PRODUCTS OF TWO CONNECTED NOWHERE REAL LINEARLY ORDERED SPACES

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Abstract. We shall show that if K_0, K_1, L_0, and L_1 are nowhere real connected linearly ordered topological spaces and $f : K_0 \times K_1 \to L_0 \times L_1$ is a continuous injective function, then f is coordinate-wise.

1. Introduction

Let $f : X_0 \times X_1 \to Y_0 \times Y_1$ be a function. We say that f is *coordinate-wise* if and only if there exist $i < 2$, $g_0 : X_i \to Y_0$, and $g_1 : X_{1-i} \to Y_1$ such that for every $\langle x_0, x_1 \rangle \in X_0 \times X_1$, $f(x_0, x_1) = \langle g_0(x_i), g_1(x_{1-i}) \rangle$.

Many homeomorphisms from $\mathbb{R}^2$ onto $\mathbb{R}^2$ are not coordinate-wise. For example, $f(x, y) = \langle x - y, x + y \rangle$.

However, K. Eda and R. Kamijo proved the following theorem that this is not necessarily the case when we replace $\mathbb{R}$ by other connected linearly ordered spaces.

Theorem 1.1 (Eda and Kamijo [1]). *Let K be a connected linearly ordered space such that, for a dense set of $x \in K$, either $\mathrm{cf}(x)$ or $\mathrm{ci}(x)$ is uncountable. Here, $\mathrm{cf}(x)$ denotes the cofinality of x and $\mathrm{ci}(x)$ the coinitiality of x. Then for every $n < \omega$, every homeomorphism $f : K^n \to K^n$ is coordinate-wise.*

Eda and Kamijo asked if it can be extended to, for example, the cut-completion of an Aronszajn line. In this article, we shall prove the following theorem that answers this question positively with some other improvements for the case $n = 2$.

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Definition 1.2. A linearly ordered space L is *nowhere real* if and only if it is uncountable, but no uncountable convex set is separable.

Fact 1.3. For every linearly ordered set K, K is nowhere real if and only if the closure of any countable subset of K is nowhere dense.

Theorem 1.4. *Let K_0, K_1, L_0, and L_1 be connected nowhere real linearly ordered spaces. Then every continuous injection $f : K_0 \times K_1 \to L_0 \times L_1$ is coordinate-wise.*

Thus, these four connected linearly ordered spaces may or may not be different, and the function only needs to be a continuous injection, instead of a homeomorphism.

The proof is done by a set-theoretic argument using countable elementary submodels. It is quite different from the argument of Eda and Kamijo, which is much more topological.

We also show this theorem can be extended to the product of any finite number of connected nowhere real linearly ordered spaces. An article about this result is now in preparation.

2. Connected Linearly Ordered Spaces

In this section, we shall state known basic facts on connected linearly ordered spaces. We shall use the standard interval notation for a linearly ordered set K, such as for $a, b \in K$ with $a \leq b$,

$$(a, b) = \{\, x \in K : a < x < b \,\}$$
$$[a, b] = \{\, x \in K : a \leq x \leq b \,\}.$$

Lemma 2.1. *Let K be a connected linearly ordered space. For every non-empty subset A of K, if A has an upper bound in K, then A has the least upper bound in K.*

Proof. Let $A \subseteq K$ be a non-empty set that has an upper bound. Suppose that A does not have the least upper bound. Let U be the set of all upper bounds of A. Then it is easy to see that U is clopen, and hence K is not connected, which is a contradiction. □

By using the same argument as the proof to show every bounded subset of $\mathbb{R}$ is compact, we can show the following lemma.

Lemma 2.2. *Every bounded closed subset of a connected linearly ordered space is compact.*

By using this lemma, we can show the following.

Lemma 2.3. *Let K and L be connected linearly ordered spaces and $g : K \to L$ a continuous function. Let $a, b \in K$ with $a < b$. Then there exist maximum and minimum values of g on $[a, b]$.*

We can also see the intermediate value theorem for continuous functions from one connected linearly ordered space to another.

Lemma 2.4. *Let K and L be connected linearly ordered spaces, and let $g : K \to L$ be a continuous function. Let $a, b \in K$ with $a < b$. If $z \in L$ is between $g(a)$ and $g(b)$, then there exists $c \in (a, b)$ such that $g(c) = z$.*

Proof. Without loss of generality, we may assume $g(a) < g(b)$. Let $c = \sup\{y \in K : a \le y \wedge \forall x \in (a, y)(g(x) < z)\}$. Clearly, we have $a < c < b$. We shall show that $g(c) = z$.

Claim. For every $x \in (a, c)$, $g(x) < z$.

$\vdash$ Let $x \in (a, c)$. By the definition of c, there exists a $y \in (x, c)$ such that, for every $x' \in (a, y)$, $g(x') < z$. Since $x \in (a, y)$, we have $g(x) < z$. $\dashv$ (Claim)

By the claim, it is easy to see $g(c) \le z$. Hence, it suffices to show $g(c) \ge z$. Suppose not, i.e., $g(c) < z$. Then there exists $b' > c$ such that, for every $x \in (c, b')$, $g(x) < z$. Then for every $x \in (a, b')$, $g(x) < z$. Thus, $c \ge b'$, which is a contradiction. $\square$

3. Continuous Functions from One Linearly Ordered Space to Another

Throughout this section, let K and L be any connected linearly ordered sets, let $g : K \to L$ be a continuous function, and let M be a countable elementary submodel of $H(\theta)$ with $K, L, f \in M$ for some regular cardinal θ with $\mathcal{P}(\mathcal{P}(K \cup L)) \in H(\theta)$. Without loss of generality, we may assume K and L are disjoint.

Definition 3.1. Let $J(K, M)$ be the set of all $x \in \operatorname{int}(K \setminus M)$ such that $\inf(K \cap M) \le x \le \sup(K \cap M)$.

For every $x \in K$, let

$$\begin{aligned} \eta(K, M, x) &= \sup\{\, y \in \operatorname{cl}(K \cap M) : y \le x \,\}, \\ \zeta(K, M, x) &= \inf\{\, y \in \operatorname{cl}(K \cap M) : y \ge x \,\}, \\ I(K, M, x) &= [\eta(K, M, x), \zeta(K, M, x)], \\ I_{\mathrm{int}}(K, M, x) &= (\eta(K, M, x), \zeta(K, M, x)), \\ C(K, M, x) &= \{\, \eta(K, M, x), \zeta(K, M, x) \,\} \end{aligned}$$

if they exist. Note that if $x \in J(K, M)$, then all of them exist. When it is clear from the context, we omit K and M.

Remark 3.2. We can easily show the following facts.

(1) $I(x)$ is a convex set.
(2) If $x \in \operatorname{cl}(K \cap M)$, then $\eta(x) = \zeta(x) = x$ and $I(x) = \{\, x \,\}$.

(3) If $x \notin \mathrm{cl}(K \cap M)$, then $\eta(x)$ is either an element of M or a limit point of $K \cap M$ from below (i.e., for every $x' < x$, there exists a $y \in K \cap M$ such that $x' < y < x$).
(4) If $x \notin \mathrm{cl}(K \cap M)$, then $\zeta(x)$ is either an element of M or a limit point of $K \cap M$ from above.

Lemma 3.3. *Let $x \in J(K, M)$. Then either $\eta(x) \notin M$ or $\zeta(x) \notin M$.*

Proof. Suppose not, i.e., $\eta(x) \in M$ and $\zeta(x) \in M$. By the elementarity of M, since K is connected, there exists an $x' \in K \cap M$ such that $\eta(x) < x' < \zeta(x)$. This is a contradiction by the definition of $\eta(x)$ and $\zeta(x)$. □

We can prove the following lemma, which means that the maximum and minimum values of g on $I(\hat{x})$ are attained at endpoints of $I(K, M, \hat{x})$.

Lemma 3.4. *Let $\hat{x} \in J(K, M)$. Then*

$$\max g^{\rightarrow}(I(\hat{x})) = \max g^{\rightarrow}(C(\hat{x}))$$
$$\min g^{\rightarrow}(I(\hat{x})) = \min g^{\rightarrow}(C(\hat{x})).$$

In particular, if $g(\eta(\hat{x})) = g(\zeta(\hat{x}))$, then g is constant on $I(\hat{x})$.

