Dimension-like invariants, their usage and computation.

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May 25–29, 2015

12-th Annual Workshop on Topology and Dynamical Systems Nipissing University

Contents

The usage of dimension-like invariants

- Extend classical dimensions on the wider class of spaces.
- Find invariants for which classical dimension theorems (sum theorem, subset theorem, compactification theorem etc.) hold.

Covering dimension dim was introduced by Čech for normal spaces.

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Definition [M. Katetov]

$\dim X \leq n$ for a Tychonoff space X if any finite cozero-set cover of X has a refinement of order $\leq n$.

Big Inductive dimension Ind was introduced by Čech for normal spaces.

Definition [E. Čech 1931]

$\operatorname{Ind} X = -1$ iff $X = \emptyset$.
$\operatorname{Ind} X \leq n$ if for any two disjoint closed sets F_1 and F_2 there exists their partition L (i.e. $X \setminus L = O_1 \cup O_2$, $F_i \subset O_i$, O_i is open in X) such that $\operatorname{Ind} L \leq n - 1$.
$\operatorname{Ind} X = \infty$ if the inequality $\operatorname{Ind} X \leq n$ doesn't hold for any n.

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[V. Filippov, M. Charalambous 1973]

Dimension Ind_0 of a Tychonoff space X is defined analogously to Ind but instead of closed and open sets zero-sets and cozero-sets are considered.

Dimension-like invariants of covering type

General approach

All spaces are Tychonoff.

Let C be the family of open covers of X.
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Theorem (equality $C - \dim X = \dim bX$)

Let for the family C of finite open covers of X and its compactification bX the following conditions hold.

(A) Any element of C can be extended to bX (i.e., for any $v = \{V_1, \ldots, V_k\} \in C$ there exists an open cover $v' = \{V_1', \ldots, V_k'\}$ of compactification bX such that $V_i' \cap X = V_i$, $i = 1, \ldots, k$).

(B) In any finite open cover of bX an extension of cover from C is refined.

Then $C - \dim X = \dim bX$.

(A) Completions with respect to precompact uniformities [P. Samuel, 1948, Yu. Smirnov, 1952].

(B) Using the one-to-one correspondence between compactifications and complete subrings of functions of $C^*(X)$ [I. Gelfand, 1941].

(C) The Wallman method constructing compactifications using normal bases [N. Shanin, O. Frink 1964].

δ-dimension of Smirnov of a proximity space (X, δ) [Yu. Smirnov 1956]

correspond to the case of $C-\dim$, where C is the family of all δ-uniform covers.

δ-dimension of Yu. Smirnov and uniform dimension δd_μ of J. Isbell

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Theorem [Yu. Smirnov 1956, J. Isbell 1964]

Let precompact reflection of uniformity μ correspond to proximity δ. Then

$$\delta d_\mu X = \delta\text{-dimension}\, X = \dim b_\mu X,$$

where $b_\mu X$ is the Samuel compactification of X with respect to μ.

$$\{\delta d_\mu X : \mu \text{ is uniformity on } X\} =$$

$$= \{\dim bX : bX \text{ is compactification of } X\}$$

Normal base

Definition [O. Frink 1964]

A base $\mathcal{F}$ for the closed subsets of X is said to be *normal* if:
(i) $\mathcal{F}$ is a *ring of sets on the set* X, i. e. the empty set $\emptyset$ and the set X are elements of $\mathcal{F}$, and $\mathcal{F}$ is closed under finite unions and finite intersections.
(ii) $\mathcal{F}$ is *disjunctive*: for a given closed set G of X and any point x not in G, there exists an element $F \in \mathcal{F}$ which contains x and is disjoint from G.
(iii) $\mathcal{F}$ is *base-normal*: for a given pair $F_1,\ F_2$ of disjoint elements of $\mathcal{F}$ there exists a pair $G_1,\ G_2$ of elements of $\mathcal{F}$ such that $F_1 \cap G_2 = \emptyset$, $F_2 \cap G_1 = \emptyset$, and $G_1 \cup G_2 = X$.

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The Wallman compactification of X with respect to normal base $\mathcal{F}$ is denoted $\omega(X, \mathcal{F})$.

Definition [S. Iliadis 2005]

Dimension $\mathrm{d}(X,\mathcal{F})$ of X by the normal base $\mathcal{F}$ correspond to the case of $C-\dim$ where C is the family of finite open covers of sets from $\mathcal{F}^c = \{X \setminus F : F \in \mathcal{F}\}$.

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Theorem [2012]

$\{\mathrm{d}(X,\mathcal{F}) : \mathcal{F}$ is normal base $X\} =$
$= \{\dim \omega X : \omega X$ is Wallman type compactification of $X\}$ and

$$\mathrm{d}(X,\mathcal{F}) = \dim \omega(X,\mathcal{F}) = \delta d_{\mu} X$$

for some uniformity μ on X.

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V. Ul'yanov [1977] has shown that not all compactifications are of Wallman type.

