On lattice of maps

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Let f and g be two maps defined on a space X. If there exists a map $h : g[X] \to f[X]$ such that $f = h \circ g$, then we will briefly write $g \prec f$.

We say that X has a *multiplicative lattice of maps* (Shchepin 1976) if there exists a family Ψ of maps with domain X such that:

(ML1) For any map $f \colon X \to f(X)$ there exists $g \in \Psi$ with $g \prec f$ and $w(g(X)) \leq w(f(X))$;

(ML2) If $\{f_\alpha : \alpha \in A\} \subset \Psi$, then the diagonal $\triangle\{f_\alpha : \alpha \in A\} \in \Psi$.

If Ψ consists of open , d-open or skeletal maps, then it is called a multiplikative lattice of open, d-open or skeletal maps, respectively.

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(L1) For any map $f\colon X \to f(X)$ there exists $g \in \Psi$ with $g \prec f$ and $w(g(X)) \leq w(f(X))$;

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If Ψ consists of open , d-open or skeletal maps, then it is called a lattice of open, d-open or skeletal maps, respectively.

If Ψ consists of maps onto second countable spaces and the following conditions are fulfilled:

(ω-ML1) For any map $f\colon X \to f[X]$, such that $w(f[X]) = \aleph_0$, then there exists $g \in \Psi$ with $g \prec f$;

(ω-ML2) If $\{f_n : 0 \leq n\} \subset \Psi$, then $\triangle\{f_n : 0 \leq n\} \in \Psi$,

then the family Ψ is called an *ω-multiplicative lattice of maps*.

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(ω-L1) For any map $f\colon X \to f[X]$, such that $w(f[X]) = \aleph_0$, then there exists $g \in \Psi$ with $g \prec f$;

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then the family Ψ is called an ω- *lattice of maps*.

In accordance with J. Mioduszewski and L. Rudolf (1969), a surjective map $f : X \to Y$ is said to be *skeletal*;(resp. *d-open*; M. Tkachenko 1981) if

$$\operatorname{Int} \overline{f[U]} \neq \varnothing$$

(resp. $f[U] \subseteq \operatorname{Int} \overline{f[U]}$) for every nonempty open $U \subseteq X$.

Obviously, every d-open map is skeletal. Moreover, any d-open map between compact Hausdorff spaces is always open

Skeletally Dugundji spaces

We say that a space X is *skeletally Dugundji* if there exists an inverse system $S = \{X_\alpha, p^\beta_\alpha, \alpha < \beta < \tau\}$ with surjective skeletal bonding maps, satisfying the following conditions:

(i) X_0 is a separable metric space and all maps $p^{\alpha+1}_\alpha$ have metrizable kernels (i.e., there exists a separable metric space M_α such that $X_{\alpha+1}$ is embedded in $X_\alpha \times M_\alpha$ and $p^{\alpha+1}_\alpha$ coincides with the restriction $\pi|X_{\alpha+1}$ of the projection $\pi\colon X_\alpha \times M_\alpha \to X_\alpha$);

(ii) for any limit ordinal $\gamma < \tau$ the space X_γ is a (dense) subset of $\varprojlim\{X_\alpha, p^\beta_\alpha, \alpha < \beta < \gamma\}$;

(iii) X is embedded in $\varprojlim S$ such that $p_\alpha(X) = X_\alpha$ for each α, where $p_\alpha\colon \varprojlim S \to X_\alpha$ is the α-th limit projection;

(iv) for every bounded continuous real-valued function f on $\varprojlim S$ there exists $\alpha \in A$ such that $p_\alpha \prec f$ (i.e. there exists a continuous function g on X_α with $f = g \circ p_\alpha$).

Theorem (E. Shchepin (1981))

For a compact Hausdorff Tychonoff space X the following are equivalent:

(i) *X is Dugundji;*

(ii) *X has a multiplicative lattice of open maps.*

Theorem (A.K., Sz. Plewik, V. Valov, (2013))

For a Tychonov space X the following are equivalent:

(i) *X is skeletally Dugundji;*

(ii) *X has a multiplicative lattice of skeletal maps.*

Open-open game

Consider two person game of countable lenght.

- player I chooses a non-empty open set $A_0 \subseteq X$
- player II chooses a non-empty open set $B_0 \subseteq A_0$

at the n-th inning

- player I chooses a non-empty open subset $A_n \subseteq X$
- player II chooses a non-empty open subset $B_n \subseteq A_n$

player I wins, whenever the union $B_0 \cup B_1 \cup \ldots \subseteq X$ is dense in X, otherwise player II wins.

Open-open game were introduced by P. Daniels, K. Kunen and H. Zhou *On the open-open game*, Fund. Math. 145 (1994), no. 3, 205 - 220.

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I-favorable space

The space X is called I-favorable, whenever player I can be insured that he wins no matter how player II plays.

Let $(X, \mathcal{T})$ be a topological space. Denote by $\mathcal{T}^n$ the set of all finite sequence of length n consisting of open non-empty set.

