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Abstract. An open chain cover $\mathcal{U} = \{U_\alpha : \alpha \in \kappa\}$ (κ a cardinal) of a space X is a systematic cover if $\overline{U_\alpha} \subset U_\beta$ when $\alpha < \beta$, and X is Type I if $\kappa = \omega_1$ and each $\overline{U_\alpha}$ is Lindelöf. A closed subspace $D \subset X$ is narrow in X if for each systematic cover $\{V_\alpha : \alpha \in \omega_1\}$ of X, either there is α such that $D \subset V_\alpha$ or $\overline{V_\alpha} \cap D$ is Lindelöf for each α. Taking systematic covers given by $s^{-1}([0, \alpha))$ for a continuous $s \colon X \to \mathbb{L}_{\geq 0}$ (where $\mathbb{L}_{\geq 0}$ is the long ray) defines functionally Type I spaces and functionally narrow subspaces. For instance, $\mathbb{L}_{\geq 0}$ and ω_1 are narrow in themselves and any other space.

We investigate these properties and relative versions, as well as their relationship, and show in particular the following. There are functionally Hausdorff Type I spaces which are not functionally Type I, while regular Type I spaces are functionally Type I. We exhibit examples of spaces which are narrow in some but not in other spaces. There are subspaces of a Tychonoff space Y that are functionally narrow but not narrow in Y, while both notions agree if Y is normal. Under PFA and using classical results, any ω_1-compact locally compact countably tight Type I space contains a non-Lindelöf subspace narrow in it (a copy of ω_1, actually), while a Suslin tree does not. There are spaces with subspaces narrow in them that are essentially discrete. Finally, we investigate natural partial orders on (functionally) narrow subspaces and when these orders are ω- or ω_1-closed.

1. Introduction and Definitions

This paper is about the notions of Type I, functionally Type I, narrow, and functionally narrow spaces (and relative versions), and their relations.

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Key words and phrases. narrow subspaces, non-Lindelöf spaces, Type I spaces.