Geometry of Scales

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- Big Idea
  - In small scale geometry, one consider collections of scales on X so that each scale can be interpreted as neighborhoods of a smaller scale.
  - In large scale geometry, one considers collections of scales on X so that each scale can be intepreted as the points of a bigger scale.

Let $\mathcal{C}$ be a collection of covers of X.

Small Scale Structure

$\mathcal{C}$ is a **small scale structure** (**uniform structure**) if

- for every $\mathcal{U}, \mathcal{V} \in \mathcal{C}$ there exists $\mathcal{W} \in \mathcal{C}$ that $st(\mathcal{W}, \mathcal{W})$ refines both $\mathcal{U}$ and $\mathcal{V}$.

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- **Optional (Anti)Hausdorff Property**: The union of any finite number of bounded sets is bounded.

Most Important Examples of these Structures

The Most Important Example: Assume X is metric. The ss-structure associated to the metric is the collection of covers which coarsen the cover by $\epsilon-$balls for some $\epsilon > 0$ (covers with positive Lebesgue number). The ls-structure associated to the metric is the collection of covers which refine the cover by $r-$balls for some $r > 0$ (covers with finite mesh).

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Topological Groups

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Let $\{U_\alpha : \alpha \in A\}$ be a neighborhood base at the identity such that $U_\beta \cdot U_\beta \subset U_\alpha$ for all $\alpha > \beta$. One can define a ss-structure by declaring the uniformly bounded covers of G to be the collections $\{gU_\alpha : g \in G\}$ for $\alpha \in A$.

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Remark

It is easy to see that this ss-structure generates the original topology on G.

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Let K be a compact subset of G. We define the uniformly bounded covers of G to be the collections $\{gK : g \in G\}$ as K ranges over all compact subsets of G.

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If G is countable and discrete then the above construction boils down to taking uniformly bounded covers to be $\{gF : g \in G\}$ where G ranges over the finite subsets of G. The resulting ls-structure is precisely the ls-structure inherited from the metric induced by the Cayley graph of G, provided G is finitely generated.

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Remark

The notion of translation is fundamental to applications of ls-geometry. For example, partial bijections of "bounded translation" on a metric space X act as partial isometries on $l^2(X)$ by shifting part of the domains of l^2 functions and killing the rest.

Small Scale Connections to Topology

- If X is metric, then $A \subset X$ is a neighborhood of a point x if there exists some $\epsilon > 0$ such that $st(x, \{B(y, \frac{\epsilon}{2}) : y \in X\}) = B(x, \epsilon) \subset A$.

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- Every small scale structure induces a topology on X as follows: $A \subset X$ is a neighborhood of a point x if there exists a uniform cover $\mathcal{U}$ such that $st(x, \mathcal{U}) \subset A$.

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- Every small scale structure induces a topology on X as follows: $A \subset X$ is a neighborhood of a point x if there exists a uniform cover $\mathcal{U}$ such that $st(x, \mathcal{U}) \subset A$.
- A compact Hausdorff space has a unique uniform structure that generates the topology: It consists of all coarsenings of finite open covers.

Reminder about Partitions of unity

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Barycentric Subdivision

A **derivative** of a continuous partition of unity $f : X \to K$ where K is a simplicial complex(with metric topology) is the induced partition of unity $X \to K \to b(K)$ where $b(K)$ is the first barycentric subdivision of K.

Dydak partitions of unity paper

Given a continuous partition of unity $f : X \to K$, the cover of X by the carriers of f are star refined by the carriers of the derivative of f.

Proposition

A topological Hausdorff space X is paracompact if and only if the collection of open covers of X forms a base for a uniform structure on X, and that uniform structure generates the original topology on X.

Compactness, paracompactness, barycentric subdivision, and topological groups are uniform concepts.

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Theorem

Let X be an ss-structure. The uniform covers on X are precisely the union of all uniform covers from pseudo-metric spaces (X, d) so that $id_X : X \to (X, d)$ is ss-continuous.

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Theorem

Let X be an ls-structure. The uniformly bounded covers on X are precisely the union of all uniformly bounded covers coming from ∞-metric spaces (X, d) so that $id_X : (X, d) \to X$ is ls-continuous.

Ostrand Type Result

Ostrand Theorem

Let $n \geq 0$. A paracompact space X has covering dimensions less than or equal to n if and only if for every open covering $\mathcal{U}$ of X, there exists an open refinement $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$ such that $\mathcal{V}_i$ is a disjoint family for each $1 \leq i \leq n+1$.

Disjointness

- Let X be metric and $R \geq 0$. A family of subsets $\mathcal{F}$ is said to be $R-$disjoint if $d(U, V) > R$ for each $U, V \in \mathcal{F}$. (They are disjoint at scale R.)

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Definition/Theorem

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$\Updownarrow$

2) X is said to have **asymptotic dimension at most** n provided that for each $R > 0$ there exists a uniformly bounded cover $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$ where $\mathcal{V}_i$ is an R-disjoint family for each $i = 1, 2, \ldots, n + 1$.

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1) There exists a uniformly bounded cover $\mathcal{V}$ which coarsens $\mathcal{U}$ with multiplicity at most $n+1$.

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2) There exists a uniformly bounded cover $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$ where $st(\mathcal{V}_i, \mathcal{U})$ is a disjoint collection for each $i = 1, 2, \ldots, n$.