Proof. We shall prove it for max as the same proof works for min. Let $v = \max g^{\rightarrow}(I(\hat{x}))$. Suppose $v \neq \max g^{\rightarrow}(C(X))$.

If $\eta(x) \in M$, let $a = \eta(x)$. Otherwise, $\eta(x)$ is a limit point of $K \cap M$ from below. So, there exists an $a \in K \cap M$ such that $a < \eta(x)$ and for every $x \in [a, \eta(x)]$, $g(x) < v$. Similarly, if $\zeta(x) \in M$, let $b = \zeta(x)$. Otherwise, let $b \in K \cap M$ such that $b > \zeta(x)$ and for every $x \in [\zeta(x), b]$, $g(x) < v$. Then we have $a, b \in K \cap M$ and for every $x \in [a, b]$, if $g(x) = v$, then $\eta(x) < x < \zeta(x)$.

Note that $v = \max g^{\rightarrow}([a, b])$. Since $a, b, g \in M$, we have $v \in M$. Since there exists an $x \in [a, b]$ such that $g(x) = v$, by the elementarity of M, there exists such an $x \in M$. By the previous paragraph, we have $\eta(x) < x < \zeta(x)$. By the definition of $\eta(x)$ and $\zeta(x)$, this implies $x \notin M$. This is a contradiction. □

Lemma 3.5. *Let $\hat{x} \in J(K, M)$ with $g(\hat{x}) \in M$. Then g is constant on $I(\hat{x})$.*

Proof. Let $v = g(\hat{x})$. By Lemma 3.4, it suffices to show that $g(\eta(\hat{x})) = g(\zeta(\hat{x})) = v$.

We shall show $g(\eta(\hat{x})) = v$. Suppose not. If $\eta(\hat{x}) \in M$, let $a = \eta(\hat{x})$. Otherwise, let $a \in K \cap M$ such that $a < \eta(\hat{x})$ and for every $y \in [a, \eta(\hat{x})]$, $g(y) \neq v$. Let $x = \min\{\, y \in K : y \geq a \text{ and } g(y) = v \,\}$. Then clearly $a \leq x \leq \hat{x}$ and $g(x) = v$. Since $a \in M$, by elementarity, $x \in M$. Hence, $x \leq \eta(\hat{x})$. But the definition of a implies $g(x) \neq v$. This is a contradiction.

Similarly, we can show $g(\zeta(\hat{x})) = v$, and hence g is constant on $I(\hat{x})$. □

Lemma 3.6. *Let $\hat{x} \in J(K, M)$. If $\eta(\hat{x}) \notin M$ and $g(\eta(\hat{x})) \in M$, then g is constant on $I(\hat{x})$. Similarly, if $\zeta(\hat{x}) \notin M$ and $g(\zeta(\hat{x})) \in M$, then g is constant on $I(\hat{x})$.*

Proof. Let $v = g(\eta(\hat{x}))$. Suppose that g is not constant on $I(\hat{x})$. Then $g(\zeta(\hat{x})) \neq g(\eta(\hat{x})) = v$. If $\zeta(\hat{x}) \in M$, then let $b = \zeta(\hat{x})$. Otherwise, let $b \in K \cap M$ such that $b \geq \zeta(\hat{x})$ and for every $x \in [\zeta(\hat{x}), b]$, $g(x) \neq v$. Let x_0 be the maximum of all $x \in K$ such that $x \leq b$ and $g(x) = v$. Then clearly we have $x_0 \leq b$ and $g(x_0) = v$. By the definition of b, we have $x_0 < \zeta(\hat{x})$. Since $g(\eta(\hat{x})) = v$, we have $x_0 \geq \eta(\hat{x})$. But by elementarity, we have $x_0 \in M$. Since $\eta(\hat{x}) \notin M$, we have $x_0 > \eta(\hat{x})$. This is a contradiction since $(\eta(\hat{x}), \zeta(\hat{x})) \cap M = \emptyset$. □

Lemma 3.7. *Let $\hat{x} \in J(K, M)$ be so that $g(\hat{x}) \in J(L, M)$. Then*

$$\{\, g(\eta(\hat{x})), g(\zeta(\hat{x}))\,\} = C(g(\hat{x})).$$

Proof. We first prove the following claim.

Claim. For every $x \in (\eta(\hat{x}), \zeta(\hat{x}))$, $g(x) \notin M$.

$\vdash$ Suppose that there exists an $x \in (\eta(\hat{x}), \zeta(\hat{x}))$ such that $g(x) \in M$. By Lemma 3.5, g is constant on $I(x) = I(\hat{x})$. Hence, $g(\hat{x}) = g(x) \in M$. This is a contradiction to $g(\hat{x}) \in J(L, M)$. $\dashv$ (Claim)

By the claim and Lemma 2.4, we have $g^{\rightarrow}(I(\hat{x})) \subseteq I(g(\hat{x}))$. In particular, $g(\eta(\hat{x})) \in I(g(\hat{x}))$. Since $\eta(\hat{x}) \in \mathrm{cl}(K \cap M)$, we have $g(\eta(\hat{x})) \in \mathrm{cl}(L \cap M)$. Hence, $g(\eta(\hat{x})) \in \mathrm{cl}(L \cap M) \cap I(g(\hat{x})) = C(g(\hat{x}))$. Similarly, we can show $g(\zeta(\hat{x})) \in C(g(\hat{x}))$.

If $g(\eta(\hat{x})) = g(\zeta(\hat{x}))$, then by Lemma 3.4, g is constant on $I(\hat{x})$. Then $g(\hat{x}) = g(\eta(\hat{x})) \in \mathrm{cl}(L \cap M)$, which is a contradiction to the assumption $g(\hat{x}) \in J(L, M)$. Thus, $g(\eta(\hat{x})) \neq g(\zeta(\hat{x}))$. Since $|C(g(\hat{x}))| = 2$, this clearly implies $\{\, g(\eta(\hat{x})), g(\zeta(\hat{x}))\,\} = C(g(\hat{x}))$. □

4. $f : K_0 \times K_1 \to L$

Throughout this section, we assume that K_0, K_1, and L are nowhere real connected linear orders, $f : K_0 \times K_1 \to L$ is a continuous function, and M is a countable elementary submodel of $H(\theta)$ with $K_0, K_1, L \in M$ for a regular cardinal θ with $\mathcal{P}(\mathcal{P}(K_0 \cup K_1 \cup L)) \in H(\theta)$.

Lemma 4.1. *Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in \mathrm{cl}(K_1 \cap M)$. Then*

$$\max f^{\rightarrow}(I(\hat{x}) \times \{\,\hat{y}\,\}) = \max\{\, f(\eta(\hat{x}), \hat{y}), f(\zeta(\hat{x}), \hat{y})\,\}$$
$$\min f^{\rightarrow}(I(\hat{x}) \times \{\,\hat{y}\,\}) = \min\{\, f(\eta(\hat{x}), \hat{y}), f(\zeta(\hat{x}), \hat{y})\,\}.$$

In particular, $x \mapsto f(x, \hat{y})$ is constant on $I(\hat{x})$ if and only if $f(\eta(\hat{x}), \hat{y}) = f(\zeta(\hat{x}), \hat{y})$.

Proof. If $\hat{y} \in M$, then $x \mapsto f(x, \hat{y})$ is a function lying in M. So Lemma 3.4 implies the conclusion.

