Dynamical properties of maps with zero-dimensional sets of periodic points

Hisao Kato

University of Tsukuba

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Abstract

Abstract : In this talk, we study several dynamical properties of maps with zero-dimensional sets of periodic points. This is a joint work with my students: Y. Ikegami, A. Ueda and M. Hiraki.

Colorings and eventual colorings

Let $f : X \to X$ be a fixed-point free map of a metric space X, i.e., $f(x) \neq x$ for each $x \in X$. A subset C of X is called a *color* of f if $f(C) \cap C = \phi$. Note that $f(C) \cap C = \phi$ if and only if $C \cap f^{-1}(C) = \phi$. We say that a cover $\mathcal{C}$ of X is a *coloring* of f if each element C of $\mathcal{C}$ is a color of f. The minimal cardinality $C(f)$ of closed (or open) colorings of f is called the *coloring number* of f. The coloring number $C(f)$ has been investigated by many mathematicians.

The following is a classical theorem of this subject.

Theorem 2.1 (Lusternik and Schnirelman)

Let $f : S^n \to S^n$ be the antipodal map of the n-dimensional sphere S^n. Then $C(f) = n + 2$.

Theorem 2.2 (van Douwen)

If $f : X \to X$ is a fixed-point free (i.e., $P_1(f) = \phi$) and closed map of a finite-dimensional (separable) metric space X with $ord(f) = \sup\{|f^{-1}(x)| \mid x \in X\} < \infty$, then f is colorable and hence the extension $\beta f : \beta X \to \beta X$ of f is fixed-point free, where βX denotes the Stone-Čech compactification of the space X.

Theorem 2.3 (Buzyakova and Chigogidze)

If $f : X \to X$ be a fixed-point free map of a locally compact space X, then f is colorable.

Theorem 2.4 (Aarts, Fokkink and Vermeer)

Let $f : X \to X$ be a fixed-point free involution of a metric space X with $\dim X = n < \infty$. Then $C(f) \leq n + 2$.

Theorem 2.5 (Aarts, Fokkink and Vermeer)

(1) Let $f : X \to X$ be a fixed-point free homeomorphism of a metric space X with $\dim X = n < \infty$ or
(2) let X be a finite-dimensional compact metric space and let $f : X \to X$ be any fixed-point free map.
Then $C(f) \leq n + 3$.

Let $f : X \to X$ be a fixed-point free map of a space X and $p \in \mathbb{N}$. A subset C of X is *eventually colored within p of f* if

$$\bigcap_{i=0}^{p} f^{-i}(C) = \phi.$$

Note that C is a color of f if and only if C is eventually colored within 1.

Proposition 2.6

Let $f : X \to X$ be a fixed-point free map of a metric space X and $p \in \mathbb{N}$. Then a subset C of X is eventually colored within p of f if and only if each point $x \in C$ wanders off C within p, i.e., for each $x \in C$, $f^i(x) \notin C$ with some $i \leq p$.

We define the eventual coloring number $C(f,p)$ as follows. A cover $\mathcal{C}$ of X is called an *eventual coloring within* p if each element C of $\mathcal{C}$ is eventually colored within p. The minimal cardinality $C(f,p)$ of all closed (or open) eventual colorings within p is called the *eventual coloring number* of f within p. Note that $C(f,1) = C(f)$. If there is some $p \in \mathbb{N}$ with $C(f,p) < \infty$, we say that f is eventually colored. If there is some $p \in \mathbb{N}$ with $C(f,p) \leq k$, we say that f is *eventually* k*-colorable* $(k \geq 2)$.

For evaluating the eventual coloring numbers of maps, we define the following index $\psi_n(k)$. Let $n \in \mathbb{N} \cup \{0\}$ and $0 \leq k \leq n+1$. Put $R(n,k) = n - (n+2-k)[\frac{n}{n+2-k}]$, where $[x] = \max\{m \in \mathbb{N} \cup \{0\} |\ m \leq x\}$ for $x \in [0, \infty)$. Note that $R(n,k)$ means the remainder of n divided by $(n+2-k)$.

