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by

Kyriakos Keremedis

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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SECOND COUNTABLE UC METRIC SPACES ARE LEBESGUE IN ZF

KYRIAKOS KEREMEDIS

ABSTRACT. We show in the Zermelo-Fraenkel set theory ZF that

(i) if $\mathbf{X}$ is second countable, then $\mathbf{X}$ is Lebesgue if and only if it is *UC space* (: every continuous real valued function on $\mathbf{X}$ is uniformly continuous) if and only if it is *normal* (: the distance of every two disjoint, non-empty, closed subsets of $\mathbf{X}$ is strictly positive);

(ii) $\mathbf{X}$ is Lebesgue if and only if every open cover $\mathcal{U}$ of $\mathbf{X}$ has a refinement $\mathcal{V} = \{B(x, \delta) : x \in K\}$ for some $\delta > 0$ and some $K \subseteq X$ if and only if for every open cover $\mathcal{U}$ of $\mathbf{X}$ consisting of open balls there exists $\varepsilon > 0$ and a subcover $\mathcal{V}$ of $\mathcal{U}$ such that for every $V \in \mathcal{V}$, $\delta(V) > 2\varepsilon$;

(iii) $\mathbf{X}$ is complete if and only if it is almost complete;

(iv) $\mathbf{X}$ is almost Cauchy if and only if every sequence in $\mathbf{X}$ admits an almost Cauchy subsequence;

(v) every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$ has a Cauchy subsequence if and only if each countable subspace of $\mathbf{X}$ is almost compact if and only if each countable subspace of $\mathbf{X}$ is totally bounded;

(vi) $\mathbf{X}$ is sequentially compact if and only if each countable subspace of $\mathbf{X}$ is almost compact and almost complete;

(vii) if each countable subspace of $\mathbf{X}$ is totally bounded, then $\mathbf{X}$ is sequentially bounded.

1. Notation and Terminology

Let $\mathbf{X} = (X, d)$ be a metric space, $x \in X$, and $\varepsilon > 0$. $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ denotes the open ball in $\mathbf{X}$ with center x and radius

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