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## 50TH SPRING TOPOLOGY AND DYNAMICAL SYSTEMS CONFERENCE CONTRIBUTED PROBLEMS IN SET-THEORETIC TOPOLOGY

by

EDITED BY JOHN E. PORTER

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**50TH SPRING TOPOLOGY AND DYNAMICAL  
SYSTEMS CONFERENCE CONTRIBUTED PROBLEMS  
IN SET-THEORETIC TOPOLOGY**

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## 1. PARTIAL METRICS

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Let  $\rho$  be a partial metric on  $X$  [40]. The sets

$$B_\epsilon^\rho(x) = \{y \in X : \rho(x, y) - \rho(x, x) < \epsilon\}$$

form a basis for a  $T_0$  space on  $X$  [14]. A  $\rho$ -gilded ball [4] (referred to as a closed ball in the metric case) is given by

$$\bar{B}_\epsilon^\rho = \{y \in X : \rho(x, y) - \rho(x, x) \leq \epsilon\}.$$

One of the challenges we come across is that, in a partial metric space  $(X, \rho)$ , a  $\rho$ -gilded ball need not be topologically closed.

**Question 1.** When is a topological space  $X$  partially metrizable?

**Question 2.** When is the partial metric topology on  $X$  compact?

In [5], Samer Assaf and Koushik Pal introduced the axioms of a strong partial metric. It is trivial to show that if the partial metric space  $(X, \rho)$  is  $T_1$ , then  $\rho$  is a strong partial metric.

**Question 3.** Can we strengthen the strong partial metric axioms to generate a  $T_2$  space that need not be  $T_3$ ? A  $T_3$  space that need not be  $T_4$ ? A  $T_4$  space that need not be metrizable?

The question below goes hand in hand with the question above.

**Question 4.** If the partial metric space is  $T_2$ , must  $(X, \rho)$  be metrizable? What if the partial metric space is  $T_3$  or  $T_4$ ?

## 2. SOME PROBLEMS ON MONOTONE NORMALITY

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The first problem involves paracompactness in monotonically normal spaces. The fundamental paper about paracompactness in monotonically normal spaces is [8], written by Z. Balogh and M. E. Rudin. In it they generalize an earlier result of R. Engelking and D. Lutzer [27] for generalized ordered spaces (GO-spaces) by proving the following theorem.

**Theorem 1.** *A monotonically normal space  $X$  is paracompact if and only if  $X$  does not contain a closed subspace that is homeomorphic to a stationary set in a regular uncountable ordinal.*

In her encyclopedia article [48], Rudin comments that Theorem 1 generalizes each of about twenty known paracompactness results for generalized ordered spaces to the class of monotonically normal spaces. The proofs have the following form. We suppose that a monotonically normal space  $X$  has a closed-hereditary property  $P$  (e.g., metacompactness or a  $G_\delta$ -diagonal). If  $X$  is not paracompact, then Theorem 1 gives a stationary subset  $S$  of a regular uncountable ordinal that is homeomorphic to a closed subspace of  $X$ . Then  $S$  inherits property  $P$  from  $X$ . Finally, one verifies that no stationary subspace of a regular uncountable ordinal can have property  $P$ , and that completes the proof. But in one case, there is a problem. It is known [10] that a GO-space with a  $\sigma$ -minimal base must be paracompact. (Recall that a collection  $\mathcal{M}$  is minimal if each  $M \in \mathcal{M}$  contains a point that is not in any other member of  $\mathcal{M}$ , and “ $\sigma$ -minimal” means “a countable union of minimal collections.”) Unfortunately, the property “ $X$  has a  $\sigma$ -minimal base” is not closed hereditary. In fact, Dennis K. Burke has shown that *any* space is a closed subspace of some space with a  $\sigma$ -minimal base [15].) Consequently, we have the following.

**Problem 2.** *Suppose  $X$  is a monotonically normal space that has a  $\sigma$ -minimal base. Is  $X$  paracompact?*

The second problem asks about metrization of spaces that have monotonically normal compactifications. (For example, any GO-space is of this type, but not every metric space [34].) In [47], Rudin establishes a deep link between compact monotonically normal spaces and compact linearly ordered topological spaces (LOTS) by proving the following theorem.