First, we put $\psi_n(0) = 1$ $(k = 0)$. Next we consider the following two cases (i) and (ii):

$$\text{(i) } R(n,k) < n + 1 - k.$$

$$\text{(ii) } R(n,k) = n + 1 - k.$$

For each $1 \leq k \leq n + 1$, we define the index $\psi_n(k)$ by

$$\psi_n(k) = \begin{cases} k(2[\frac{n}{n+2-k}] - 1) + 2 & (\text{if } R(n,k) < n + 1 - k), \\ k(2[\frac{n}{n+2-k}] + 1) + 1 & (\text{if } R(n,k) = n + 1 - k). \end{cases}$$

$$\dim X = n$$

$$C(f, \psi_n(k)) \leq n + 3 - k \ (k = 0, 1, 2, ..., n + 1)$$

Table of $\psi_n(k)$

nk	0	1	2	3	4	5	6
0	1	2	-	-	-	-	-
1	1	2	7	-	-	-	-
2	1	2	4	16	-	-	-
3	1	2	4	10	29	-	-
4	1	2	4	5	14	46	-
5	1	2	4	5	13	26	67

For any map $f : X \to X$, $P(f)$ denotes the set of all periodic points of f.

Theorem 2.7

Let $f : X \to X$ be a fixed-point free closed map of a metric space X with $ord(f) = \sup\{|f^{-1}(x)|\ x \in X\} < \infty$ and $\dim X = n < \infty$. If $\dim P(f) \leq 0$, then

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n + 1$. In particular, if f is a homeomorphism, then

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n + 1$.

For the locally compact case, we do not need the condition "$\mathrm{ord}(f) < \infty$". In fact, we have the following.

Theorem 2.8

Let $f : X \to X$ be any fixed-point free map of a locally compact metric space X with $\dim X = n < \infty$. If $\dim P(f) \leq 0$, then

$$C(f, \psi_n(k)) \leq n + 3 - k$$

for each $k = 0, 1, 2, ..., n + 1$.

Corollary 1

Suppose that
(1) $f : X \to X$ is a fixed-point free closed map of a metric space X such that $ord(f) < \infty$, $\dim X = n < \infty$ and $\dim P(f) \leq 0$ or
(2) $f : X \to X$ is a fixed-point free map of a locally compact metric space X such that $\dim X = n < \infty$ and $\dim P(f) \leq 0$.
Then the followings hold.
(a) If $\dim X = 0$, then $C(f, 2) = 2$.
(b) If $\dim X = 1$, then $C(f, 7) = 2$.
(c) If $\dim X = 2$, then $C(f, 16) = 2$.
(d) If $\dim X = 3$, then $C(f, 29) = 2$.
(e) If $\dim X = 4$, then $C(f, 46) = 2$.
(f) If $\dim X = 5$, then $C(f, 67) = 2$.

Example 2.9

If we do not assume $\dim P(f) \leq 0$, the above theorems are not true. Let $f : S^n \to S^n$ be the antipodal map of the n-dimensional sphere S^n. Note that $P(f) = S^n$ and $C(f,p) = C(f,1) = n+2$ for any $p \in \mathbb{N}$.

In the above resulst for noncompact case, we need the condition; $\mathrm{ord}(f) < \infty$.

Example 2.10

For the space P of all irrational numbers, there is a fixed-point free map $f : P \to P$ such that $P(f) = \phi$ and
(1) f is closed,
(2) f is finite-to-one, and
(3) f cannot be eventually colored within any $p \in \mathbb{N}$.
Let $\mathcal{U}$ be any finite open cover of P. Then there exist some $U \in \mathcal{U}$ and a point $x \in U$ such that $f^p(x) \in U$ for any $p \in \mathbb{N}$. This implies that $\mathcal{U}$ is not an eventual coloring within any $p \in \mathbb{N}$.