A winning strategy for player I is a function

$$\sigma : \bigcup\{\mathcal{T}^n : n < \omega\} \to \mathcal{T}$$

such that for any sequence $\{B_n : n < \omega\} \subseteq \mathcal{T}$ satisfying the property $B_0 \subseteq \sigma(\emptyset)$ and $B_{n+1} \subseteq \sigma(\{B_0, B_1, \ldots, B_n\})$, for any $n \in \omega$, then the union $\bigcup\{B_n : n < \omega\}$ is dense in X.

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Game and club families

Denote by $[\mathrm{coZ}(X)]^\omega$ the collection of all countable and infinite families of co-zero sets. According to P. Daniels, K. Kunen and H. Zhou (1994), a family $\mathcal{C} \subseteq [\mathrm{coZ}(X)]^\omega$ is said to be a *club* if:

(i) $\mathcal{C}$ is closed under increasing ω-chains, i.e., if $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$ is an increasing ω-chain from $\mathcal{C}$, then $\bigcup\{\mathcal{A}_n : 0 < n\} \in \mathcal{C}$;

(ii) for any $\mathcal{B} \in [\mathrm{coZ}(X)]^{\leq\omega}$ there exists $\mathcal{A} \in \mathcal{C}$ with $\mathcal{B} \subseteq \mathcal{A}$,

We say that a family $\mathcal{P}$ of open subset of a topological space X has the property Seq if:

(Seq) For every $W \in \mathcal{P}$ there exist sequences $\{U_n : 0 \leq n\} \subseteq \mathcal{P}$ and $\{V_n : 0 \leq n\} \subseteq \mathcal{P}$ such that $U_k \subseteq X \setminus V_k \subseteq U_{k+1}$, for each k, and $\bigcup\{U_n : 0 \leq n\} = W$.

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A club $\mathcal{C}$ is said to be a *c-club* if it satisfies the following condition:

(iii) If $\mathcal{P} \in \mathcal{C}$, then $\mathcal{P} \subset_c \mathrm{coZ}(X)$, $\mathcal{P}$ has the property Seq and $\mathcal{P}$ is closed under finite unions and finite intersections.

The relationship $\mathcal{P} \subset_c \mathrm{coZ}(X)$ means that $\mathcal{P} \subseteq \mathrm{coZ}(X)$ and

(c) For any nonempty $V \in \mathrm{coZ}(X)$ there exists $W \in \mathcal{P}$ such that if $U \in \mathcal{P}$ and $U \subseteq W$, then $U \cap V \neq \varnothing$.

The condition (c) may be replaced by the following:

(c^*) For any $\mathcal{W} \subseteq \mathcal{P}$, the family $\mathcal{W}$ is predense in $\mathcal{P}$ if and only if $\mathcal{W}$ is predense in $\mathrm{coZ}(X)$.

A family $\mathcal{W} \subseteq \mathcal{P}$ is *predense in* $\mathcal{P} \subseteq \mathrm{coZ}(X)$, whenever for each $P \in \mathcal{P}$ there exist $V \in \mathcal{W}$ and $Q \in \mathcal{P}$ such that $Q \subseteq V \cap P$.

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(c^*) For any $\mathcal{W} \subseteq \mathcal{P}$, the family $\mathcal{W}$ is predense in $\mathcal{P}$ if and only if $\mathcal{W}$ is predense in $\mathrm{coZ}(X)$.

A family $\mathcal{W} \subseteq \mathcal{P}$ is *predense in* $\mathcal{P} \subseteq \mathrm{coZ}(X)$, whenever for each $P \in \mathcal{P}$ there exist $V \in \mathcal{W}$ and $Q \in \mathcal{P}$ such that $Q \subseteq V \cap P$.

A club $\mathcal{C}$ is said to be a *c-club* if it satisfies the following condition:

(iii) If $\mathcal{P} \in \mathcal{C}$, then $\mathcal{P} \subset_c \text{coZ}(X)$, $\mathcal{P}$ has the property Seq and $\mathcal{P}$ is closed under finite unions and finite intersections.

The relationship $\mathcal{P} \subset_c \text{coZ}(X)$ means that $\mathcal{P} \subseteq \text{coZ}(X)$ and

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(d) for any $\mathcal{S} \subset \mathcal{P}$ and $x \notin \overline{\bigcup \mathcal{S}}$, there exists $W \in \mathcal{P}$ such that $x \in W$ and $W \cap \bigcup \mathcal{S} = \varnothing$.

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I has winning strategy in open-open game if and only if $\{\mathcal{P} \in [\mathrm{coZ}(X)]^\omega : \mathcal{P} \subset_c \mathrm{coZ}(X)\}$ *contains a club.*

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Additive club

For every family $\mathcal{A}$ of open subsets of X we define

$$\langle\mathcal{A}\rangle = \bigcup\{\langle\mathcal{A}\rangle^n : 0 \leq n\},$$

where $\langle\mathcal{A}\rangle^0 = \mathcal{A}$ and $\langle\mathcal{A}\rangle^{k+1}$ is equal to

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If $\mathcal{C}$ is a c-clubs, (a d-club) and $\langle\mathcal{A}_1 \cup \mathcal{A}_2\rangle \in \mathcal{C}$ for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{C}$, then $\mathcal{C}$ is called a *additive c-club* (*additive d-club*).

