Cell Structures

E.D. Tymchatyn
(with Wojciech Debski)

tymchat@math.usask.ca

May, 2015

Basic idea: Approximate spaces and continuous maps between spaces by simpler spaces and maps between those spaces .

Traditionally: Polyhedral or ANR approximations.

Simpler: Use discrete approximation.

Examples:

Bing-Convex metric:

Anderson-Menger curve characterization and homogeneity.

Outline of talk;
I.Simple example.

II. Polyhedral approximations and some of difficulties which arise.

III. Discrete approximations for complete metric spaces and their mappings.

IV. Extension to topologically complete spaces.

I. Simple Example

Theorem

X compact metric space $\Longrightarrow \exists\, f : C \twoheadrightarrow X$ (onto).

Outline of proof:

Construct discrete approximations $\mathcal{V}_i$ of X.

$\varprojlim(\mathcal{V}_i)$ is compact 0-dimensional metric space.

$[\varprojlim(\mathcal{V}_i)/\sim] \approx X$

Construct discrete approximations $\mathcal{U}_i$ of C

Maps $f_i : \mathcal{U}_i \to \mathcal{V}_i$ that commute with bonding maps.

Proof.

$$\mathcal{V}_0 = \{X\}$$
$$\uparrow \pi_0^1$$
$$\mathcal{V}_1 = \{V_{1,1}, \ldots, V_{1,n_1}\}, \quad \text{mesh } < 1$$
closed cover
$$\uparrow \pi_1^2$$
$$\mathcal{V}_2 = \mathcal{V}_{2,1} \cup \cdots \cup \mathcal{V}_{2,n_1}, \text{ mesh } < 1/2$$
$\mathcal{V}_{2,i}$ closed cover of $V_{1,i}$

$$\uparrow \pi_2^3$$
$$\ldots\ldots$$
$$\mathcal{V}_\infty = \varprojlim(\mathcal{V}_i, \pi_j^k)$$

□

Proof.

Define $\sim$ on $\mathcal{V}_\infty$ by $x = (x(0), x(1), \dots) \sim y = (y(0), y(1), \dots)$ iff $x(i) \cap y(i) \neq \varnothing$ for all i

$[x] = \{y \in \mathcal{V}_\infty : x \sim y\}$.

$x \sim y \Rightarrow \bigcap_{i=0}^{\infty} x(i) = \bigcap_{i=0}^{\infty} y(i)$ is a point of X

$$\pi \colon \mathcal{V}_\infty \to \mathcal{V}_\infty / \sim \ \approx X$$

□

Proof.

$$
\begin{array}{ccc}
\mathcal{U}_0 = \{C\} & \xrightarrow{f_0} & \mathcal{V}_0 = \{X\} \\
\uparrow \varphi_0^1 & & \uparrow \pi_0^1 \\
\mathcal{U}_1 = \{U_{1,1}, \dots, U_{1,n_1}\} & \xrightarrow{f_1} & \mathcal{V}_1 = \{V_{1,1}, \dots, V_{1,n_1}\}, \ \text{mesh} \ < 1 \\
\text{discrete} & & \text{closed cover} \\
\uparrow \varphi_1^2 & & \uparrow \pi_1^2 \\
\mathcal{U}_2 = \mathcal{U}_{2,1} \cup \dots \cup \mathcal{U}_{2,n_1} & \xrightarrow{f_2} & \mathcal{V}_2 = \mathcal{V}_{2,1} \cup \dots \cup \mathcal{V}_{2,n_1}, \ \text{mesh} \ < 1/2 \\
 & & \mathcal{V}_{2,i} \text{ closed cover of } V_{1,i} \\
\uparrow \varphi_2^3 & & \uparrow \pi_2^3 \\
\dots\dots & \dots\dots & \dots\dots \\
C \approx \varprojlim(\mathcal{U}_i, \varphi_i^j) & \xrightarrow{\tilde{f}=(f_0,f_1,\dots)} & \mathcal{V}_\infty = \varprojlim(\mathcal{V}_i, \pi_j^k)
\end{array}
$$

□

II. Approximation by polyhedra and ANRs

Alexandroff(1926): Approximate metric compacta by finite polyhedra.

Freudenthal (1937):Every compact metric space X admits a polyhedral inverse sequence with surjective bonding maps whose inverse limit is X. Dimension of approximating polyhedra $\leq$ dimension X.

Pasynkov(1958), Mardesic (1960): 1-dimensional chainable continuum not inverse limit of 1-dimensional polyhedra.

For mappings inverse limits are more problematic.

Mioduszewski (1963).