Now we have the following general problem for eventual coloring numbers.

Problem 2

For each $n \geq 0$ *and each* $1 \leq k \leq n+1$*, determine the minimal number* $p_n(k)$ *of natural numbers* p *satisfying the condition; if* $f : X \to X$ *is any fixed-point free homeomorphism of a separable metric space* X *such that* $\dim X = n$ *and* $\dim P(f) \leq 0$*, then* $C(f, p) \leq n + 3 - k$*.*

Compactifications and extensions of maps

In this section, we investigate metric compactifications preserving some properties of periodic points of maps.

For a map $f : X \to X$, let

$$P_i(f) = \{x \in X|\ f^j(X) = x \text{ for some } j\ (1 \leq j \leq i)\}.$$

Theorem 3.1

(1) Let X be a finite-dimensional separable metric space and let $f : X \to X$ be a closed map with ord $(f) < \infty$ or
(2) let X be a finite-dimensional locally compact metric space and let $f : X \to X$ be any map.
Then there exist a metric compactification αX of X and an extension $\alpha f : \alpha X \to \alpha X$ of f such that for each $i \in \mathbb{N}$,

$$\dim \alpha X = \dim X,$$

$$Cl_{\alpha X} P_i(f) = P_i(\alpha f),$$

$$\dim P_i(f) = \dim P_i(\alpha f).$$

Corollary 3

(1) Let X be a finite-dimensional separable metric space and let $f : X \to X$ be a closed map with ord $(f) < \infty$ or
(2) let X be a finite-dimensional locally compact metric space and let $f : X \to X$ be any map.
If $P(f) = \phi$, then there exist a (metric) compactification αX of X and an extension $\alpha f : \alpha X \to \alpha X$ of f such that $\dim \alpha X = \dim X$ and $P(\alpha f) = \phi$.

Corollary 4

(1) Let X be a finite-dimensional separable metric space and let $f : X \to X$ be a closed map with ord $(f) < \infty$ or
(2) let X be a finite-dimensional locally compact metric space and let $f : X \to X$ be any map.
Then the extension $\beta f : \beta X \to \beta X$ of f over the Cech-Stone compactification βX satisfies the condition $Cl_{\beta X}(P_i(f)) = P_i(\beta f)$ for each $i \in \mathbb{N}$.

Zero-dimensional covers of dynamical systems

In this section, we investigate zero-dimensional covers of dynamical systems with zero-dimensional sets of periodic points.

A pair (X, f) is called a *dynamical system* if X is a metric space and f is a homeomorphism of X onto itself. The dynamical system (Y, g) *covers* (X, f) *via* an onto map $p : Y \to X$ provided that $p \circ g = f \circ p$.

There is a classical theorem by W. Hurewicz.

Theorem 4.1 (W. Hurewicz, K. Kuratowski and K. Morita)

A metric space X is at most n-dimensional if and only if there is a metric zero-dimensional space Z with a closed onto map $p : Z \to X$ whose fibers have cardinality at most $n+1$.

In the theory of dynamical systems, we have the following problem:

Problem 5

Is every finite-dimensional dynamical system covered by a zero-dimensional dynamical system via a finite-to-one (more generally, countable-to-one) closed map?

J. Kulesza proved the following theorem:

Theorem 6 (J. Kulesza)

For each n-dimensional compact dynamical system (X, f) with zero-dimensional set of periodic points, there is a compact zero-dimensional dynamical system covering (X, f) via an at most $(n+1)^n$-to-one map.

In this talk, we generalize Kulesza's theorem in the case of nonseparable metric spaces, and improve the theorem. In fact, we prove that every (nonseparable) n-dimensional dynamical system with zero-dimensional set of periodic points can be covered by a zero-dimensional dynamical system via an at most 2^n-to-one closed map. Moreover, we study also periodic dynamical systems.

