SATURATED LINEAR ORDERS and MARTIN'S AXIOM

Francisco Kibedi, May 2012

ABSTRACT

Let $\mathcal{P} = \{A \subseteq \omega : A$ is infinite and co-infinite$\}$, partially ordered by strict almost containment – i.e, $A \subset^* B$ iff $A \setminus B$ is finite and $B \setminus A$ is infinite. Also, let $(\omega)^\omega = \{f | f : \omega \longrightarrow \omega\}$, and consider two partial orders on $(\omega)^\omega$: divergence – i.e., $f \prec g$ iff $\lim g(n) - f(n) = +\infty$; and strict eventual domination – i.e., $f <^* g$ iff there is an M such that for all $n > M$, $f(n) \leq g(n)$, and $\{n \in \omega | f(n) < g(n)\}$ is infinite. Which linear orders of size up to continuum ($\mathfrak{c}$) embed into one of these three partial orders? This question (in the context of $(\omega)^\omega$) was raised originally by Solovay in connection with his and Woodin's work on homomorphisms of Banach Algebras, in particular with regard to the question of automatic continuity. Richard Laver showed in 1979 that it is consistent with the axioms of Set Theory (ZFC) that $\mathfrak{c}$ can be arbitrarily large, and every linear order of size up to continuum embeds into $(\omega)^\omega$. His method involves constructing (in a forcing extension) a saturated linear order of size $\mathfrak{c}$ in $(\omega)^\omega$ (i.e., a linear order of size $\mathfrak{c}$ in which every cut of size less than $\mathfrak{c}$ is filled). Is it possible to construct a saturated linear ordering in one of these partial orders that is also *maximal*? (Such maximal saturated linear orders are called *pantachies*, a term first introduced by du Bois-Reymond, and studied also by Hausdorff.) We answer this question in the affirmative.

Furthermore, we explore the question of whether it is possible to have Martin's Axiom (MA) and a saturated linear order in $\mathcal{P}$. Since the proper forcing axiom (PFA) implies that there is no saturated linear order in $\mathcal{P}$, this is a natural question. We show that it is consistent with ZFC that $\mathfrak{c} > \omega_1$, Martin's Axiom for σ-linked partial orders (MA(σ-linked)) holds, and there is a saturated linear order of size continuum in $\mathcal{P}$; moreover, this saturated linear order is maximal (i.e., a pantachie). Our method for the former result turns out to be similar to the method used by Baumgartner, Frankiewicz and Zbierski in their proof of Con($\neg$CH + MA(σ-linked) + every Boolean algebra of cardinality $\leq \mathfrak{c}$ can be embedded into $\mathcal{P}(\omega)$/fin).