**Theorem 3.** *A compact space  $X$  is monotonically normal if and only if there is a compact LOTS  $L$  and a continuous mapping  $f$  from  $L$  onto  $X$ .*

Standard techniques prove the first sentence of the next corollary and some order-space trickery (see [11]) gives the second sentence.

**Corollary 4.** *If  $X$  is a space with a monotonically normal compactification, then there is a GO-space  $Y$  and a perfect irreducible mapping  $g$  from  $Y$  onto  $X$ . In addition, one can arrange that if  $a < b$  are points of  $Y$  with  $(a, b) = \emptyset$ , (i.e., if  $a$  and  $b$  are jump points of  $Y$ ), then  $g(a) \neq g(b)$ .*

Corollary 4 led us to wonder whether certain metrization theorems for GO-spaces might be generalized to spaces with monotonically normal compactifications. Perhaps the most important such theorem (that

does not mention the ordering of the GO-space) asserts that any semi-stratifiable GO-space is metrizable. Attempting to generalize that result, in [11] we proved the following.

**Proposition 5.** *Suppose  $X$  is semi-stratifiable and has a monotonically normal compactification. Then  $X$  is metrizable if any one of the following holds:*

- (a) *There is a  $\sigma$ -locally finite cover of  $X$  by closed metrizable subspaces.*
- (b) *There is a  $\sigma$ -locally finite cover of  $X$  by compact subspaces.*
- (c)  *$X = \bigcup\{Y_n : n \geq 1\}$  where each  $Y_n$  is a closed discrete subspace of  $X$ .*
- (d)  *$X$  is scattered, or  $\sigma$ -compact, or countable.*

Proposition 5 leads to the following problem.

**Problem 6.** *Suppose  $X$  is a semi-stratifiable space with a monotonically normal compactification. Must  $X$  be metrizable?*<sup>1</sup>

If one is searching for counterexamples to Problem 6, it might be useful to know that any semi-stratifiable space with a monotonically normal compactification must be first countable and must be a union of dense, metrizable subspaces [11]. This narrows the scope of possible counterexamples because any counterexample must be a Nagata space (= first-countable stratifiable space) in the sense of Jack G. Ceder [18] and Carlos J. R. Borges [12].

There are many properties other than semi-stratifiability that give metrizability in a GO-space, and this suggests a more general question.

**Problem 7.** *Suppose  $P$  is a topological property and suppose that any GO-space with property  $P$  must be metrizable. Is it true that if  $X$  has property  $P$  and has a monotonically normal compactification, then  $X$  must be metrizable?*

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<sup>1</sup>This problem was recently answered in the affirmative by Gary Gruenhage and David J. Lutzer [29].

## 3. 2-MARKOV STRATEGIES IN SELECTION GAMES

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Let  $S_{fin}^\omega(\mathcal{A}, \mathcal{B})$  be the statement that whenever  $A_n \in \mathcal{A}$  for  $n < \omega$ , there exist  $B_n \in [A_n]^{<\omega}$  such that  $\bigcup\{B_n : n < \omega\} \in \mathcal{B}$ . This *selection principle* characterizes a property of a topological space  $X$  when  $\mathcal{A}$  and  $\mathcal{B}$  are defined in terms of  $X$ . For example, if  $\mathcal{O}_X$  is the collection of open covers of  $X$ , then  $S_{fin}^\omega(\mathcal{O}_X, \mathcal{O}_X)$  is the well-known Menger covering property.

This property may be made stronger by considering the following two-player game of length  $\omega$ :  $G_{fin}^\omega(\mathcal{A}, \mathcal{B})$ . During each round  $n < \omega$ , the first player  $\mathcal{A}$  chooses  $A_n \in \mathcal{A}$ , followed by  $\mathcal{B}$  choosing  $B_n \in [A_n]^{<\omega}$ .  $\mathcal{B}$  wins the game if  $\bigcup\{B_n : n < \omega\} \in \mathcal{B}$ ; otherwise,  $\mathcal{A}$  wins. If  $\mathcal{B}$  has a *winning strategy* for the game (a function which defines a move for each finite sequence of previous moves by  $\mathcal{A}$  and beats every possible response by  $\mathcal{A}$ ), then we write  $\mathcal{B} \uparrow G_{fin}^\omega(\mathcal{A}, \mathcal{B})$ .