Suppose $\hat{y}$ is a limit point of $K_1 \cap M$. Let $v = \max f^{\rightarrow}(I(\hat{x}) \times \{ \hat{y} \})$. Suppose $f(\eta(\hat{x}), \hat{y}) < v$ and $f(\zeta(\hat{x}), \hat{y}) < v$. Then there exists $x_0 \in (\eta(\hat{x}), \zeta(\hat{x}))$ such that $f(x_0, \hat{y}) = v$. Then there exist $v_0 \in L$ such that $\max \{ f(\eta(\hat{x}), \hat{y}), f(\zeta(\hat{x}), \hat{y}) \} < v_0 < v$. Then there exists an open neighborhood U of $\hat{y}$ such that for every $y \in U$, $f(\eta(\hat{x}), y) < v_0$, $f(\zeta(\hat{x}), y) < v_0$, and $f(x_0, y) > v_0$. Let $y \in U \cap M$. Then by Lemma 3.4, since the map $x \mapsto f(x, y)$ belongs to M, $v_0 < f(x_0, y) \leq \max \{ f(\eta(\hat{x}), y), f(\zeta(\hat{x}), y) \} < v_0$. This is a contradiction. □

Lemma 4.2. *Let $\hat{x} \in J(K_0, M)$ and let $\hat{y}$ be a limit point of $K_1 \cap M$ from above. Suppose that $f(\hat{x}, \hat{y}) \in J(L, M)$. Then there exists a $c \in K_1$ such that $c > \hat{y}$ and for every $y \in [\hat{y}, c]$, $f(\eta(\hat{x}), y) = f(\eta(\hat{x}), \hat{y})$ and $f(\zeta(\hat{x}), y) = f(\zeta(\hat{x}), \hat{y})$.*

Proof. Since $|C(f(\hat{x}, \hat{y}))| \leq 2$ and $f(\hat{x}, \hat{y}) \in J(L, M)$, there exists a $c \in K_1 \cap M$ such that $c > \hat{y}$ such that

$$|f^{\rightarrow}(\{ \eta(\hat{x}) \} \times [\hat{y}, c]) \cap C(f(\hat{x}, \hat{y}))| \leq 1$$
$$|f^{\rightarrow}(\{ \zeta(\hat{x}) \} \times [\hat{y}, c]) \cap C(f(\hat{x}, \hat{y}))| \leq 1$$
$$f^{\rightarrow}(\{ \hat{x} \} \times [\hat{y}, c]) \subseteq \operatorname{int} I(f(\hat{x}, \hat{y})).$$

Then there exist $v, w \in C(f(\hat{x}, \hat{y}))$ such that for every $y \in [\hat{y}, c]$, if $f(\eta(\hat{x}), y) \in C(f(\hat{x}, \hat{y}))$, then $f(\eta(\hat{x}), y) = v$, and if $f(\zeta(\hat{x}), y) \in C(f(\hat{x}, \hat{y}))$, then $f(\zeta(\hat{x}), y) = w$.

Claim 1. For every $y \in [\hat{y}, c] \cap M$, $\{ f(\eta(\hat{x}), y), f(\zeta(\hat{x}), y) \} = C(f(\hat{x}, \hat{y}))$.

$\vdash$ Let $y \in [\hat{y}, c] \cap M$. Then we have $f(\hat{x}, y) \in \operatorname{int} I(f(\hat{x}, \hat{y})) \subseteq J(L, M)$. Since $y \in M$, the map $x \mapsto f(x, y)$ belongs to M. By Lemma 3.7, we have $\{ f(\eta(\hat{x}), y), f(\zeta(\hat{x}), y) \} = C(f(\hat{x}, y))$. Since $f(\hat{x}, y) \in \operatorname{int}(I(f(\hat{x}, \hat{y})))$, we have $C(f(\hat{x}, y)) = C(f(\hat{x}, \hat{y}))$. Thus,

$$\{ f(\eta(\hat{x}), y), f(\zeta(\hat{x}), y) \} = C(f(\hat{x}, \hat{y})). \qquad \dashv \text{(Claim 1)}$$

Claim 2. For every $y \in [\hat{y}, c] \cap M$, $f(\eta(\hat{x}), y) = v$ and $f(\zeta(\hat{x}), y) = w$.

$\vdash$ By Claim 1, both $f(\eta(\hat{x}), y)$ and $f(\zeta(\hat{x}), y)$ belong to $C(f(\hat{x}, \hat{y}))$. By the definition of v and w, we have $f(\eta(\hat{x}), y) = v$ and $f(\zeta(\hat{x}), y) = w$. $\dashv$ (Claim 2)

Claim 3. For every $y \in [\hat{y}, c] \cap \operatorname{cl}(K_1 \cap M)$, $f(\eta(\hat{x}), y) = v$ and $f(\zeta(\hat{x}), y) = w$.

$\vdash$ If $y \in M$, then it is clear from Claim 2. Otherwise, y is a limit point of $K_1 \cap M$. By Claim 2, for every $y_1 \in K_1 \cap M$, $f(\eta(\hat{x}), y_1) = v$ and $f(\zeta(\hat{x}), y_1) = w$. Since y is a limit point of $K_1 \cap M$, we have $f(\eta(\hat{x}), y) = v$ and $f(\zeta(\hat{x}), y) = w$. $\dashv$ (Claim 3)

In particular, since $\hat{y}$ is a limit point of $K_1 \cap M$, $f(\eta(\hat{x}), \hat{y}) = v$ and $f(\zeta(\hat{x}), \hat{y}) = w$.

Now we shall show that for every $y \in [\hat{y}, c]$, $f(\eta(\hat{x}), y) = v$ and $f(\zeta(\hat{x}), y) = w$. If $y \in \mathrm{cl}(K_1 \cap M)$, then this is clear from Claim 3. Suppose not. It is easy to see $y \in J(K_1, M)$. Since $\eta(y), \zeta(y) \in \mathrm{cl}(K_1 \cap M)$, we have $f(\eta(\hat{x}), \eta(y)) = f(\eta(\hat{x}), \zeta(y)) = v$ and $f(\zeta(\hat{x}), \eta(y)) = f(\zeta(\hat{x}), \zeta(y)) = w$. By Lemma 4.1, we have $f(\eta(\hat{x}), y) = v$ and $f(\zeta(\hat{x}), y) = w$. □

Lemma 4.3. *Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in \mathrm{cl}(K_1 \cap M)$. Suppose that $f(\hat{x}, \hat{y}) \in J(L, M)$. Then*

$$\{ f(\eta(\hat{x}), \hat{y}), f(\zeta(\hat{x}), \hat{y}) \} = C(f(\hat{x}, \hat{y})).$$

Proof. First, suppose $\hat{y} \in M$. Then the map $x \mapsto f(x, \hat{y})$ is a function from K_0 into L lying in M. The conclusion follows from Lemma 3.7.

Now suppose $\hat{y}$ is a limit point of $K_1 \cap M$. Without loss of generality, we may assume $\hat{y}$ is a limit point of $K_1 \cap M$ from above. By Lemma 4.2, there exists a $c > \hat{y}$ such that for every $y \in [\hat{y}, c]$, $f(\eta(\hat{x}), y) = f(\eta(\hat{x}), \hat{y})$ and $f(\zeta(\hat{x}), y) = f(\zeta(\hat{x}), \hat{y})$. Let $y \in [\hat{y}, c] \cap M$ such that $f(\hat{x}, y) \in \mathrm{int}(I(f(\hat{x}, \hat{y}))$. Then, since the map $x \mapsto f(x, y)$ belongs to M and $f(\hat{x}, y) \in J(L, M)$, by Lemma 3.7, we have $\{ f(\eta(\hat{x}), y), f(\zeta(\hat{x}), y \} = C(f(\hat{x}, y)) = C(f(\hat{x}, \hat{y}))$. Hence, $\{ f(\eta(\hat{x}), \hat{y}), f(\zeta(\hat{x}), \hat{y}) \} = C(f(\hat{x}, \hat{y}))$. □

Lemma 4.4. *Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in \mathrm{cl}(K_1 \cap M)$. If $f(\eta(\hat{x}), \hat{y}) \in M$ and $\eta(\hat{x}) \notin M$, then for every $x \in I(\hat{x})$, $f(x, \hat{y}) = f(\hat{x}, \hat{y})$.*

Proof. If $\hat{y} \in M$, then the map $x \mapsto f(x, \hat{y})$ belongs to M. So, by Lemma 3.6, for every $x \in I(\hat{x})$, $f(x, \hat{y}) = f(\hat{x}, \hat{y})$.

Now suppose $\hat{y} \notin M$. Since $\hat{y} \in \mathrm{cl}(K_1 \cap M)$, $\hat{y}$ is a limit point of $K_1 \cap M$. Without loss of generality, we may assume $\hat{y}$ is a limit point of $K_1 \cap M$ from above.

