Monotone Classes of Dendrites

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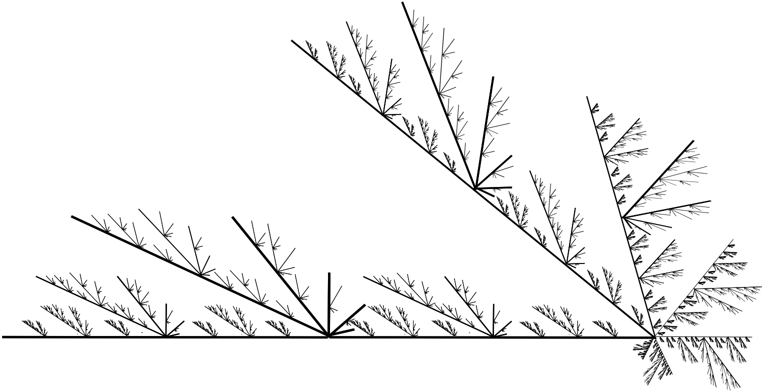
A *continuum* is a compact connected metric space.

A *dendrite* is a locally connected continuum without simple closed curves.

A map $f : X \longrightarrow Y$ is said to be *monotone* if $f^{-1}(y)$ is connected for all $y \in f(X)$.

Hence, there is a natural *quasi-order* placed on the set of dendrites $\mathcal{D}$ by $X \leq Y$ iff there exists a monotone onto map $f : Y \longrightarrow X$.

Two dendrites are said to be *monotone equivalent* if there exists monotone onto maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$.



Note: If T can be embedded in X, then $T \leq X$. Hence universal dendrite $D_\omega \geq T$ for every dendrite T..

A dendrite X is monotonically isolated if whenever Y is monotone equivalent to X implies that X is homeomorphic to Y.

Theorem

(Matrinez-de-la-Vega,M) A dendrite X is isolated with respect to monotone maps if and only if the set of ramification points of X is finite.

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If $A \subset D$ then $R(A)$ is the set of ramification points of X intersected with A.

A *tree* , T is a dendrite such that for each subarc $I \subset T$, $R(I)$ is finite.

Theorem

(Nash-Williams) If $\{T_i\}$ is a sequence of trees then there exists an N such that for every $i \geq N$, there is a $j_i > i$ such that $T_i \leq T_{j_i}$.

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Suppose that there exists an arc $A \subset D$ such that $R(A)$ is infinite. Then D is called a *comb* and A is called a *spine* of D.

Suppose that there exists an arc $A \subset X$ such that $\overline{R(A)}$ is homeomorphic to $\overline{\{1/n\}}_{n=1}^{\infty}$. Then D is called a *harmonic comb*.

A comb D is a *countable comb* if $\overline{R(A)}$ is countable for every arc $A \subset D$.

On the other hand, if there exists a spine A such that $\overline{R(A)}$ is uncountable, then A is called a *wild spine* and D is called a *wild comb*.

Let X be a wild comb with wild spine A. A is *perfect* if for every $y \in R(A)$ and arc $B \subset A$ such that $\overline{R(B)}$ is uncountable, there exists $x \in R(B)$ such that $T_y \preceq_r T_x$.

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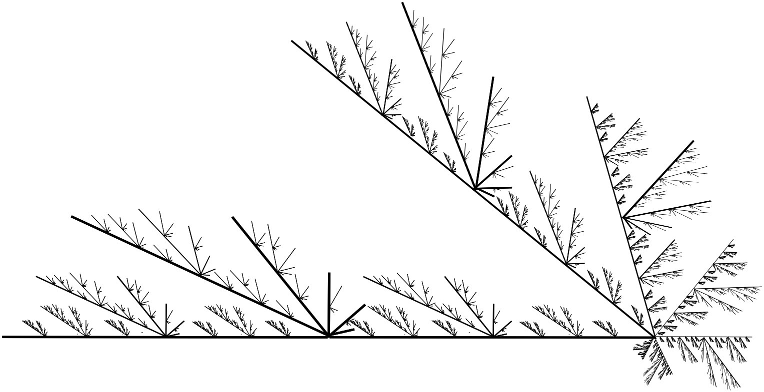
Wild Combs

1. If X is a wild comb with a perfect spine that contains a free countable comb, then X is not monotonically isolated.
2. If X is a wild comb with a perfect spine such that no perfect spine contains a free arc, then X is not monotonically isolated.
3. If X is a wild comb with a perfect spine such that contains a free arc, then X is not monotonically isolated.
4. If X is a wild comb that contains no perfect spine and no free countable comb, then X is monotonically equivalent to D_3.



The universal dendrite $D_\omega \geq D$ is monotone equivalent to D_3 (Charatonik).

WQO and BQO

A quasi-ordered set Q is *well-quasi-ordered (wqo)* if every strictly descending sequence is finite and every antichain (collection of pairwise incomparable elements) is finite.

Let Q be quasi-ordered under $\leqq$ and define the following quasi-ordering, $\leqq_1$, on the power set $\mathcal{P}(Q)$ by $X \leqq_1 Y$ if and only if there exists a function $f : X \longrightarrow Y$ such that $x \leqq f(x)$ for each $x \in X$, where $X, Y \in \mathcal{P}(Q)$.

Rado [**?**] constructed a quasi-ordered set Q such that Q was wqo but $\mathcal{P}(Q)$ was not. So a stronger notion of well-quasi-ordering called *better-quasi-ordered (bqo)* was constructed by Nash-Williams that preserved the property under the power set.

The definition of bqo we give is due to Laver [?] and is equivalent to but less technical than Nash-Williams [?]: Q is bqo if $\mathcal{P}^{\omega_1}(Q)$ is wqo. Here $\mathcal{P}^{\omega_1}(Q)$ is defined inductively by:

1. $\mathcal{P}^0(Q) = Q$.
2. if α is a successor ordinal then $\mathcal{P}^{\alpha+1}(Q) = \mathcal{P}(\mathcal{P}^{\alpha}(Q))$
3. if β is a limit ordinal then define $\mathcal{P}^{\beta} = \bigcup_{\alpha<\beta} \mathcal{P}^{\alpha}(Q)$.

Also, $\mathcal{P}^{\omega_1}(Q)$ is quasi-ordered by $\leqq_{\omega_1}$, which is a natural extension of both $\leqq$ and $\leqq_1$, and is defined inductively on $\alpha, \beta < \omega_1$ in the following way: Suppose that $X \in \mathcal{P}^{\alpha}(Q)$, $Y \in \mathcal{P}^{\beta}(Q)$, then $X \leqq_{\omega_1} Y$ if and only if

1. If $\alpha = 0$, $\beta = 0$ then $X \leqq Y$ since $X, Y \in Q$.
2. If $\alpha = 0$, $\beta > 0$ then there exists $Y' \in Y$ such that $X \leqq_{\omega_1} Y'$.
3. If $\alpha > 0$, $\beta > 0$ then then for every $X' \in X$ there exists $Y' \in Y$ such that $X' \leqq_{\omega_1} Y'$.

Question: Is the set of dendrites wqo under monotone onto maps?

Is it bqo?

Thank You!