These concepts were first introduced by Marion Scheepers in [49]. Of course,  $\mathcal{B} \uparrow G_{fin}^\omega(\mathcal{A}, \mathcal{B}) \Rightarrow S_{fin}^\omega(\mathcal{A}, \mathcal{B})$ , but the converse need not hold since each  $B_n$  may be defined in  $S_{fin}^\omega(\mathcal{A}, \mathcal{B})$  using knowledge of all  $A_n$ , not just those “previously played.” Thus, for each topological property  $P$  characterized by  $S_{fin}^\omega(\mathcal{A}, \mathcal{B})$ , we denote the (possibly) stronger property  $\mathcal{B} \uparrow G_{fin}^\omega(\mathcal{A}, \mathcal{B})$  as *strategic*  $P$ .

Such notions may be made even stronger using *limited information strategies*. A  $k$ -Markov strategy for  $\mathcal{B}$  uses only the last  $k$  moves of  $\mathcal{A}$  and the round number. When  $\mathcal{B}$  has a winning  $k$ -Markov strategy for  $G_{fin}^\omega(\mathcal{A}, \mathcal{B})$ , we write  $\mathcal{B} \uparrow_{k\text{-mark}} G_{fin}^\omega(\mathcal{A}, \mathcal{B})$ . Similarly, for each topological property  $P$  characterized by  $S_{fin}^\omega(\mathcal{A}, \mathcal{B})$ , we denote property  $\mathcal{B} \uparrow_{k\text{-mark}} G_{fin}^\omega(\mathcal{A}, \mathcal{B})$  as  $k$ -Markov  $P$ .

In the case of the selection game  $G_{fin}^\omega(\mathcal{A}, \mathcal{B})$ , it may be shown that a  $(k+2)$ -Markov strategy may always be improved to a 2-Markov strategy, as shown in [21] with regards to  $G_{fin}^\omega(\mathcal{O}_X, \mathcal{O}_X)$ .

The following natural question is open.

**Question 1.** Do there exist (interesting/topological)  $\mathcal{A}$  and  $\mathcal{B}$  such that  $\mathcal{B} \uparrow G_{fin}^\omega(\mathcal{A}, \mathcal{B})$  but  $\mathcal{B} \not\uparrow_{2\text{-mark}} G_{fin}^\omega(\mathcal{A}, \mathcal{B})$ ?

Consider the case that  $\mathcal{A} = \mathcal{B} = \mathcal{O}_X$ , i.e., the Menger game. The following summarize results from [50], [21], and [22].

**Definition 2.** For any cardinal  $\kappa$ , let  $\kappa^\dagger = \kappa \cup \{\infty\}$  denote the *one-point Lindelöfication* of discrete  $\kappa$ , where points in  $\kappa$  are isolated and the neighborhoods of  $\infty$  are co-countable.

**Proposition 3.**  $\mathcal{B} \uparrow G_{fin}^\omega(\mathcal{O}_{\kappa^\dagger}, \mathcal{O}_{\kappa^\dagger})$ .

**Definition 4.** For two functions  $f$  and  $g$ , we say  $f$  is *almost compatible* with  $g$  if  $|\{x \in \text{dom}(f) \cap \text{dom}(g) : f(x) \neq g(x)\}| < \omega$ .

**Definition 5.**  $\mathcal{A}'(\kappa)$  states that there exists a collection of pairwise almost compactible finite-to-one functions  $\{f_A \in \omega^A : A \in [\kappa]^{\leq \omega}\}$ .

**Theorem 6.**  $\mathcal{A}'(\omega_n)$  holds for all  $n < \omega$ .

**Theorem 7.**  $\mathcal{A}'(\kappa)$  implies  $\mathcal{B} \uparrow_{2\text{-mark}} G_{fin}^\omega(\mathcal{O}_{\kappa^\dagger}, \mathcal{O}_{\kappa^\dagger})$ .

**Theorem 8.** For any cardinal  $\kappa$ ,  $\kappa$  Cohen reals may be added to a model of ZFC + CH while preserving  $\mathcal{A}'(\mathfrak{c})$ .

**Theorem 9.** There exists a model of ZFC where  $\mathcal{A}'(\omega_\omega)$  fails.

**Theorem 10.**  $\mathcal{B} \uparrow_{2\text{-mark}} G_{fin}^\omega(\mathcal{O}_{\omega_\omega^\dagger}, \mathcal{O}_{\omega_\omega^\dagger})$ .