Suppose that there exists an $x \in I(\hat{x})$ such that $f(x, \hat{y}) \neq f(\hat{x}, \hat{y})$. Since L is nowhere real, there exists an $x_0 \in I(\hat{x})$ such that $f(x_0, \hat{y}) \in J(L, M)$. By Lemma 4.2, there exists a $c > \hat{y}$ such that for every $y \in [\hat{y}, c]$, $f(\eta(\hat{x}), y) = f(\eta(\hat{x}), \hat{y})$ and $f(\zeta(\hat{x}), y) = f(\zeta(\hat{x}), \hat{y})$. Let $y \in [\hat{y}, c] \cap M$. Then the map $x \mapsto f(x, y)$ belongs to M. Moreover, $\eta(\hat{x}) \notin M$ and $f(\eta(\hat{x}), y) \in M$. By Lemma 3.6, we have $f(\eta(\hat{x}), y) = f(\zeta(\hat{x}), y)$. Hence, $f(\eta(\hat{x}), \hat{y}) = f(\zeta(\hat{x}), \hat{y})$. Then Lemma 4.1 implies for every $x \in I(\hat{x})$, $f(x_0, \hat{y}) = f(\eta(\hat{x}), \hat{y}) \notin J(L, M)$. This is a contradiction. □

Lemma 4.5. *Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in J(K_1, M)$. Let U be an open subset of L such that $f^{\rightarrow}\partial(I(\hat{x}) \times I(\hat{y})) \subseteq U$. Then there exist $a, b \in K_0 \cap M$ and $c, d \in K_1 \cap M$ such that $a \leq \eta(\hat{x}) < \zeta(\hat{x}) \leq b$, $c \leq \eta(\hat{y}) <$*

$\zeta(\hat{y}) \leq d$, *and for every* $x \in (a,b)$ *and* $y \in (c,d)$, *if* $f(x,y) \notin U$, *then* $\langle x,y\rangle \in I_{int}(\hat{x}) \times I_{int}(\hat{y})$.

Note that for every $\hat{x} \in J(K_0,M)$ and $\hat{y} \in J(K_1,M)$, $\partial(I(\hat{x}) \times I(\hat{y}))$ is the set of all $\langle x,y\rangle \in I(\hat{x}) \times I(\hat{y})$ such that either $x \in C(\hat{x})$ or $y \in C(\hat{y})$.

Proof. For every $\langle x,y\rangle \in \partial(I(\hat{x}) \times I(\hat{y}))$, let $a_{\langle x,y\rangle}, b_{\langle x,y\rangle} \in K_0$ and let $c_{\langle x,y\rangle}, d_{\langle x,y\rangle} \in K_1$ be so that $a_{\langle x,y\rangle} < x < b_{\langle x,y\rangle}$, $c_{\langle x,y\rangle} < y < d_{\langle x,y\rangle}$, and $f^{\rightarrow}([a_{\langle x,y\rangle}, b_{\langle x,y\rangle}] \times [c_{\langle x,y\rangle}, d_{\langle x,y\rangle}]) \subseteq U$. Let $U_{\langle x,y\rangle} = (a_{\langle x,y\rangle}, b_{\langle x,y\rangle}) \times (c_{\langle x,y\rangle}, d_{\langle x,y\rangle})$. Since $\partial(I(\hat{x}) \times I(\hat{y}))$ is compact, there exists a finite subset $\{ \langle x_0,y_0\rangle, \dots, \langle x_{k-1},y_{k-1}\rangle \}$ of $\partial(I(\hat{x}) \times I(\hat{y}))$ such that $\partial(I(\hat{x}) \times I(\hat{y})) \subseteq \bigcup_{l<k} U_{\langle x_l,y_l\rangle}$. For each $l < k$, define $U'_l = U_{\langle x_l,y_l\rangle}$, $a'_l = a_{\langle x_l,y_l\rangle}$, $b'_l = b_{\langle x_l,y_l\rangle}$, $c'_l = c_{\langle x_l,y_l\rangle}$, and $d'_l = d_{\langle x_l,y_l\rangle}$.

Claim 1. None of $\{ a'_l \mid l < k \text{ and } a'_l < \eta(\hat{x}) \}$, $\{ b'_l \mid l < k \text{ and } b'_l > \zeta(\hat{x}) \}$, $\{ c'_l \mid l < k \text{ and } c'_l < \eta(\hat{y}) \}$, and $\{ d'_l \mid l < k \text{ and } d'_l > \zeta(\hat{y}) \}$ is empty.

$\vdash$ Since $\langle \eta(\hat{x}), \eta(\hat{y})\rangle \in \bigcup_{l<k} U'_l$, there exists an $l < k$ such that $\langle \eta(\hat{x}), \eta(\hat{y})\rangle \in U'_l$. Thus, $\{ a'_l \mid l < k \text{ and } a'_l < \eta(\hat{x}) \}$ is non-empty. Similarly, we can show that other three sets are non-empty. $\dashv$ (Claim 1)

Claim 2. There exist $a, b \in K_0 \cap M$ and $c, d \in K_1 \cap M$ such that

- $\max\{ a'_l \mid l < k \text{ and } a'_l < \eta(\hat{x}) \} < a \leq \eta(\hat{x})$,
- $\zeta(\hat{x}) \leq b < \max\{ b'_l \mid l < k \text{ and } b'_l > \zeta(\hat{x}) \}$,
- $\max\{ c'_l \mid l < k \text{ and } c'_l < \eta(\hat{y}) \} < c \leq \eta(\hat{y})$,
- $\zeta(\hat{y}) \leq d < \max\{ d'_l \mid l < k \text{ and } d'_l > \zeta(\hat{y}) \}$,
- if $\eta(\hat{x}) \in M$, then $a = \eta(\hat{x})$,
- if $\zeta(\hat{x}) \in M$, then $b = \zeta(\hat{x})$,
- if $\eta(\hat{y}) \in M$, then $c = \eta(\hat{y})$, and
- if $\zeta(\hat{y}) \in M$, then $d = \zeta(\hat{y})$.

$\vdash$ We shall only define a. Similar constructions work for b, c, and d. If $\eta(\hat{x}) \in M$, then let $a = \eta(\hat{x})$.

Suppose $\eta(\hat{x}) \notin M$. Then $\eta(\hat{x})$ is a limit point of $K_0 \cap M$ from below. By Claim 1, $\{ a'_l \mid l < k \text{ and } a'_l < \eta(\hat{x}) \}$ is non-empty. Clearly, $\max\{ a'_l \mid l < k \text{ and } a'_l < \eta(\hat{x}) \} < \eta(\hat{x})$. Since $\eta(\hat{x})$ is a limit point of $K_0 \cap M$ from below, there exists an $a \in K_0 \cap M$ such that

$$\max\{ a'_l \mid l < k \text{ and } a'_l < \eta(\hat{x}) \} < a \leq \eta(\hat{x}). \qquad \dashv \text{(Claim 2)}$$

Claim 3. $(a,b) \times (c,d) \subseteq (I_{\mathrm{int}}(\hat{x}) \times I_{\mathrm{int}}(\hat{y})) \cup \bigcup_{l<k} U'_l$.

$\vdash$ Let $\langle x,y\rangle \in [a,b] \times [c,d]$. Suppose $\langle x,y\rangle \notin I_{\mathrm{int}}(\hat{x}) \times I_{\mathrm{int}}(\hat{y})$. It suffices to show that there exists an $l < k$ such that $\langle x,y\rangle \in U'_l$.

We shall only prove the case in which $x \leq \eta(\hat{x})$ as other cases can be proved similarly.

Case 1. $y \leq \eta(\hat{x})$.

There exists an $l < k$ such that $\langle \eta(\hat{x}), \eta(\hat{y}) \rangle \in U'_l$. Then we have $a'_l < \eta(\hat{x}) < b'_l$ and $c'_l < \eta(\hat{y}) < d'_l$. So $a'_l \leq a$ and $c'_l \leq c$. Thus, we have $a'_l \leq a \leq x < \eta(\hat{x}) < b'_l$ and $c'_l \leq c \leq y < \eta(\hat{y}) < d'_l$. So $\langle x, y \rangle \in U'_l$.

Case 2. $\eta(\hat{y}) < y < \zeta(\hat{y})$.

There exists an $l < k$ such that $\langle \eta(\hat{x}), y \rangle \in U'_l$. Then we have $a'_l < \eta(\hat{x}) < b'_l$, and $c'_l < y < d'_l$. Thus, we have $a'_l \leq a < x \leq \eta(\hat{x}) < b'_l$. So $\langle x, y \rangle \in U'_l$.

Case 3. $y \geq \zeta(\hat{y})$.