If X, Y metric compacta and $f\colon X = \varprojlim(P_i, p_i^j) \longrightarrow Y = \varprojlim(Q_i, q_i^j)$ then for each sequence $\{\varepsilon_i\}$ of positive numbers $\varepsilon_i \to 0$ there exists

$$
\begin{array}{ccccccccccc}
P_{m_1} & \xleftarrow{p_{m_1}^{m_2}} & P_{m_2} & \xleftarrow{p_{m_2}^{m_3}} & P_{m_3} & \longleftarrow & \dots & \longleftarrow & P_{m_k} & \xleftarrow{p_{m_k}^{m_{k+1}}} & \dots \\
f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & \dots & & f_k \downarrow & & \dots \\
Q_{n_1} & \xleftarrow{q_{n_1}^{n_2}} & Q_{n_2} & \xleftarrow{q_{n_2}^{n_3}} & Q_{n_3} & \longleftarrow & \dots & \longleftarrow & Q_{n_k} & \xleftarrow{q_{n_k}^{n_{k+1}}} & \dots
\end{array}
$$

so each diagram

$$
\begin{array}{ccccc}
 & & P_{m_k} & \longleftarrow & P_{m_r} \\
 & & \downarrow & & \downarrow \\
Q_{n_i} & \longleftarrow & Q_{n_k} & \longleftarrow & Q_{n_r}
\end{array}
$$

is ε_k commutative for $i < k < r$.

$f(x) = \{y_i\}_{i \in N}$ where $y_i = \lim_k q_i^{n_k} \circ f_k \circ p_{m_k}(x)$.

Mardesic (1981) Resolutions - inverse systems with restrictions to study non-compact cases.
For topologically complete spaces resolutions are inverse limits. Not all inverse limits are resolutions.

Mardesic - Watanabe (1989). Approximate inverse systems and approximate resolutions of arbitrary spaces and mappings and filling in the deficiencies indicated above.

III. Cell structures

Definition

- *Graph* is ordered pair (G, r).
- G is discrete set and r is reflexive and symmetric relation on G.
- *Cells* are points of G.
- a, b are adjacent if $(a, b) \in r$.
- $\mathrm{st_r}(a) = \{b \in G \mid (a, b) \in r\}$.

Let $\{(G_i, r_i) \mid i = 1, 2, \dots\}$ be graphs and $g_i^j\colon G_j \to G_i$ be functions for $j \geq i$ satisfying

i) $g_i^i =$ identity on G_i

ii) $g_i^k = g_i^j \circ g_j^k$ for $i < j < k$ and

iii) $(a, b) \in r_{i+1} \implies (g_i^{i+1}(a), g_i^{i+1}(b)) \in r_i$

$$G_1 \xleftarrow{g_1^2} G_2 \xleftarrow{g_2^3} G_3 \xleftarrow{g_3^4} \dots$$

If $a \in G_i$, say $\deg(a) = i$.
Let $\Pi = \Pi G_i$ topological product.
Π is complete, 0-dimensional, metric space.
$G_\infty = \varprojlim(G_i, g_i^j)$.
If $x = (x(1), x(2), \dots) \in \Pi$ then $x \in G_\infty$ iff $g_i^{i+1}(x(i+1)) = x(i) \forall i$.
G_∞ is set of *threads*.

If $a \in G_i$ let $< a >= \{x \in G_\infty \mid x(i) = a\}$.
$\{< x(i) >: x \in G_\infty, i = 1, 2, \ldots\}$ is basis of open and closed sets for G_∞.

Proposition

G_∞ is closed in Π.Hence,G_∞ is topologically complete, 0-dimensional, metric space.
eg. if each G_i is countable, G_∞ is closed subset of irrationals.

Definition

Set $x \sim y$ in G_∞ if $(x(i), y(i)) \in r_i$ for each i.
$\sim$ is reflexive and symmetric relation on G_∞.

Proposition

$\sim$ is closed in $G_\infty \times G_\infty$.

Proof.

$\sim = \bigcap R_i$ where $R_i = \{(x, y) \in G_\infty \times G_\infty \mid (x(i), y(i)) \in r_i\}$.
R_i is closed since r_i is closed in discrete space $G_i \times G_i$. □

Definition

Cells $a \in G_m$ and $b \in G_n$ are *close* if $(g_k^m(a), g_k^n(b)) \in r_k$ for $k = \min\{m, n\}$.

Cauchy Sequence is a sequence of cells $\{u(j)\}$ in $\bigcup G_i$ such that

1) $\lim \mathrm{deg}(\mathrm{u(j)}) = \infty$ and
2) $u(i)$ and $u(j)$ are close for all i and j sufficiently large.

Cauchy sequence $\{u(i)\}$ *converges* to thread $x \in G_\infty$ if $x(i)$ and $u(j)$ are close for all i and sufficiently large j.

Note. A Cauchy sequence may converge to different threads.

