Asymptotic dimensions and absolute extensors

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Asymptotic dimensions

All metrics are assumed to be proper, i.e., all closed balls are compact.

Definition

Given a family $\mathcal{A}$ of subsets of a metric space X, we say that $\mathcal{A}$ is uniformly bounded if

$$\mathrm{mesh}(\mathcal{A}) = \sup\{\mathrm{diam}(A) \mid A \in \mathcal{A}\} < \infty.$$

Definition

Given $D > 0$, we say that a family $\mathcal{A}$ of subsets of a metric space X is D-disjoint if

$$d(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\} > D$$

for all distinct $A, B \in \mathcal{A}$.

Asymptotic category

The objects: proper metric spaces.

The morphisms: proper asymptotically Lipschitz maps.

Given metric spaces (X, d) and (Y, ϱ), a map $f\colon X \to Y$ is called asymptotically Lipschitz if there exist $\lambda > 0$ and $s \geq 0$ such that

$$\varrho(f(x), f(y)) \leq \lambda d(x, y) + s$$

for all $x, y \in X$.

Asymptotic dimension

By the definition (M. Gromov), the asymptotic dimension of a metric space (X, d) does not exceed n ($\mathrm{asdim} X \leq n$) if for all $R > r_0$ one can find R-disjoint uniformly bounded families $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$ of subsets of X such that $\cup_{i=0}^{n} \mathcal{U}_i$ covers X.

Asymptotic Assouad-Nagata dimension

By the definition, the asymptotic Assouad-Nagata dimension of a metric space (X, d) does not exceed n ($\mathrm{asdim}_{AN} X \leq n$) if there exist $c > 0$ and $r_0 > 0$ such that for all $R > r_0$ one can find R-disjoint and cR-bounded families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ of subsets of X such that $\cup_{i=0}^{n} \mathcal{U}_i$ covers X.

Symmetric powers

Recall that, for any subgroup G of the symmetric group S_n, the G-symmetric product $SP^n_G X$ is defined as follows. Denote by $\sim$ the following equivalence relation on X^n:

$$(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$$

if there exists $\sigma \in G$ such that $y_i = x_{\sigma(i)}$, for every $i = 1, \ldots, n$. Denote by $[x_1, \ldots, x_n]$ the equivalence class containing $(x_1, \ldots, x_n)$. Then

$$SP^n_G X = \{[x_1, \ldots, x_n] \mid (x_1, \ldots, x_n) \in X^n\} = X^n/G.$$

The metric $\hat{d}$ on $SP^n_G X$ is defined as follows:

$$\hat{d}([x_1, \ldots, x_n], [y_1, \ldots, y_n]) = \min_{\sigma \in G} \max_{1 \leq i \leq n} d(x_i, y_{\sigma(i)}).$$

A multiple-valued function in the sense of Almgren is a map of the form $f\colon X \to SP^n(Y)$ where X and Y are metric spaces.
According to J. Goblet, F. J. Almgren introduced these functions to tackle the problem of estimating the size of the singular set of mass-minimizing integral currents.
Almgren's multiple-valued functions are a fundamental tool for understanding geometric variational problems in codimension higher than 1.

Asymptotic dimension of symmetric powers

[J. Goblet] (essentially):

$$\mathrm{asdim}_{AN} SP^n(X) \leq (\mathrm{asdim}_{AN}(X)) + 1)^n - 1.$$

[Shukel-Z.] (using the mappings into polyhedra):

$$\mathrm{asdim}_{AN} SP^n(X) \leq n \mathrm{asdim}_{AN}(X)$$

[Shukel-Radul] (using Kolmogorov's trick):

$$\mathrm{asdim} SP^n_G(X) \leq n \mathrm{asdim}(X)$$

[Kucab-Z.]:

$$\mathrm{asdim}_{AN} SP^n_G(X) \leq n \mathrm{asdim}_{AN}(X)$$

Generalizations

This can be generalized over some other functorial constructions.
E.g., hypersymmetric powers

$$\exp_n(X) = \{A \in \exp X \mid A \neq \emptyset,\ |A| \leq n\}.$$

Asymptotic power dimension

We use the notation (λ, B)-dim $X \leq n$ in the meaning that there exist λ-disjoint and B-bounded families $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ of subsets of X such that $\cup_{i=0}^{n} \mathcal{U}_i$ covers X.
We say that the asymptotic power dimension of X does not exceed n ($\mathrm{asdim}_P X \leq n$) if there exists $\alpha > 0$ and $r_0 > 0$ such that (r, r^α)-dim $X \leq n$ for all $r > r_0$. As usual we say that $\mathrm{asdim}_P X = n$ if $\mathrm{asdim}_P X \leq n$ and it is not true that $\mathrm{asdim}_P X \leq n - 1$.

Inequalities

$$\operatorname{asdim} X \leq \operatorname{asdim}_P X \leq \operatorname{asdim}_{AN} X$$

There exists a metric space X with $\operatorname{asdim} X = 0$, $\operatorname{asdim}_P X = 1$ and $\operatorname{asdim}_{AN} X = 2$.
We let $X = \cup_{i=1}^{\infty} X_i$, where X_i is isometric to $\{(im, in) \mid m, n \in \mathbb{N},\ m \leq 2^i,\ n \leq i^2\}$, $i \in \mathbb{N}$. We place the "rectangles" X_i consequently above the positive x-axis so that their lower side is on this axis and the distance between X_i and X_{i-1} is i. The metric on X is inherited from $\mathbb{R}^2$.