$\vdash$ Similar to Case 1. $\dashv$ (Claim 3)

Now, we shall show that a, b, c, and d witness the conclusion. Let $x \in (a, b)$ and $y \in (c, d)$ be so that $f(x, y) \notin U$. By the definition of U'_l, $\langle x, y \rangle \notin \bigcup_{l<k} U'_l$. By Claim 3, we have $\langle x, y \rangle \in I_{\text{int}}(\hat{x}) \times I_{\text{int}}(\hat{y})$. □

Lemma 4.6. *Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in J(K_1, M)$. Then*

$$\max f^{\rightarrow}(I(\hat{x}) \times I(\hat{y})) = \max f^{\rightarrow}(C(\hat{x}) \times C(\hat{y}))$$
$$\min f^{\rightarrow}(I(\hat{x}) \times I(\hat{y})) = \min f^{\rightarrow}(C(\hat{x}) \times C(\hat{y})).$$

Proof. Let $v = \max f^{\rightarrow}(I(\hat{x}) \times I(\hat{y}))$.

CLAIM. $v \in f^{\rightarrow}(\partial(I(\hat{x}) \times I(\hat{y})))$

$\vdash$ Suppose not, i.e., $v \notin f^{\rightarrow}(\partial(I(\hat{x}) \times I(\hat{y})))$. Then by Lemma 4.5, there exist $a, b \in K_0 \cap M$ and $c, d \in K_1 \cap M$ with $a \leq \eta(\hat{x}) < \zeta(\hat{x}) \leq b$ and $c \leq \eta(\hat{y}) < \zeta(\hat{y}) \leq d$ such that for every $\langle x, y \rangle \in [a, b] \times [c, d]$, if $f(x, y) = v$, then $\langle x, y \rangle \in I_{\text{int}}(\hat{x}) \times I_{\text{int}}(\hat{y})$. Note that there exists $\langle x, y \rangle \in [a, b] \times [c, d]$ such that $f(x, y) = v$. By the elementarity of M, there exists such an $\langle x, y \rangle \in M$. By the definition of a, b, c and d, we have $\langle x, y \rangle \in I_{\text{int}}(\hat{x}) \times I_{\text{int}}(\hat{y})$. This is a contradiction to the definition of $I_{\text{int}}(\hat{x})$ and $I_{\text{int}}(\hat{y})$. $\dashv$ (Claim)

Let $\langle x_0, y_0 \rangle \in \partial(I(\hat{x}) \times I(\hat{y}))$ be so that $f(x_0, y_0) = v$. Then at least one of $x_0 = \eta(\hat{x})$, $x_0 = \zeta(\hat{x})$, $y_0 = \eta(\hat{y})$ and $y_0 = \zeta(\hat{y})$ holds. For example, suppose $y_0 = \eta(\hat{y})$. Then $v = \max f^{\rightarrow}(I(\hat{x}) \times \{\, \eta(\hat{y}) \,\})$. By Lemma 4.1, we have either $f(\eta(\hat{x}), \eta(\hat{y})) = v$ or $f(\zeta(\hat{x}), \eta(\hat{y})) = v$. By a similar argument, we can see that $v \in f^{\rightarrow}(C(\hat{x}) \times C(\hat{y}))$ in the other three cases, too. □

Lemma 4.7. *Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in J(K_1, M)$ be such that $f(\hat{x}, \hat{y}) \in M$. Then for every $x \in I(\hat{x})$ and $y \in I(\hat{y})$, $f(x, y) = f(\hat{x}, \hat{y})$.*

Proof. Let $v = f(\hat{x}, \hat{y})$.

Case 1. $\eta(\hat{x}), \eta(\hat{y}) \in M$.

Then $\zeta(\hat{x})$ is a limit point of $K_0 \cap M$ from above, and $\zeta(\hat{x})$ is a limit point of $K_1 \cap M$ from above.

CLAIM 1. $f(\zeta(\hat{x}), \zeta(\hat{y})) = v$.

$\vdash$ It suffices to show that for every $b \in K_0$ and $d \in K_1$, if $\zeta(\hat{x}) < b$ and $\zeta(\hat{y}) < d$, then there exist $x \in [\zeta(\hat{x}), b)$ and $y \in [\zeta(\hat{y}), d)$ such that $f(x, y) = v$. Let $b \in K_0$ and $d \in K_1$ with $\zeta(\hat{x}) < b$ and $\zeta(\hat{y}) < d$. Since $\zeta(\hat{x})$ is a limit point of $K_0 \cap M$ from above and $\zeta(\hat{y})$ is a limit point of $K_1 \cap M$ from above, there exist $b' \in (\zeta(\hat{x}), b) \cap M$ and $d' \in (\zeta(\hat{y}), d) \cap M$. Since '$\exists\, x \in (\eta(\hat{x}), b') \exists\, y \in (\eta(\hat{y}), d')\ (f(x, y) = v)$' holds in V, it also holds in M. So there exist $x \in (\eta(\hat{x}), b') \cap M$ and $y \in (\eta(\hat{y}), d') \cap M$ such $f(x, y) = v$. However, by the definition of $\zeta(\hat{x})$ and $\zeta(\hat{y})$, we have $\zeta(\hat{x}) \leq x$ and $\zeta(\hat{y}) \leq y$. $\dashv$ (Claim 1.)

By Lemma 4.4, since $\zeta(\hat{y}) \notin M$ and $f(\zeta(\hat{x}), \zeta(\hat{y})) = v \in M$, for every $y \in I(\hat{y})$, $f(\zeta(\hat{x}), y) = v$. In particular, $f(\zeta(\hat{x}), \eta(\hat{y})) = v$.

By a similar argument, we can show that for every $x \in I(\hat{x})$, $f(x, \zeta(\hat{y})) = v$. In particular, $f(\eta(\hat{x}), \zeta(\hat{y})) = v$.

Since $f(\zeta(\hat{x}), \eta(\hat{y})) = v$, by Lemma 4.4, $f(\eta(\hat{x}), \eta(\hat{y})) = v$. By Lemma 4.6, for every $x \in I(\hat{x})$ and $y \in I(\hat{y})$, $f(x, y) = v$.

Case 2. $\eta(\hat{x}) \in M$ and $\eta(\hat{y}), \zeta(\hat{y}) \notin M$.

CLAIM 2. For every $b \in K_0$ and $c, d \in K_1$ with $\zeta(\hat{x}) < b$ and $c < \eta(\hat{y}) < \zeta(\hat{y}) < d$, there exists $\langle x, y\rangle \in K_0 \times K_1$ such that $\zeta(\hat{x}) < x < b$, $y \in (c, \eta(\hat{y})) \cup (\zeta(\hat{y}), d)$, and $f(x, y) = v$.

$\vdash$ By Lemma 3.3, we have $\zeta(\hat{x}) \notin M$, and hence $\zeta(\hat{x})$ is a limit point of $K_0 \cap M$ from above. So, without loss of generality, we may assume $b \in M$. Moreover, since $\eta(\hat{y}), \zeta(\hat{y}) \notin M$, $\eta(\hat{y})$ is a limit point of $K_1 \cap M$ from below and $\zeta(\hat{y})$ is a limit point of $K_1 \cap M$ from above. Thus, without loss of generality, we may assume $c, d \in M$.

Since '$\exists\, \langle x, y\rangle \in K_0 \times K_1(\eta(\hat{x}) < x < b \wedge c < y < d \wedge f(x, y) = v)$' is true in V, there exist $\langle x, y\rangle \in (K_0 \times K_1) \cap M$ such that $\eta(\hat{x}) < x < b$, $c < y < d$, and $f(x, y) = v$. By the definition of $\eta(\hat{x})$ and $\zeta(\hat{x})$, since $x \in M$, we have $x \notin (\eta(\hat{x}), \zeta(\hat{x})]$, and hence $\zeta(\hat{x}) < x < b$. Similarly, by the definition of $\eta(\hat{y})$ and $\zeta(\hat{y})$, since $y \in M$, we have $y \notin [\eta(\hat{y}), \zeta(\hat{y})]$; hence, $y \in (c, \eta(\hat{y})) \cup (\zeta(\hat{y}), d)$. $\dashv$ (Claim 2.)