The following theorem is the main theorem in this section.

Theorem 7

Let $f : X \to X$ be a homeomorphism of a metric space X such that $\dim X = n < \infty$ and $\dim P(f) \leq 0$. Then, there is a homeomorphism $\tilde{f} : C \to C$ of a zero-dimensional metric space C and an at most 2^n-to-one closed surjective map $p : C \to X$ such that $p \circ \tilde{f} = f \circ p$. Moreover, if X is a compact (resp. separable) space, then C can be taken as a compact (resp. separable) space. In particular, $h(f) = h(\tilde{f})$.

As a special case we consider the case that $f : X \to X$ is a continuum-wise expansive homeomorphism of a compact metric space X. A homeomorphism $f : X \to X$ of a compact metric space (X, d) is *expansive* if there is $c > 0$ such that for any $x, y \in X$ with $x \neq y$, there is an integer $k \in \mathbb{Z}$ such that $d(f^k(x), f^k(y)) \geq c$. Let

$$I_0(f) = \bigcup \{M \mid M \text{ is a zero dimensional f-invariant closed set of } f\}.$$

Let $Y_k = \{1, 2, ..., k\}$ $(k \in \mathbb{N})$ be the discrete space having k-elements and let $Y_k^{\mathbb{Z}} = \prod_{-\infty}^{\infty} Y_k$ be the product space. Then the shift homeomorphism $\sigma : Y_k^{\mathbb{Z}} \to Y_k^{\mathbb{Z}}$ is defined by $\sigma((x_j)_j) = (x_{j+1})_j$.

Theorem 8

Let $f : X \to X$ be an expansive homeomorphism of a compact metric space X with $\dim X = n$. Then there exist $k \in \mathbb{N}$, a closed σ-invariant set Σ of $\sigma : Y_k^{\mathbb{Z}} \to Y_k^{\mathbb{Z}}$ and an at most 2^n-to-one onto map $p : \Sigma \to X$ such that $p \circ \sigma = f \circ p$ and $|p^{-1}(x)| = 1$ for any $x \in I_0(f)$.

J. Kulesza showed that if we do not assume $\dim P(f) \leq 0$, there is a compact one-dimensional dynamical system which can not be covered by a compact zero-dimensional dynamical system via a finite-to-one closed map.

Dynamical decomposition theorems of homeomorphisms with zero-dimensional sets of periodic points

In this section, we study some dynamical decomposition theorems of spaces related to given homeomorphisms. We introduce new notions of 'bright spaces' and 'dark spaces' of homeomorphisms except n times, and by use of the notions we prove some dynamical decomposition theorems of spaces related to given homeomorphisms.

For a homeomorphism $f : X \to X$ of a space X and $k \in \mathbb{N}$, let $P_k(f)$ denote the set of points of period $\leq k$. Also, $P(f)$ denotes the set of all periodic poins of f. A subset Z of X is a *bright space* of f except n times ($n \in \{0\} \cup \mathbb{N}$) if for any $x \in X$,

$$|\{p \in \mathbb{Z}|\ f^p(x) \notin Z\}| \leq n,$$

where $|A|$ denotes the cardinality of a set A. Also we say that $L = X - Z$ is a *dark space* of f except n times. Note that for any $x \in X$, $|O_f(x) \cap L| \leq n$, where $O_f(x) = \{f^p(x)|\ p \in \mathbb{Z}\}$ denotes the orbit of x, and also note that $L \cap P(f) = \phi$.

