http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS

Volume 61, 2023

Pages 77–99

Bi-topological spaces and the Continuity Problem

by

Dieter Spreen

Electronically published on December 24, 2021

Topology Proceedings

Web:	http://topology.nipissingu.ca/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 61 (2023)
Pages 77-99

E-Published on December 24, 2021

BI-TOPOLOGICAL SPACES AND THE CONTINUITY PROBLEM

DIETER SPREEN

Abstract. The *Continuity Problem* is the question whether effective operators are continuous, where an effective operator F is a function on a space of constructively given objects x, defined by mapping construction instructions for x to instructions for $F(x)$ in a computable way. In the present paper the problem is dealt with in a bi-topological setting. To this end the topological setting developed by the author [22] is extended to the bi-topological case. Under very natural conditions it is shown that an effective operator F between bi-topological spaces $\mathcal{T} = (T, \tau, \sigma)$ and $\mathcal{T}' = (T', \tau', \sigma')$ is (effectively) continuous, if τ' is (effectively) regular with respect to σ'. A central requirement on $\mathcal{T}'$ is that bases of the neighbourhood filters of the points in T' can computably be enumerated in a uniform way, not only with respect to topology τ', but also with respect to σ'. As follows from an example by Friedberg, the last condition is indispensable. Conversely, it is proved that (effectively) bi-continuous operators are effective. Prominent examples of bi-topological spaces are quasi-metric spaces. Under a very reasonable computability requirement on the quasi-metric it is shown that all effectivity assumptions made in the general results are satisfied in the quasi-metric case.

2020 *Mathematics Subject Classification.* Primary 03F60; Secondary 03D45, 54D65, 54E35, 54E55.

Key words and phrases. Bi-topological space, pairwise regular, bi-continuous, quasi-pseudo-metric, constructive mathematics, recursive mathematics, numbering, effective operator, continuity problem.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 731143.