By Claim 2, it is easy to see that either $f(\zeta(\hat{x}), \eta(\hat{y})) = v$ or $f(\zeta(\hat{x}), \zeta(\hat{y})) = v$. By the same argument as in Case 1, we can show that for every $y \in I(\hat{y})$, $f(\zeta(\hat{x}), y) = v$. Since $f(\zeta(\hat{x}), \eta(\hat{y})) = f(\zeta(\hat{x}), \eta(\hat{y})) = v$ and $\zeta(\hat{x}) \notin M$, by Lemma 4.4, for every $x \in I(\hat{x}) \cap M$, we have $f(x, \eta(\hat{y})) = f(x, \zeta(\hat{y})) = v$. In particular, we have $f(\eta(\hat{x}), \eta(\hat{y})) = f(\eta(\hat{x}), \zeta(\hat{y})) = f(\zeta(\hat{x}), \eta(\hat{y})) = f(\zeta(\hat{x}), \zeta(\hat{y})) = v$. By Lemma 4.6, for every $x \in I(\hat{x})$ and $y \in I(\hat{y})$, we have $f(x, y) = v$.

Case 3. $\eta(\hat{x}), \zeta(\hat{x}), \eta(\hat{y}), \zeta(\hat{y}) \in M$.

As in Case 2, we can show the following claim.

CLAIM 3. For every $a, b \in K_0$ and $c, d \in K_1$ with $a < \eta(\hat{x}) < \zeta(\hat{x}) < b$ and $c < \eta(\hat{y}) < \zeta(\hat{y}) < d$, there exist $x \in (a, \eta(\hat{x})) \cup (\zeta(\hat{x}), b)$ and $y \in (c, \eta(\hat{y})) \cup (\zeta(\hat{y}), d)$ such that $f(x, y) = v$.

This implies that at least one of $f(\eta(\hat{x}), \eta(\hat{y}))$; $f(\eta(\hat{x}), \zeta(\hat{y}))$; $f(\zeta(\hat{x}), \eta(\hat{y}))$; and $f(\zeta(\hat{x}), \zeta(\hat{y}))$ is equal to v. In all cases, we can apply Lemma 4.4 several times to show that $f(\eta(\hat{x}), \eta(\hat{y})) = f(\eta(\hat{x}), \zeta(\hat{y})) = f(\zeta(\hat{x}), \eta(\hat{y})) = f(\zeta(\hat{x}), \zeta(\hat{y})) = v$. $\square$

By the previous lemma, we can easily see the following lemma.

Lemma 4.8. *Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in J(K_1, M)$. If f is not constant on $I(\hat{x}) \times I(\hat{y})$, then*

$$f^{\rightarrow}(I(\hat{x}) \times I(\hat{y})) = I(f(\hat{x}, \hat{y}))$$
$$f^{\rightarrow}(C(\hat{x}) \times C(\hat{y})) = C(f(\hat{x}, \hat{y})).$$

Proof. Suppose that f is not constant on $I(\hat{x}) \times I(\hat{y})$. Then without loss of generality, we may assume $f(\hat{x}, \hat{y}) \in J(L, M)$. By Lemma 4.7, for every $\langle x, y \rangle \in I(\hat{x}) \times I(\hat{y})$, $f(x, y) \notin M$. Hence, we have $f^{\rightarrow}(I(\hat{x}) \times I(\hat{y})) \subseteq I(f(\hat{x}, \hat{y}))$.

It is easy to see $f^{\rightarrow}(C(\hat{x}) \times C(\hat{y})) \subseteq C(f(\hat{x}, \hat{y}))$. But if $f^{\rightarrow}(C(\hat{x}) \times C(\hat{y}))$ is a singleton, then by Lemma 4.6, f is constant on $I(\hat{x}) \times I(\hat{y})$, which is a contradiction to the assumption. So $f^{\rightarrow}(C(\hat{x}) \times C(\hat{y})) = C(f(\hat{x}, \hat{y}))$. It follows that $f^{\rightarrow}(I(\hat{x}) \times I(\hat{y})) = I(f(\hat{x}, \hat{y}))$. $\square$

5. $f : K_0 \times K_1 \rightarrow L_0 \times L_1$

Let K_0, K_1, L_0, and L_1 be connected linearly ordered spaces. Let $f : K_0 \times K_1 \rightarrow L_0 \times L_1$ be an injective continuous function. Let g_0 and g_1 be so that $f(x, y) = \langle g_0(x, y), g_1(x, y) \rangle$ for every $\langle x, y \rangle \in K_0 \times K_1$. Let M be a countable elementary submodel of $H(\theta)$ for some regular cardinal θ with $\mathcal{P}(\mathcal{P}(K_0 \cup K_1 \cup L_0 \cup L_1)) \in H(\theta)$.

Lemma 5.1. *Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in J(K_1, M)$. Then for every $i < 2$, g_i is not constant on $I(\hat{x}) \times I(\hat{y})$.*

Proof. Suppose not, i.e., there exists $i < 2$ such that g_i is constant on $I(\hat{x}) \times I(\hat{y})$.

Since f is injective, we have either $g_{1-i}(\eta(\hat{x}), \eta(\hat{y})) < g_{1-i}(\zeta(\hat{x}), \eta(\hat{y}))$ or $g_{1-i}(\eta(\hat{x}), \eta(\hat{y})) > g_{1-i}(\zeta(\hat{x}), \eta(\hat{y}))$. Without loss of generality, we may assume $g_{1-i}(\eta(\hat{x}), \eta(\hat{y})) < g_{1-i}(\zeta(\hat{x}), \eta(\hat{y}))$.

CLAIM. $g_{1-i}(\zeta(\hat{x}), \eta(\hat{y})) < g_{1-i}(\zeta(\hat{x}), \zeta(\hat{y}))$

$\vdash$ Suppose not. Then $g_{1-i}(\zeta(\hat{x}), \eta(\hat{y})) > g_{1-i}(\zeta(\hat{x}), \zeta(\hat{y}))$. If $g_{1-i}(\zeta(\hat{x}), \eta(\hat{y})) < g_{1-i}(\eta(\hat{x}), \eta(\hat{y}))$, then by Lemma 2.4, there exists $y \in$

$(\eta(\hat{y}), \zeta(\hat{y}))$ such that $g_{1-i}(\zeta(\hat{y}), y) = g_{1-i}(\eta(\hat{x}), \eta(\hat{y}))$. This is a contradiction to the assumption that f is injective. Similarly, we can derive a contradiction when $g_{1-i}(\zeta(\hat{x}), \eta(\hat{y})) > g_{1-i}(\eta(\hat{x}), \eta(\hat{y}))$. $\dashv$ (Claim.)

Similarly, we can prove $g_{1-i}(\eta(\hat{x}), \eta(\hat{y})) < g_{1-i}(\zeta(\hat{x}), \eta(\hat{y})) < g_{1-i}(\zeta(\hat{x}), \zeta(\hat{y})) < g_{1-i}(\eta(\hat{x}), \zeta(\hat{y})) < g_{1-i}(\eta(\hat{x}), \eta(\hat{y}))$. This is a contradiction. $\square$

Lemma 5.2. *Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in J(K_1, M)$. Then there exists $i < 2$ such that for every $x \in I(\hat{x})$,*

$$g_i(x, \eta(\hat{y})) = g_i(\eta(\hat{x}), \eta(\hat{y})) \neq g_i(\eta(\hat{x}), \zeta(\hat{y})) = g_i(x, \zeta(\hat{y}))$$

and for every $y \in I(\hat{y})$,

$$g_{1-i}(\eta(\hat{x}), y) = g_{1-i}(\eta(\hat{x}), \eta(\hat{y})) \neq g_{1-i}(\zeta(\hat{x}), \eta(\hat{y})) = g_{1-i}(\zeta(\hat{x}), y).$$

Proof. By Lemma 4.1, it suffices to show that there exists $i < 2$ such that

$$g_i(\eta(\hat{x}), \eta(\hat{y})) = g_i(\eta(\hat{x}), \zeta(\hat{y})) \neq g_i(\zeta(\hat{x}), \eta(\hat{y})) = g_i(\zeta(\hat{x}), \zeta(\hat{y}))$$
$$g_{1-i}(\eta(\hat{x}), \eta(\hat{y})) = g_{1-i}(\zeta(\hat{x}), \eta(\hat{y})) \neq g_{1-i}(\eta(\hat{x}), \zeta(\hat{y})) = g_{1-i}(\zeta(\hat{x}), \zeta(\hat{y})).$$