Definition

A *cell structure* is an inverse sequence of graphs satisfying

iv) $\forall$ thread $x \in G_\infty$, $\forall i$, $\exists j \geq i$ s.t. $g_i^j(\mathrm{st}_{r_j}^2(x(j))) \subset \mathrm{st}_{r_j}(x(i))$

v) $\forall$ thread x, $\forall i$, $\exists j \geq i$ s.t. $g_i^j(\mathrm{st}_{\mathrm{r_j}}(\mathrm{x(j)}))$ is finite.

A cell structure is *complete* if

vi) each Cauchy sequence of cells converges

Note. In general if iv) and v) are satisfied and each $st_{r_i}(a)$ is finite then vi) is satisfied.

Let $(*) = \{((G_i, r_i), g_i^j)\}$ be a cell structure. $\sim$ is transitive by iv). $x \sim y \sim z$ implies $x \sim z$.

Definition

For $x \in G_\infty$ let $[x] = \{y \in G_\infty \mid x \sim y\}$.
By v) $[x]$ is compact.
Define $\pi : G_\infty \to G_\infty / \sim = G^*$. π is perfect mapping.
G^*, the *space determined by the cell structure*, is topologically complete metrizable space.

Proposition

If $(*)$ *is cell structure then* $\{G^* \setminus \pi(G_i \setminus < A >) \mid A \subset G_i, i = 1, 2, \dots\}$ *is basis for topology on* G^*.

eg. $G_i = \{p10^{-i} \mid p \textit{ integer }\}$
$(x, y) \in r_i$ *iff* $|x - y| \leq 10^{-i}$.
g_i^j *an order preserving retraction.*
$G^* \equiv \mathbb{R}$.

Theorem (1)

Each complete metric space is homeomorphic to a space determined by some cell structure.

Hint: Let $\mathcal{V}_i$ be locally finite closed covers of X....

Cell maps.

Let

$$*)\quad G_0 \xleftarrow{g_0^1} G_1 \xleftarrow{g_1^2} G_2 \dots$$

$$*')\quad H_0 \xleftarrow{h_0^1} H_1 \xleftarrow{h_1^2} H_2 \dots$$

be cell structures

A function $f\colon \bigcup G_i \to \bigcup H_i$ is called a *cell map* of $(*)$ to $(*')$ if f takes close cells to close cells and Cauchy sequences to Cauchy sequences.

Proposition

The composition of cell maps is a cell map.

Theorem (2)

Let $f\colon \bigcup G_i \to \bigcup H_i$ *be a cell map of cell structure* $(*)$ *to complete cell structure* $(*')$*. Then* f *induces a continuous function* $f^*\colon G^* \to H^*$ *defined as follows. For* x*, a thread in* G_∞ $\quad f^*(\pi(x)) = \pi^{'}(y)$ *where* y *is a thread in* H_∞ *such that* $f(x)$ *converges to* y*.*

Theorem (3)

Let (X, d) *and* (Y, ρ) *be complete metric spaces. Let* $\{\mathcal{U}_i\}$ *and* $\{\mathcal{V}_i\}$ *sequences of locally finite closed covers of* X *and* Y *respectively such that* $\mathrm{mesh}(\mathcal{U}_i) < 1/i$, $\mathrm{mesh}(\mathcal{V}_i) < 1/i$, $\mathcal{U}_{i+1}$ *refines* $\mathcal{U}_i$ *and* $\mathcal{V}_{i+1}$ *refines* $\mathcal{V}_i$ *for each* i *and* $\mathcal{U}_0 = \{X\}$ *and* $\mathcal{V}_0 = \{Y\}$*. Then each continuous function* $F\colon X \to Y$ *is induced by a cell map of* $\bigcup \mathcal{U}_i$ *to* $\bigcup \mathcal{V}_i$*.*

Some classes of spaces. Let $\{((G_i, r_i), g_i^j)\}$ be a cell structure.

1) If each $r_i = \Delta$ then G^* yield all topologically complete 0-dimensional metric spaces.
2) If graphs G_i are finite then G^* yield all compact metric spaces.
3) If graphs G_i are finite and connected then G^* yield all metric continua.
4) If all G_i are finite trees then G^* yield all metric treelike continua.
5) If all G_i have each mutually adjacent set of cells of cardinality $\leq n+1$ then G^* yield all at most n-dimensional complete metric spaces.

Space X is *topologically complete* if it admits a complete uniformity.

Proposition

X is topologically complete iff it is homeomorphic to a closed subset of product of completely metrizable spaces.

Remark

Above results extend to spaces which are perfect images of closed subsets of a product of discrete spaces.
In particular, topologically complete spaces, paracompact spaces, metric spaces.

THANK YOU