It remains open whether  $\mathcal{B} \uparrow_{2\text{-mark}} G_{fin}^\omega(\mathcal{O}_{\omega_{\omega+1}^\dagger}, \mathcal{O}_{\omega_{\omega+1}^\dagger})$  might fail when  $\mathcal{A}'(\omega_\omega)$  fails. Due to the above, any attempt to show  $\mathcal{B} \not\uparrow_{2\text{-mark}} G_{fin}^\omega(\mathcal{O}_{\kappa^\dagger}, \mathcal{O}_{\kappa^\dagger})$  cannot happen solely within ZFC.

#### 4. SYMMETRIZABLE $L$ -SPACES

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S. Nedevev [42] showed every Lindelöf symmetrizable space is hereditarily Lindelöf. I. Juhász, Z. Nagy, and Z. Szentmiklóssy [32] showed there is a Hausdorff nonseparable Lindelöf symmetrizable space assuming CH, while Balogh, Burke and S. W. Davis [7] showed there is a Hausdorff nonseparable Lindelöf symmetrizable space in ZFC. D. B. Shakhmatov [51] showed it is consistent with ZFC that there is a regular nonseparable Lindelöf symmetrizable space.

**Question 1.** Is there a regular symmetrizable  $L$ -space in ZFC?

Justin Tatch Moore [41] showed that L-spaces exist in ZFC, while Stevo Todorčević showed it is consistent with ZFC that there are no S-spaces. It is consistent that both S-spaces and L-spaces exist, and if there is an S-space, then there is an uncountable right separated S-space, and if there is an L-space, then there is an uncountable left separated L-space. There are no uncountable symmetrizable Lindelöf left separated spaces [7]. Nedev [42] showed there are no symmetrizable S-spaces.

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## 5. COUNTABLY COMPACT, SEQUENTIALLY COMPACT, SEQUENTIAL, NONCOMPACT SPACES OF COUNTABLE TIGHTNESS

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A space has countable tightness if the closure of any set is the union of the closures of its countable subsets. A set is sequentially closed if no sequence from the set converges to a point outside the set. A space is sequential if every sequentially closed set is closed. A countably compact sequential space is sequentially compact.

**Problem 1.** *Is there a ZFC example of a Tychonoff countably compact noncompact space of countable tightness which does not contain a copy of  $\omega_1$ ? Can the space even be sequential or just sequentially compact?*

**Problem 2.** *Does the proper forcing axiom (PFA) imply that every infinite countably compact space of countable tightness contains a non-trivial converging sequence?*

One might well rephrase the first question to ask if PFA implies that a noncompact countably compact space will contain a copy of  $\omega_1$  if the space has countable tightness or is even sequential. Clearly, a negative answer to the second problem would be a stronger result. It has proven very useful to be able to prove the existence of copies of  $\omega_1$  [43]. A noncompact countably compact space will carry a countably complete free maximal filter  $\mathcal{F}$  of closed subsets. There is a proper poset that forces that there is an uncountable free sequence  $S = \{x_\alpha : \alpha \in \omega_1\}$  with the property that  $S \setminus F$  is countable for all  $F \in \mathcal{F}$  (see [6]). If  $X$  has character at most  $\omega_1$ , then PFA implies this free sequence can be chosen to be a copy of  $\omega_1$  [23]. It is even sufficient that  $X$  has a sufficiently rich supply of points of countable character for this to hold under PFA. For example, PFA implies that if  $X$  is  $\aleph_0$ -bounded, then  $X$  will contain a copy of  $\omega_1$ .



[26]. Recall that PFA implies that compact spaces of countable tightness are sequential and have a rich supply of points of countable character [6], [23].

It follows from PFA that there are countably compact spaces of countable tightness that are not sequential nor sequentially compact [24]. However, it is not clear just how far from sequential a countably compact countably tight space can be. In particular, while we do not know if there is a rich enough supply of points of countable character, it is known to be consistent with CH that there is quite a rich supply of converging sequences in any countably compact space of countable tightness [25].