By Lemma 5.1, for every $i < 2$, we may assume $g_i(\hat{x}, \hat{y}) \in J(L_i, M)$. For each $i < 2$, let $v_i = g_i(\hat{x}, \hat{y})$. By Lemma 4.8, for every $i < 2$, $g_i{}^{\to}(C(\hat{x}) \times C(\hat{y})) = C(v_i)$, which means that

$$\begin{aligned}\{\, &f(\eta(\hat{x}), \eta(\hat{y})), f(\eta(\hat{x}), \zeta(\hat{y})), f(\zeta(\hat{x}), \eta(\hat{y})), f(\zeta(\hat{x}), \zeta(\hat{y})) \,\} \\ &= \{\, \langle \eta(v_0), \eta(v_1) \rangle, \langle \eta(v_0), \zeta(v_1) \rangle, \langle \zeta(v_0), \eta(v_1) \rangle, \langle \zeta(v_0), \zeta(v_1) \rangle \,\}.\end{aligned}$$

By reversing the order of L_0 and/or L_1 if necessary, we may assume $f(\eta(\hat{x}), \eta(\hat{y})) = \langle \eta(v_0), \eta(v_1) \rangle$.

Claim 1. If for every $i < 2$, $g_i(\eta(\hat{x}), \eta(\hat{y})) \neq g_i(\zeta(\hat{x}), \eta(\hat{y}))$, then $\eta(\hat{y}) \in M$

$\vdash$ Suppose $\eta(\hat{y}) \notin M$. Then $\eta(\hat{y})$ is a limit point of $K_1 \cap M$ from below. By Lemma 4.2, there exists $c \in K_1$ such that $c < \eta(\hat{y})$ and for every $y \in K_1$ and $i < 2$, if $c < y < \eta(\hat{y})$, then $g_i(\eta(\hat{x}), y) = g_i(\eta(\hat{x}), \eta(\hat{y}))$ and $g_i(\zeta(\hat{x}), y) = g_i(\zeta(\hat{x}), \eta(\hat{y}))$. Clearly, this contradicts the assumption that f is injective. $\dashv$ (Claim 1.)

Claim 2. There exists $i < 2$ such that $g_i(\eta(\hat{x}), \eta(\hat{y})) = g_i(\zeta(\hat{x}), \eta(\hat{y}))$.

$\vdash$ Suppose not. By Claim 1, $\eta(\hat{y}) \in M$. We also have $f(\zeta(\hat{x}), \eta(\hat{y})) = \langle \zeta(v_0), \zeta(v_1) \rangle$. Then we have either '$f(\eta(\hat{x}), \zeta(\hat{y})) = \langle \eta(v_0), \zeta(v_1) \rangle$ and $f(\zeta(\hat{x}), \zeta(\hat{y})) = \langle \zeta(v_0), \eta(v_1) \rangle$' or vice versa. In either case, we have $g_i(\eta(\hat{x}), \zeta(\hat{y})) \neq g_i(\zeta(\hat{x}), \zeta(\hat{y}))$ for every $i < 2$. However, by the same argument as in Claim 1, we can show $\zeta(\hat{y}) \in M$. This is a contradiction to Lemma 3.3. $\dashv$ (Claim 2.)

By the same argument, there exist $j, k, l < 2$ such that

$$g_j(\zeta(\hat{x}), \eta(\hat{y})) = g_j(\zeta(\hat{x}), \zeta(\hat{y})),$$
$$g_k(\zeta(\hat{x}), \zeta(\hat{y})) = g_k(\eta(\hat{x}), \zeta(\hat{y})),$$
$$g_l(\eta(\hat{x}), \zeta(\hat{y})) = g_l(\eta(\hat{x}), \eta(\hat{y})).$$

But since f is injective, it is easy to see $i = k \neq j = l$. □

Lemma 5.3. *Let $\hat{x} \in J(K_0, M)$ and $\hat{y} \in J(K_1, M)$. Then there exists $i < 2$ such that for every $y \in K_1$ with $\hat{y} < y \leq \sup(K_1 \cap M)$, we have $g_i(\eta(\hat{x}), \hat{y}) = g_i(\eta(\hat{x}), y)$, $g_i(\zeta(\hat{x}), \hat{y}) = g_i(\zeta(\hat{x}), y)$, and $g_i(\eta(\hat{x}), \hat{y}) \neq g_i(\zeta(\hat{x}), \hat{y})$.*

Proof. By Lemma 5.2, there exists $i < 2$ such that for every $y \in K_1$, if $\hat{y} \leq y \leq \zeta(y_0)$, then $g_i(\eta(\hat{x}), y) = g_i(\eta(\hat{x}), \hat{y})$, and $g_i(\zeta(\hat{x}), y) = g_i(\zeta(\hat{x}), \hat{y})$. Moreover, $g_i(\eta(\hat{x}), \hat{y}) \neq g_i(\zeta(\hat{x}), \hat{y})$.

Now, we shall show that for every $y \in K_1$, if $\hat{y} \leq y < \sup(K_1 \cap M)$, then $g_i(\eta(\hat{x}), y) = g_i(\eta(\hat{x}), \hat{y})$ and $g_i(\zeta(\hat{x}), y) = g_i(\zeta(\hat{x}), \hat{y})$. Suppose not. Let y_1 be the infimum of $y \in K_1$ with $y \geq \hat{y}$ such that either $g_i(\eta(\hat{x}), y) \neq g(\eta(\hat{x}), \hat{y})$ or $g_i(\zeta(\hat{x}), y) \neq g(\zeta(\hat{x}), y_0)$. Note $g_i(\eta(\hat{x}), y_1) = g(\eta(\hat{x}), \hat{y})$ and $g_i(\zeta(\hat{x}), y_1) = g(\zeta(\hat{x}), \hat{y})$.

Case 1. y_1 is a limit point of $K_1 \cap M$ from above.

Since $g_i(\eta(\hat{x}), y_1) = g_i(\eta(\hat{x}), \hat{y}) \neq g_i(\zeta(\hat{x}), \hat{y}) = g_i(\zeta(\hat{x}), y_1)$, there exists $x_0 \in I(\hat{x})$ such that $g_i(x_0, y_1) \in J(L_i, M)$. So, by Lemma 4.2, there exists $y_2 \in K_1$ such that $y_1 < y_2$ and for every $y \in [y_1, y_2]$, $g_i(\eta(\hat{x}), y) = g_i(\eta(\hat{x}), y_1)$ and $g_i(\eta(\hat{x}), y) = g_i(\eta(\hat{x}), y_1)$. This is a contradiction to the definition of y_1.

Case 2. y_1 is a limit point of $K_1 \cap M$ from below, but not from above.

Let $y_2 = \inf \{ y \in K_1 \cap M \mid y > y_1 \}$. By Lemma 5.2, there exists $j < 2$ so that $g_j(\eta(\hat{x}), y_1) = g_j(\zeta(\hat{x}), y_1)$, and for every $y \in [y_1, y_2]$, $g_{1-j}(\eta(\hat{x}), y) = g_{1-j}(\eta(\hat{x}), y_1)$ and $g_{1-j}(\zeta(\hat{x}), y) = g_{1-j}(\zeta(\hat{x}), y_1)$. Since $g_i(\eta(\hat{x}), y_1) \neq g_i(\zeta(\hat{x}), y_1)$, we have $i \neq j$, and hence $1 - j = i$. So for every $y \in [y_1, y_2]$, $g_i(\eta(\hat{x}), y) = g_i(\eta(\hat{x}), y_1)$ and $g_i(\zeta(\hat{x}), y) = g_i(\zeta(\hat{x}), y_1)$. This is a contradiction to the definition of y_1.

Case 3. y_1 is a limit point of $K_1 \cap M$ neither from above nor from below.