For a dark space L of f except n times and $0 \leq j \leq n$, we put

$$A_f(L,j) = \{x \in X| \ |\{p \in \mathbb{Z}| \ f^p(x) \in L\}| = j\},$$

$(= \{x \in X| \ |O_f(x) \cap L| = j\})$. $A_f(L,j)$ denotes the set of all point $x \in X$ whose orbit $O_f(x)$ appears in L just j times. Note that $P(f) \subset A_f(L,0)$ and $A_f(L,j)$ is f-invariant, i.e. $f(A_f(L,j)) = A_f(L,j)$ and $A_f(L,i) \cap A_f(L,j) = \phi$ if $i \neq j$. Hence we have the f-invariant decomposition related to the dark space L as follows;

$$X = A_f(L,0) \cup A_f(L,1) \cup \cdots \cup A_f(L,n).$$

The following theorem is well-known.

Theorem 5.1

A space X has at most dimension n (i.e. $\dim X \leq n$) if and only if X can be represented as a union of $(n+1)$ zero-dimensional subspaces of X.

The following fact may be known.

Proposition 5.2

Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism. Then there exist f-invariant zero-dimensional dense G_δ-sets $A_f(j)$ $(j = 0, 1, 2, ..., n)$ of X such that

$$X = A_f(0) \cup A_f(1) \cup \cdots \cup A_f(n).$$

Related to this section, Arts, Fokkink and Vermeer proved the following interesting theorem of dynamical systems of homeomorphisms under some dimensional conditions of periodic points.

Theorem 5.3 (Arts, Fokkink and Vermeer)

Suppose that $f : X \to X$ *is a homeomorphism of a (metric) space* X *with* $\dim X \leq n$ $(< \infty)$.
Then there exists a dense G_δ*-set* Z *of* X *such that* $\dim Z = 0$ *and*

$$X = Z \cup f(Z) \cup f^2(Z) \cup \cdots \cup f^n(Z)$$

if and only if $\dim P_k(f) < k$ *for each* $1 \leq k \leq n$.

Theorem 5.4

Suppose that X is a space with $\dim X = n\ (< \infty)$ and $f : X \to X$ is a homeomorphism. Then there exists a bright space Z of f except n times such that Z is a zero-dimensional dense G_δ-set of X and the dark space $L = X - Z$ of f is a $(n-1)$-dimensional F_σ-set of X if and only if $\dim P(f) \leq 0$.

Corollary 9

Suppose that X *is a space with* $\dim X = n$ $(< \infty)$ *and* $f : X \to X$ *is a homeomorphism. Then there exists a zero-dimensional* G_δ*-dense set* Z *of* X *such that for any* $(n+1)$ *integers* $k_0 < k_1 < \cdots < k_n$,

$$X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \cdots \cup f^{k_n}(Z)$$

if and only if $\dim P(f) \leq 0$.

By use of F_σ-dark spaces, we have the following decomposition theorem.

Theorem 5.5

Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism with $\dim P(f) \leq 0$. If L is a dark space of f except n times such that L is an F_σ-set of X and $\dim\ (X - L) \leq 0$, then $\dim A_f(L, j) = 0$ for each $j = 0, 1, 2, ..., n$. In particular, there is the f-invariant zero-dimensional decomposition of X related to the dark space L:

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

A homeomorphism $f : X \to X$ of a compact metric space (X, d) is *continuum-wise expansive* if there is $c > 0$ such that for any nondegenerate subcontinuum A of X, there is an integer $k \in \mathbb{Z}$ such that diam $f^k(A) \geq c$. Note that every expansive homeomorphism is continuum-wise expansive. Such $c > 0$ is called an *expansive constant* for f. It is known that if a compact metric space X admits a continuum-wise expansive homeomorphism f on X, then $\dim X < \infty$ and every minimal set of f is zero-dimensional. Moreover, $\dim I_0(f) \leq 0$, where

$$I_0(f) = \bigcup \{M |\ M \text{ is a zero-dimensional } f\text{-invariant closed set of } X\}.$$

In particular, $\dim P(f) \leq 0$.

In the case of continuum-wise expansive homeomorphisms, by use of compact dark spaces we obtain the following decomposition theorem.