## 6. PRESERVATION OF A NEIGHBORHOOD BASE OF A CLOSED DISCRETE SET

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Let  $\langle X, \tau \rangle$  be a topological space. Let  $\mathbf{V}$  be a ground model and let  $\mathbf{V}^{\mathbb{P}}$  be the forcing extension of  $\mathbf{V}$  by the forcing  $\mathbb{P}$ . We define a topological space  $\langle X, \tau^{\mathbb{P}} \rangle$  in  $\mathbf{V}^{\mathbb{P}}$  such that  $\tau^{\mathbb{P}}$  is the topology generated by  $\tau$ . Let us give some key definitions.

**Definition 1.** We say that a forcing  $\mathbb{P}$  *destroys* a neighborhood base of a closed set  $A$  if there is  $W \in \tau^{\mathbb{P}}$  such that  $A \subseteq W$  and for every  $U \in \tau$  with  $A \subseteq U$ , we have  $U \not\subseteq W$ . If a forcing  $\mathbb{P}$  does not destroy a neighborhood base of  $A$ , then we say that  $\mathbb{P}$  *preserves* a neighborhood base of  $A$ .

Note that if a closed set  $A$  is a singleton, then a neighborhood base of  $A$  is preserved by any forcing. For  $x \in X$ , let  $\mathcal{N}(x)$  be the set of all neighborhoods of  $x$ .

**Definition 2.** Let  $\langle X, \tau \rangle$  be a topological space. We define a cardinal function  $\varphi$  such that for a non-isolated point  $x \in X$ ,

$$\varphi(x) = \min \left\{ |\mathcal{U}| : \mathcal{U} \subseteq \mathcal{N}(x), \bigcap \mathcal{U} \notin \mathcal{N}(x) \right\};$$

for an isolated point  $x$ , let  $\varphi(x)$  be undefined.

Note that if  $X$  is a first countable  $T_1$ -space, then for every non-isolated  $x \in X$ , we have  $\varphi(x) = \aleph_0$ .

**Definition 3.** A forcing  $\mathbb{P}$  is said to satisfy the *countable covering property* if for every countable set  $A \in \mathbf{V}^{\mathbb{P}}$  such that  $A \subseteq \mathbf{V}$ , there is a countable set  $B \in \mathbf{V}$  such that  $A \subseteq B$ .

It is not difficult to show the following proposition.

**Proposition 4.** *Let  $\langle X, \tau \rangle$  be a regular topological space and let  $I$  be the set of all isolated points in  $X$ . Let  $A$  be a closed discrete subset of  $X$  such that*

- $|A \setminus I| = \aleph_0$ , and
- $(\forall a \in A \setminus I)(\varphi(a) = \aleph_0)$ .

*Then a forcing that satisfies the countable covering property destroys a neighborhood base of  $A$ .*

In fact, adding a Cohen real destroys a neighborhood base of  $A$  in the above proposition. If a closed discrete set  $A$  is uncountable, then I do not know if a forcing that satisfies the countable covering property and destroys a neighborhood base of  $A$  can always be found.

**Question 5.** Let  $\langle X, \tau \rangle$  be a regular topological space and let  $I$  be the set of isolated points of  $X$ . Let  $A$  be a closed discrete subset of  $X$  such that

- $|A \setminus I| > \aleph_0$ , and
- $(\forall a \in A \setminus I)(\varphi(a) = \aleph_0)$ .

Does some forcing that satisfies the countable covering property destroy a neighborhood base of  $A$ ?

## 7. COMPACT JÓNSSON–TARSKI ALGEBRAS

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To be more precise, a *Jónsson–Tarski algebra* [31] is  $(X, m, u, v)$  with  $X$  a set and  $m : X \times X \rightarrow X$  and  $u, v : X \rightarrow X$  operations which satisfy the equations

$$\begin{aligned} m(ux, vx) &= x \\ u m(x, y) &= x \\ v m(x, y) &= y. \end{aligned}$$

These equations simply state that  $m$  is bijective with inverse  $x \mapsto (ux, vx)$ . This equational class was invented to provide an example wherein all finitely generated free algebras are isomorphic.

For any equational class  $\mathcal{E}$  there is the class  $\mathcal{KE}$  of all  $\mathcal{E}$ -algebras with a compact Hausdorff topology in such a way that the  $\mathcal{E}$ -operations are

continuous. The morphisms are those continuous maps which additionally commute with the  $\mathcal{E}$ -operations. Thus, if  $\mathcal{JT}$  is the class of Jónsson–Tarski algebras, the morphisms in  $\mathcal{KJT}$  are continuous maps  $f$  which also satisfy  $m(fx, fy) = f m(x, y)$ ,  $f(ux) = u fx$ , and  $f(vx) = v fx$ . The objects are just compact spaces homeomorphic to their square, but the morphisms are more than continuous.