It is well-known that, for any metric d on a set X, the function $d' = \ln(1 + d)$ is also a metric on X. The following is, in some sense, a special case of a result from [Damian Sawicki, Remarks on coarse triviality of asymptotic Assouad-Nagata dimension, Topology Appl. Volume 167, 2014, 69–75.].

Theorem

Let (X, d) be a metric space. Then $\mathrm{asdim}_P(X, d) = \mathrm{asdim}_{AN}(X, d')$.

Theorem

Let (X, d) *be a metric space of finite asymptotic power dimension. Then*

$$\mathrm{asdim}_P SP^n_G(X) \leq n\,\mathrm{asdim}_P X.$$

Question

Is there an invariant metric d on the group of integers $\mathbb{Z}$ such that

$$\operatorname{asdim}(\mathbb{Z}, d) \leq \operatorname{asdim}_P(\mathbb{Z}, d) \leq \operatorname{asdim}_{AN}(\mathbb{Z}, d)?$$

Absolute extensors

Let $\mathcal{C}$ be a subcategory of the asymptotic category.

Definition

A space X is an absolute extensor for a class $\mathcal{C}$ ($X \in AE(\mathcal{C})$) if for every morphism $f\colon A \to X$ in $\mathcal{C}$, where A is a closed subobject of Y there exists an extension $\bar{f}\colon Y \to X$ of f.

Let $\mathcal{AN}(\omega)$ denote the subcategory of the asymptotic category whose objects are spaces of finite asymptotic Assouad-Nagata dimension.

Theorem (O. Shukel)

Let X be a proper metric space and $X \in AE(\mathcal{AN}(\omega))$. Then $SP^n(X) \in AE(\mathcal{AN}(\omega))$.

The proof is based on results of the paper [Goblet J. Lipschitz extension of multiple Banach-valued functions in the sense of Almgren // arXiv: math/0609606v1 [math.MG] 21 sep. 2006.]

Theorem

Let X be a proper metric space and $X \in AE(\mathcal{AN}(\omega) \cap \mathcal{P}(\alpha))$, where $\alpha \leq \omega$. Then $SP^n(X) \in AE(\mathcal{AN}(\omega) \cap \mathcal{P}(\alpha))$.

Hölder maps

Let X, Y be metric spaces. A map $f\colon X \to Y$ is said to be Hölder if there are $C, \alpha > 0$ such that

$$\varrho(f(x), f(y)) \leq Cd(x, y)^{\alpha}, \ x, y \in X.$$

A map $f\colon X \to Y$ is said to be asymptotically bi-Hölder if there exist $X' \subset X$ and $Y' \subset Y$ such that the restrictions $f|X'\colon X' \to Y'$ and $f^{-1}|Y'\colon Y' \to X'$ are Hölder and X', Y' are metrically dense in X, Y respectively.

Hölder invariance

Theorem

If X and Y are bi-Hölder equivalent, then $\mathrm{asdim}_P X = \mathrm{asdim}_P Y$.

Higson corona

Let (X, d) be a proper metric space. A bounded function $\varphi\colon X \to \mathbb{R}$ is said to be *slowly oscillating* if

$$\lim_{x\to\infty} \operatorname{diam}(B_r(x)) = 0$$

for all $r > 0$.

The algebra of all bounded slowly oscillating functions determines the *Higson compactification* of X; its remainder is called the *Higson corona* of X and is denoted by $\nu(X)$.

Sublinear corona

A. Dranishnikov and J. Smith introduced the *sublinear corona*.
Let $x_0 \in X$ be a base point. For $x \in X$ let $|x| = d(x, x_0)$.
The *sublinear corona* $\nu_L(X)$ is the remainder of the compactification of X generated by the algebra of bounded functions φ satisfying

$$\lim_{x \to \infty} \mathrm{diam}(B_{\alpha(|x|)}(x)) = 0$$

for all sublinear functions α.
(The latter means that $\lim_{t \to \infty}(\alpha(t)/t) = 0$.)

Asymptotic sublinear inductive dimension

Two closed subsets A, B of X are *asymptotically sublinearly* disjoint if $\bar{A} \cap \bar{B} \cap \nu_L(X) = \emptyset$.
A closed subset C of X is an *asymptotically sublinear* separator between asymptotically sublinearly disjoint subsets A and B if $\bar{C} \cap \nu_L(X)$ is a separator between $\bar{A} \cap \nu_L(X)$ and $\bar{B} \cap \nu_L(X)$ in $\nu_L(X)$.
Using this notion one can define, in a standard way, the *asymptotic sublinear inductive dimension* asInd_L.

Questions

1. $$\mathrm{asdim}_{AN}(X) = \mathrm{asInd}_L(X)?$$
2. Does the asymptotic power dimension of a proper metric space and the covering dimension of the subpower corona of this space coincide?

Thank you!