Then $y_1 \in J(K_1, M)$. We have $g_i(\eta(\hat{x}), \eta(y_1)) = g_i(\eta(\hat{x}), \hat{y}) \neq g_i(\zeta(\hat{x}), \hat{y}) = g_i(\zeta(\hat{x}), \eta(y_1))$. By Lemma 5.2, there exists $j < 2$ such that $g_j(\eta(\hat{x}), \eta(y_1)) = g_j(\zeta(\hat{x}), \eta(y_1)$, and for every $y \in I(y_1)$, $g_{1-j}(\eta(\hat{x}), y) = g_{1-j}(\eta(\hat{x}), \eta(y_1)) = g_{1-j}(\eta(\hat{x}), \hat{y})$ and $g_{1-j}(\zeta(\hat{x}), y) = g_{1-j}(\zeta(\hat{x}), \eta(y_1)) = g_{1-j}(\zeta(\hat{x}), \hat{y})$. Since $g_i(\eta(\hat{x}), \eta(y_1)) \neq g_i(\zeta(\hat{x}), \eta(y_1))$, we have $j \neq i$, and hence $1 - j = i$. So for every $y \in I(y_1)$, $g_i(\eta(\hat{x}), y) = g_i(\eta(\hat{x}), \hat{y})$ and $g_i(\zeta(\hat{x}), y) = g_i(\zeta(\hat{x}), \hat{y})$. This is a contradiction to the definition of y_1. □

Lemma 5.4. *Let $\hat{x} \in J(K_0, M)$. Then there exists $i < 2$ such that, for every $y, y' \in (\inf(K_1 \cap M), \sup(K_1 \cap M))$, we have $g_i(\eta(\hat{x}), y) = g_i(\eta(\hat{x}), y')$ and $g_i(\zeta(\hat{x}), y) = g_i(\zeta(\hat{y}), y')$*

Proof. Let $\hat{y} \in J(K_1, M)$ be arbitrary. By Lemma 5.3, there exists $i < 2$ such that for every $y \in (\hat{y}, \sup(K_1 \cap M))$, $g_i(\eta(\hat{x}), y) = g_i(\eta(\hat{x}), \hat{y})$, $g_i(\zeta(\hat{x}), y) = g_i(\zeta(\hat{x}), \hat{y})$, and $g_i(\eta(\hat{x}), \hat{y}) \neq g_i(\zeta(\hat{x}), \hat{y})$.

Now, it suffices to show that for every $y_0 \in (\inf(K_1 \cap M), \sup(K_1 \cap M))$, we have $g_i(\eta(\hat{x}), y_0) = g_i(\eta(\hat{x}), \hat{y})$ and $g_i(\zeta(\hat{x}), y_0) = g_i(\zeta(\hat{y}), \hat{y})$. It is trivial if $y_0 \geq \hat{y}$. So assume $y_0 < \hat{y}$.

Let $\hat{y}' \in J(K_1, M)$ be so that $\hat{y}' < y_0$. By Lemma 5.3, there exists $j < 2$ such that for every $y \in (\hat{y}', \sup(K_1 \cap M))$, we have $g_j(\eta(\hat{x}), y) = g_j(\eta(\hat{x}), \hat{y}')$, and $g_j(\zeta(\hat{x}), y) = g_j(\zeta(\hat{x}), \hat{y}')$. Since $\hat{y}' < y_0 < \hat{y}$, we have $g_j(\eta(\hat{x}), y_0) = g_j(\eta(\hat{x}), \hat{y}) = g_j(\eta(\hat{x}), \hat{y}')$ and $g_j(\zeta(\hat{x}), y_0) = g_j(\zeta(\hat{x}), \hat{y}) = g_j(\zeta(\hat{x}), \hat{y}')$.

Now it suffices to show that $i = j$. Suppose not. Let $y \in (\hat{y}', \sup(K_1 \cap M))$. Then $g_j(\eta(\hat{x}), y) = g_j(\eta(\hat{x}), \hat{y}) = g_j(\eta(\hat{x}), \hat{y}')$ and $g_i(\eta(\hat{x}), y) = g_i(\eta(\hat{x}), \hat{y})$. So $f(\eta(\hat{x}), y) = f(\eta(\hat{x}), \hat{y})$. This is a contradiction to the assumption that f is injective. □

6. Proof of the Main Theorem

Let K_0, K_1, L_0, L_1, f, g_0, and g_1 be as in the previous section. Now we do not fix M.

Lemma 6.1. *For every $\langle \hat{x}, \hat{y} \rangle \in K_0 \times K_1$, there exists $i < 2$ such that, for every $y \in K_1$, $g_i(\hat{x}, y) = g_i(\hat{x}, \hat{y})$.*

Proof. Let $\hat{x} \in K_0$ and $\hat{y} \in K_1$. By way of contradiction, we assume that for every $i < 2$, there exists $y_i \in K_1$ such that $g_i(\hat{x}, y_i) \neq g_i(\hat{x}, \hat{y})$.

Let M be a countable elementary submodel of $H(\theta)$ with K_0, K_1, L_0, L_1, f, g_0, g_1, $\hat{x}$, $\hat{y}$, y_0, and $y_1 \in M$.

Since K_0 is nowhere real, $\hat{x}$ is a limit point of $J(K_0, M)$ from above. Hence, there exists $\hat{x}' \in J(K_0, M)$ such that for every $i < 2$, $g_i(\hat{x}', y_i) \neq g_i(\hat{x}', \hat{y})$.

By Lemma 5.4, there exists $i < 2$ such that, for every $y \in (\inf(K_1 \cap M), \sup(K_1 \cap M))$, we have $g_i(\hat{x}', y) = g_i(\hat{x}', \hat{y})$. But then $g_i(\hat{x}', y_i) = g_i(\hat{x}', \hat{y})$, which is a contradiction. □

We can finally prove the main theorem.

Proof of Theorem 1.4. By Lemma 6.1, for every $x \in K_0$, there exists $i_x < 2$ such that for every $y, y' \in K_1$, $g_{i_x}(x, y) = g_{i_x}(x, y')$. Similarly, for every $y \in K_1$, there exists $j_y < 2$ such that, for every $x, x' \in K_0$, $g_{j_y}(x, y) = g_{j_y}(x', y)$. Now it suffices to show there exists $i \neq j < 2$ such that, for $x \in K_0$, $i_x = i$, and, for every $y \in K_1$, $j_y = j$.

Let $y \in K_1$ be arbitrary and let $j = j_y$. We shall show that for every $x \in K_0$, $i_x \neq j$. Suppose not, i.e., $i_x = j$. Let $a, b \in K_0$ be so that $a < x < b$ and $c, d \in K_1$ so that $c < y < d$. Since $j = j_y = i_x$, we have $g_j(a, y) = g_j(b, y) = g_j(x, c) = g_j(x, d) = g_j(x, y)$. Since f is injective, none of $g_{1-j}(a, y)$, $g_{1-j}(b, y)$, $g_{1-j}(x, c)$, and $g_{1-j}(x, d)$ is equal to $g_{1-j}(x, y)$. So either there exist two of them that are greater than $g_{1-j}(x, y)$, or there exist two of them that are smaller than $g_{1-j}(x, y)$. For example, suppose $g_{1-j}(a, y) > g_{1-j}(x, c) > g_{1-j}(x, y)$. By the intermediate value theorem, there exists $a' \in K_0$ such that $a < a' < x$ and $g_{1-j}(a', y) = g_{1-j}(x, c)$. Then $f(a', y) = f(x, c)$, which is a contradiction to the injectivity of f. $\square$

7. Open questions

We can also prove the following theorem that extends Theorem 1.4 to any finite number of connected linearly ordered spaces though the proof is complicated.

Theorem 7.1. *Let K_0, K_1, $\ldots$, K_{n-1}, L_0, L_1, $\ldots$, L_{n-1} be connected nowhere real linearly ordered spaces, and let $f : \Pi_{i<n} K_i \to \Pi_{i<n} L_i$ be a continuous injection. Then f is coordinate-wise.*

A paper about this result is in preparation.

Theorem 1.4 demonstrates that the situation is totally different when we use connected nowhere real linearly ordered spaces instead of $\mathbb{R}$. Hence, we may ask the following broad question.

Question 7.2. What theorems about $\mathbb{R}$ can be extended to connected linearly ordered spaces?

Considering the lemmas proved in this article, it seems very unlikely that any similar theorems about $\mathbb{R}$ can be generalized to connected linearly ordered spaces.

While we heavily relied on linear orders in this article, we have no evidence that they are essential. This means that the following question is still wide open.

Question 7.3. Can we weaken the assumption that K_0, K_1, L_0, and L_1 are linearly ordered spaces? For example, what if they are 1-dimensional in some sense?

References

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