Any category  $\mathcal{KE}$  is isomorphic to the category of algebras of a monad of sets [37]. Such a category has lots of constructions. For example, it has coequalizers of pairs of maps and arbitrary coproducts.

**Question 1.** How are coproducts constructed in  $\mathcal{KJT}$ ?

For  $X$  in  $\mathcal{KJT}$ ,  $x \in X$ , the *orbit* of  $x$  is the set of all  $tx$  as  $t$  ranges over all unary operations derived from  $m$ ,  $u$ , and  $v$ . Examples of such  $tx$  are  $m(x, x)$ ,  $m(ux, x)$ , and  $m(uvux, m(vx, vvx))$ . Notice that  $m(ux, ux)$  is idempotent and  $m(vx, ux)$  is invertible.  $X$  is *minimal* if every orbit is dense. The Cantor set  $2^\omega$  admits a minimal structure [36].

Let  $F$  be the free algebra on one generator in  $\mathcal{KJT}$ . It is tempting to speculate that  $F = \beta M$  where  $M$  is the monoid of all unary operations derived from  $m$ ,  $u$ , and  $v$ . This is not possible since  $M$  is countable, whereas  $\beta\omega$  is not homeomorphic to its square.

**Question 2.** What is a concrete construction for  $F$ ?

$F$  contains a minimal algebra  $I$  and it is unique up to isomorphism [39]. Every minimal admits a surjective map from  $I$ .  $F$  is separable. Therefore,  $c \leq |F| \leq 2^c$ .

**Question 3.** What is the largest cardinality of a minimal algebra in  $\mathcal{KJT}$ ?

In effect, Question 3 asks for the cardinality of the minimal algebra in  $F$ .

## 8. WEIGHTS OF DYADIC SPACES

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A topological space  $X$  is *dyadic* if  $X$  is the continuous image of some  $2^\kappa$ . A topological space  $X$  is *supercompact* if  $X$  has a subbase  $\mathcal{B}$  where if  $\mathcal{U} \subset \mathcal{B}$  is a cover of  $X$ , then there are  $U_0, U_1 \in \mathcal{U}$  which cover  $X$ .

**Question 1.** What is the least weight of a dyadic non-supercompact space?

**Remark 2.** The only known example, due to Murray G. Bell [9], has weight  $\aleph_3$ .

The motivation is finding interesting “phase transitions” at weight  $\aleph_3$  in analogy with E. V. Ščepin’s results [52] about how several natural functors, including the Vietoris hyperspace functor and the symmetric power functors, behave very differently on spaces of weight less than  $\aleph_2$  from the way they behave on space of weight  $\geq \aleph_2$ .

## 9. PROBLEMS ON MONOTONE COVERING PROPERTIES

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Given a covering property  $\mathcal{P}$ , one can define a monotone version of the covering property by requiring an operator  $r$  that assigns the right kind of refinement to each acceptable open covering  $\mathcal{U}$  in such a way that  $r(\mathcal{U})$  refines  $r(\mathcal{V})$  whenever  $\mathcal{U}$  refines  $\mathcal{V}$ . For example, Strashimir G. Popvassilev [44] calls a space *monotonically (countably) metacompact* if one can assign to every (countable) open cover  $\mathcal{U}$  a point-finite open cover  $r(\mathcal{U})$  that refines  $\mathcal{U}$  so that  $r(\mathcal{U})$  refines  $r(\mathcal{V})$  whenever  $\mathcal{U}$  refines  $\mathcal{V}$ .

Timothy Chase and Gary Gruenhage [19] show that compact monotonically countably metacompact spaces are metrizable, and Gruenhage announced at the conference that he and Chase [20] have proven that separable monotonically countably metacompact spaces are also metrizable. Chase and Gruenhage’s results simultaneously generalize similar results for proto-metrizable spaces and Moore spaces. Can Chase and Gruenhage’s results be further generalized? Recall that a space is orthocompact if every open cover has an interior preserving open refinement. That is, every open cover has an open refinement, with the further property that at any point, the intersection of all open sets in the refinement containing that point is also open.

