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HOMOGENEOUS TOPOLOGICAL SPACES

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Electronically published on October 18, 2020

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A SURVEY OF CARDINALITY BOUNDS ON HOMOGENEOUS TOPOLOGICAL SPACES

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Abstract. In this survey we catalogue the many results of the past several decades concerning bounds on the cardinality of a topological space with homogeneous or homogeneous-like properties. These results include van Douwen’s Theorem, which states $|X| \leq 2^{\pi w(X)}$ if $X$ is a power homogeneous Hausdorff space [26], and its improvements $|X| \leq d(X)^{\pi \chi(X)}$ [44] and $|X| \leq 2^{\ell(X)\pi f(X)}$ [19] for spaces $X$ with the same properties. We also discuss de la Vega’s Theorem, which states that $|X| \leq 2^{f(X)}$ if $X$ is a homogeneous compactum [25], as well as its recent improvements and generalizations to other settings. This reference document also includes a table of strongest known cardinality bounds on spaces with homogeneous-like properties. The author has chosen to give some proofs if they exhibit typical or fundamental proof techniques. Finally, a few new results are given, notably (1) $|X| \leq d(X)^{\pi \chi(X)}$ if $X$ is homogeneous and Hausdorff, and (2) $|X| \leq \pi \chi(X)^{\psi(X)}$ if $X$ is a regular homogeneous space. The invariant $\pi_{\chi}(X)$, defined in this paper, has the property $\pi_{\chi}(X) \leq \pi \chi(X)$ and thus (1) improves the bound $d(X)^{\pi \chi(X)}$ for homogeneous Hausdorff spaces. The invariant $\psi(X)$, defined in [33], has the properties $\psi(X) \leq \pi \chi(X)$ and $\psi(X) \leq \psi(X)$ if $X$ is Hausdorff, thus (2) improves the bound $2^{\ell(X)^{\pi \chi(X)}}$ in the regular, homogeneous setting.

1. Introduction

A topological space $X$ is homogeneous if for every $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. Roughly, $X$ is homogeneous if the topology at every point is "identical" to that of every other point. $X$ is power homogeneous if there exists a cardinal $\kappa$ such that $X^\kappa$ is homogeneous. Many commonly studied spaces are homogeneous (for example, $\mathbb{R}^2$, the unit circle, all connected manifolds in general, and topological groups) and as such are ubiquitous across fields of mathematics. In particular, those homogeneous

2020 Mathematics Subject Classification. 54D20, 54A25, 54D10.
Key words and phrases. cardinality bounds, cardinal invariants.
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