Theorem 5.6

Suppose that X is a compact metric space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a continuum-wise expansive homeomorphism. Then there exists a compact $(n-1)$-dimensional dark space L of f except n times such that $\dim A_f(L, j) = 0$ for each $j = 0, 1, 2, ..., n$. In particular, there is the f-invariant zero-dimensional decomposition of X related to the compact dark space L:

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

Remark 5.7

(1) In the above theorem, the bright space $Z = X - L$ of f is open in X and n-dimensional.
(2) Suppose that $\dim X = 1$. Then L is a compact zero-dimensional dark space of f except 1 time such that $\dim A_f(L, j) = 0$ for each $j = 0, 1$ if and only if L is a zero-dimensional compactum such that $f^i(L) \cap L =$ for any $i \in \mathbb{N}$ and dim $(X - \cup_{i \in \mathbb{Z}} f^i(L)) = 0$.

Example 10

Let $f : I = [0,1] \to I$ be the 'tent' map of the unit interval I defined by $f(x) = 2x$ for $0 \leq x \leq 1/2$ and $f(x) = 2 - 2x$ for $1/2 \leq x \leq 1$. Consider the inverse limit

$$X = \{(x_i)_{i=1}^{\infty} \in I^{\infty} \mid f(x_{i+1}) = x_i \text{ for } i \in \mathbb{N}\} \subset I^{\infty}$$

of f. Then the shift map $\tilde{f}$ is a continuum-wise exapnsive homeomorphism of the Knaster continuum X. Consider the subset

$$L = \{(x_i)_{i=1}^{\infty} \in X \mid x_1 = 1\}.$$

Then L is a compact zero-dimensional dark space L of $\tilde{f}$ except 1 time such that $\dim A_{\tilde{f}}(L,0) = 0$. In fact, $X = A_{\tilde{f}}(L,0) \cup A_{\tilde{f}}(L,1)$ is a zero-dimensional decomposition of the Knaster continuum X.

Key Lemmas

To prove the above results, we need the following key lemmas.

Lemma 11

Let $\mathcal{C} = \{C_i |\ 1 \le i \le m\}$ be an open cover of a metric space X with $\dim X = n < \infty$ *and let $\mathcal{B} = \{B_i |\ 1 \le i \le m\}$ be a closed shrinking of $\mathcal{C}$. Suppose that O is an open set in X and Z is a zero-dimensional subset of O. Then there is an open shrinking $\mathcal{C}' = \{C'_i |\ 1 \le i \le m\}$ of $\mathcal{C}$ such that for each $i \le m$,*

(0) $B_i \subset C'_i$,

(1) $C'_i = C_i$ *if* $bd(C_i) \cap O = \phi$,

(2) $C'_i \cap (X - O) = C_i \cap (X - O)$,

(3) $bd(C'_i) \cap (X - O) \subset bd(C_i) \cap (X - O)$,

(4) $bd(C'_i) \cap Z = \phi$, *and*

(5) $\{bd(C') \cap O |\ C' \in \mathcal{C}'\}$ *is in general position.*

Lemma 12

Suppose that $f : X \to X$ *is a fixed-point free homeomorphism of a metric space* X *such that* $\dim X = n < \infty$ *and* $\dim P(f) \leq 0$. *Let* $\mathcal{C} = \{C_i |\ 1 \leq i \leq m\}$ *be an open cover of* X *and let* $\mathcal{B} = \{B_i |\ 1 \leq i \leq m\}$ *be a closed shrinking of* $\mathcal{C}$. *Then for any* $k \in \mathbb{N}$, *there is an open shrinking* $\mathcal{C}' = \{C'_i |\ 1 \leq i \leq m\}$ *of* $\mathcal{C}$ *such that*

(0) $B_i \subset C'_i$,

(1) $\{f^j(bd(C'))|\ C' \in \mathcal{C}', -k \leq j \leq k\}$ *is in general position,*

(2) $bd(C') \cap P(f) = \phi$ *for each* $C' \in \mathcal{C}'$.