**Question 1.** Are compact monotonically orthocompact spaces metrizable?

**Remark 2.** The separable version of Question 1 has a negative answer. Popvassilev [45] has shown that the Sorgenfrey line is monotonically orthocompact.

Recall that a space  $X$  is proto-metrizable if  $X$  is paracompact and has an ortho-base. P. M. Gartside and P. J. Moody [28] show that a space is proto-metrizable if and only if  $X$  has a monotone operator  $r$  such that  $r(\mathcal{U})$  star-refines  $\mathcal{U}$ . Popvassilev and John E. Porter [46] show that proto-metrizable spaces possess a monotone locally finite operator.

**Question 3.** Are metacompact spaces with an ortho-base monotonically (countably) metacompact?

In general, it seems that little is known about the class of metacompact spaces with an ortho-base.

## 10. PROBLEMS ON SEQUENTIAL GROUPS

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Recall that a countable topological space is *analytic* if its topology, viewed as a subspace of the Cantor cube  $2^\omega$ , is analytic (i.e., a continuous image of the irrationals); see [56].

While the precise definitions of sequential and sequential order are somewhat lengthy, intuitively, a space is *sequential* if iteratively adding limits of convergent subsequences eventually produces the closure of any given set and the *sequential order* is the minimal “number of steps” required to produce the closure of any subset in the space in such manner.

**Problem 1.** *Is it consistent with ZFC that every countable sequential group is analytic?*

The following is a stronger version of a question in [30] (see also [13]). Recall that a group is called *Boolean* if it is a subgroup of some product of  $\mathbb{Z}_2$ 's.

**Problem 2.** *Is it consistent with ZFC that the countable Boolean group does not have a Fréchet nonmetrizable topology while another countable group does?*

A similar question can be asked about countable sequential groups.

**Problem 3.** *Is it consistent with ZFC that some countable abelian (or any topologizable) group does not have a sequential topology with an intermediate (i.e.,  $> 1$  and  $< \omega_1$ ) sequential order while another group does?*

How strong is the interaction between the convergence and the group structure in sequential groups?

**Problem 4.** *Given a “natural” set of ordinals  $A \subset \omega_1$ , is it consistent with ZFC that a group with sequential order  $\alpha$  exists if and only if  $\alpha \in A$ ? Examples of  $A$  are finite ordinals, infinite ordinals, limit ordinals,  $\{\omega\}$ , etc.*

The combinatorics of pathological Fréchet group constructions and that of compact Fréchet spaces seem to be very different; hence, the following seemingly artificial question is asked.

**Problem 5.** *Is it consistent with ZFC that there exist a Fréchet group and a Fréchet compact space whose product is not Fréchet?*

## 11. $G_\delta$ PARTITIONS OF COMPACT SPACES

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**Question 1.** Does there exist a compact  $T_2$  space that can be partitioned into more than continuum  $G_\delta$  sets? Is there a bound on the size of such partitions?

The question arose from investigations by Santi Spadaro and Paul J. Szeptycki [53] into Arhangel'skii's question whether  $2^\omega$  is a bound on the weak Lindelöf degree of a compact space in its  $G_\delta$  topology. Indeed, they show that if there is a compact space that can be partitioned into  $\kappa$  many  $G_\delta$  sets, then there is a compact space whose weak Lindelöf degree is  $\kappa^+$  in its  $G_\delta$ -topology. Recall that Arhangel'skii did prove that no compact  $T_2$  space can be partitioned into more than continuum *closed*  $G_\delta$ 's.

## 12. DISCRETE REFLEXIVITY IN SOME NEW CONTEXTS

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All spaces under consideration are assumed to be Tychonoff. If  $Z$  is a space, then  $\Delta_Z = \{(z, z) : z \in Z\} \subset Z \times Z$  is its diagonal. Given a topological property  $\mathcal{P}$ , a space  $X$  is called *discretely*  $\mathcal{P}$  if the closure of every discrete subspace of  $X$  has  $\mathcal{P}$ . The property  $\mathcal{P}$  is *discretely reflexive* (in a class  $\mathcal{Q}$ ) if any space (from the class  $\mathcal{Q}$ ) which is discretely

$\mathcal{P}$  must have the property  $\mathcal{P}$ . V. V. Tkachuk proves in [54] that every discretely compact space is compact; however, it is an open question of A. V. Arhangel'skii and R. Z. Buzyakova [3] whether every discretely Lindelöf space is Lindelöf. This question is open even for function spaces.