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Main results

Theorem (AK, Sz. Plewik, V. Valov)

For a Tychonoff space X the following conditions are equivalent:

(1) *X has an ω-multiplicative lattice of skeletal maps;*

(2) *There exists an additive c-club for X;*

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Theorem (AK, Sz. Plewik, V. Valov)

For a Tychonoff space X the following conditions are equivalent:

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Assume that $\mathcal{C}$ is an additive c-club.

If we have the following property:

$\langle \bigcup \mathcal{R} \rangle \subset_c \operatorname{coZ}(X)$, for each family $\mathcal{R} \subseteq \mathcal{C}$.

Then we can construct maps using the following operation:

Let $\mathcal{P}$ be a family of open subsets of a topological space X. For every $x \in X$ consider the set

$$[x]_{\mathcal{P}} = \{y \in X : y \in V \Leftrightarrow x \in V \text{ for all } V \in \mathcal{P}\}.$$

Let $X/\mathcal{P}$ be the family of all classes $[x]_{\mathcal{P}}$ and $q_{\mathcal{P}} : X \to X/\mathcal{P}$ be the map $x \mapsto [x]_{\mathcal{P}}$.

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Assume that $\mathcal{W} \subseteq \langle \bigcup \mathcal{R} \rangle$ is predense in $\langle \bigcup \mathcal{R} \rangle$.

Next, choose a countable subfamily $\mathcal{W}^* \subseteq \mathcal{W}$ such that $\overline{\cup \mathcal{W}^*} = \overline{\cup \mathcal{W}}$.

Because, the space X satisfies the countable chain condition, so the choice of $\mathcal{W}^*$ is possible.

For every $W \in \mathcal{W}^*$ there exists countably many elements $\mathcal{P}_{\alpha_1}, \ldots, \mathcal{P}_{\alpha_k}$ of $\mathcal{R}$ such that $W \in \langle \mathcal{P}_{\alpha_1} \cup \ldots \cup \mathcal{P}_{\alpha_k} \rangle$.

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So, there exist families $\mathcal{P}_n \in \mathcal{R}$ such that

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Lemma

Suppose $\mathcal{P} \subset_c \mathrm{coZ}(X)$ is closed under finite intersections and $\mathcal{W} \subseteq \mathcal{P}$. If the family $\mathcal{W} \subseteq \mathcal{P}$ is predense in $\mathcal{P}$, then $\mathcal{W}$ is predense in $\mathrm{coZ}(X)$.

Suppose $\mathcal{W}$ is not predense in $\mathrm{coZ}(X)$. So, we can fix $A \in \mathrm{coZ}(X)$ such that $A \cap V = \varnothing$ for each $V \in \mathcal{W}$.

Using the condition (c), fix $B \in \mathcal{P}$ such that if $B^* \in \mathcal{P}$ and $B^* \subseteq B$, then $A \cap B^* \neq \varnothing$.

Since $\mathcal{W}$ is predense in $\mathcal{P}$, we choose $V \in \mathcal{W}$ such that $V \cap B \neq \varnothing$ and $V \cap B \in \mathcal{P}$.

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Using the condition (c), fix $B \in \mathcal{P}$ such that if $B^* \in \mathcal{P}$ and $B^* \subseteq B$, then $A \cap B^* \neq \varnothing$.

Since $\mathcal{W}$ is predense in $\mathcal{P}$, we choose $V \in \mathcal{W}$ such that $V \cap B \neq \varnothing$ and $V \cap B \in \mathcal{P}$.

Again by (c), we get $A \cap V \cap B \neq \varnothing$, which contradicts $A \cap V = \varnothing$.

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In view of the previous Lemma the family $\mathcal{W}^*$, being predense in $\langle \bigcup \{\mathcal{P}_n : 0 \leq n\} \rangle \subset_c \mathrm{coZ}(X)$, has to be predense in $\mathrm{coZ}(X)$.

Since predensity of any family in $\mathrm{coZ}(X)$ is equivalent to the density of the family in X, $\mathcal{W}$ is also predense in $\mathrm{coZ}(X)$.

Finally, $\langle \bigcup \mathcal{R} \rangle \subset_c \mathrm{coZ}(X)$.

Suppose that it is not true. So, we can fix $V \in \mathrm{coZ}(X)$ such that for each $U \in \langle \bigcup \mathcal{R} \rangle$ there exists a co-zero set $A \subseteq U$, which satisfies $A \in \langle \bigcup \mathcal{R} \rangle$ and $A \cap V = \varnothing$.

In such a case, the family

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