**Problem 1** (Arhangel'skii and Buzyakova). *Suppose that  $\overline{D}$  is Lindelöf for every discrete  $D \subset C_p(X)$ . Must  $C_p(X)$  be Lindelöf?*

If we cannot prove that every discretely Lindelöf function space is Lindelöf, then it would be progress to show that a discretely Lindelöf  $C_p(X)$  has some property implied by Lindelöfness.

**Problem 2** ([55]). *Suppose that  $\overline{D}$  is Lindelöf for every discrete  $D \subset C_p(X)$ . Must  $C_p(X)$  be realcompact?*

Discrete Lindelöfness of the complement of the diagonal in the square is a much stronger property because if  $X$  is a space for which  $(X \times X) \setminus \Delta_X$  is Lindelöf, then  $X$  is Lindelöf and has a weaker metrizable topology. Therefore, weakening Lindelöfness of  $(X \times X) \setminus \Delta_X$  to discrete Lindelöfness, we can also expect strong consequences. For example, it was proved in [16] that if  $X$  is a countably compact space and  $(X \times X) \setminus \Delta_X$  is discretely Lindelöf, then  $X$  is compact and metrizable. This motivates the respective questions for function spaces asked in [55].

**Problem 3.** *Given a space  $X$ , assume that  $(C_p(X) \times C_p(X)) \setminus \Delta_{C_p(X)}$  is discretely Lindelöf. Must  $C_p(X)$  be Lindelöf?*

**Problem 4.** *Given a space  $X$ , assume that  $(C_p(X) \times C_p(X)) \setminus \Delta_{C_p(X)}$  is discretely Lindelöf. Must  $X$  be separable?*

Quite a few properties that fail to be discretely reflexive behave better in compact spaces. For example, it was proved in [2] that a compact space  $X$  must be first countable (Fréchet–Urysohn) if  $\overline{D}$  is for any discrete  $D \subset X$ . Since spaces  $C_p(X)$  have a rich algebraic structure compatible with their topology, we can also hope that more properties might be discretely reflexive in  $C_p(X)$ .

**Problem 5** ([55]). *Suppose that  $\overline{D}$  has countable pseudocharacter for any discrete set  $D \subset C_p(X)$ . Must the space  $C_p(X)$  have countable pseudocharacter?*

**Problem 6** ([55]). *Suppose that  $\overline{D}$  has the Fréchet–Urysohn property for any discrete set  $D \subset C_p(X)$ . Must the space  $C_p(X)$  have the Fréchet–Urysohn property?*

It is a classical theorem of Miroslaw Katětov [35] that a compact space  $X$  is metrizable if  $X^3$  is hereditarily normal. Therefore, a positive answer to the following question will be a strengthening of Katětov's theorem.

**Problem 7 ([1]).** *Suppose that  $X$  is a compact space such that  $\overline{D}$  is hereditarily normal for any discrete  $D \subset X^3$ . Must  $X$  be metrizable?*

István Juhász and Zoltán Szentmiklóssy establish in [33] that, for every compact space  $X$ , there exists a discrete subspace  $D \subset X \times X$  such that  $|D| = d(X)$ . Dennis Burke and Tkachuk [17] extend this result to Lindelöf  $\Sigma$ -spaces. They also establish in [17] that, for any Lindelöf  $p$ -space  $X$ , there exists a discrete subset  $D \subset X \times X$  such that  $\Delta_X \subset \overline{D}$ ; this implies, in particular, that the projection of  $D$  to the first coordinate is dense in  $X$ . An easy implication of the above-mentioned results is the existence, for any Lindelöf  $\Sigma$ -space  $X$ , of a discrete set  $D \subset X^3$  such that the projection of  $D$  onto the first coordinate is dense in  $X$ . However, the following question remains open.

**Problem 8 ([1]).** *Suppose that  $X$  is a  $\sigma$ -compact space. Does there exist a discrete  $D \subset X \times X$  such that the projection of  $D$  onto the first coordinate is dense in  $X$ ?*

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