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Definition [2012]

For the uniformly closed ring of bounded functions $\mathcal{R}$ on X the dimension $\mathrm{d}(X, \mathcal{R})$ is $C - \dim$ for the family C of all finite covers by elements of $CZ(\mathcal{R})$ (coset-sets of functions from $\mathcal{R}$).

Definition [S. Mrowka 1973]

bX is the β-like compactification of X if $bX = \omega(X, \mathcal{F})$, where $\mathcal{F}$ is the normal base of zero-sets of functions on X which are extended to bX.

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$\{\mathrm{d}(X, \mathcal{R}) : \mathcal{R}$ is uniformly closed ring of bounded functions on $X\} =$
$= \{\dim bX : bX \; - \; \beta$-like compactification of $X\}$

For a uniform space (X, μ) the preimage of open (closed) set under the uniformly continuous function $f : X \to [0, 1]$ is called uniformly open (closed).

Uniform dimension of M. Charalambous and relative dimension of A. Chigogidze

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For arbitrary space Y and its subset $X \subset Y$ denote by $CZ(X, Y) = X \cap CZ(Y)$ $(Z(X, Y) = X \cap Z(Y))$ where $CZ(Y)$ $(Z(Y))$ is the set of cozero-sets (zero-sets) of Y.

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Definition. Relative dimension $\mathrm{d}(X, Y)$ [A. Chigogidze 1977]

Relative dimension $\mathrm{d}(X, Y)$ of Chigogidze of $X \subset Y$ correspond to the case $C - \dim$ where C is the set of finite covers of sets of $CZ(X, Y)$.

Theorem [2012]

- For a uniform space (X, μ),

$$\mu\text{-}\dim X = \mathrm{d}(X, \mathcal{R}_\mu),$$

  where $\mathcal{R}_\mu$ is the ring of all bounded uniformly continuous functions on X.
- For a subset X of a space Y,

$$\mathrm{d}(X, Y) = \mathrm{d}(X, C^*(X, Y)).$$

Dimension dim [E. Čech 1933, M. Katetov]

Classical dimension dim introduced by Čech for normal and by Katetov for Tychonoff space X is a special case of dimension δd for maximal uniformity μ on X. Therefore,

$$\dim X = \mathrm{d}(X, C^*(X)) = \mathrm{d}(X, Z(X)) = \mu - \dim X = \delta d_\mu(X).$$

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Coincidence of all dimensions on compactum

If X is a compact space then there is the unique uniformity μ on X; there is the unique uniformly closed ring of bounded functions is $C^*(X)$; dimension $\mathrm{d}(X, \mathcal{F})$ of X by the normal base $\mathcal{F}$ doesn't depend on the choice of normal base $\mathcal{F}$. Thus

$$\dim X = \mathrm{d}(X, C^*(X)) = \mathrm{d}(X, Z(X)) = \mu - \dim X = \delta d_\mu X.$$

Examples of usage of dimension-like invariants of covering type

Definition [B. Pasynkov 1965]

Elementary partial product $P = P(X, O, Z)$ of base X on fiber Z relatively open subset O is a topological space which is the disjoint union of $X \setminus O$ and $O \times Z$, i.e. $P = (X \setminus O) \cup (O \times Z)$, with the base of topology on P:

(a) open subsets of the product $O \times Z$;

(b) preimages of open in X sets under the natural projection $\pi : P \to X$, where
$\pi(x, z) = x$ for $(x, z) \in O \times Z$,
$\pi(x) = x$ for $x \in X \setminus O$.

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$\pi(x, z) = x$ for $(x, z) \in O \times Z$,
$\pi(x) = x$ for $x \in X \setminus O$.

Theorem [1992]

If a subset O of X is a cozero-set then for any space Z the inequality is valid:

$$\dim\ P(X, O, Z) \leq \dim\ X \times Z.$$

Proof

(1) Since $O = \cup\{F_i : i \in \mathbb{N}\}$, $F_i \in Z(X)$,
$P(X, O, Z) = \cup\{F_i \times Z : i \in \mathbb{N}\} \cup (X \setminus O)$,
$F_i \times Z \in Z(P(X, O, Z))$, $X \setminus O \in Z(P(X, O, Z))$.

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(2) By the countable sum theorem for the relative dimension d of Chigogidze:
$\dim P(X, O, Z) \leq \max\{\mathrm{d}(F_i \times Z, P(X, O, Z)), \mathrm{d}(X \setminus O, P(X, O, Z))\}$.

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(3) $CZ(F_i \times Z, X \times Z) = CZ(F_i \times Z, O \times Z) = CZ(F_i \times Z, P(X, O, Z))$,
$CZ(X \setminus O, X) = CZ(X \setminus O, P(X, O, Z))$.
Hence, $\mathrm{d}(F_i \times Z, P(X, O, Z)) = \mathrm{d}(F_i \times Z, X \times Z)$,
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Hence, $\mathrm{d}(F_i \times Z, P(X, O, Z)) = \mathrm{d}(F_i \times Z, X \times Z)$,
$\mathrm{d}(X \setminus O, P(X, O, Z)) = \mathrm{d}(X \setminus O, X)$.

(4) By the subset theorem for the relative dimension d of Chigogidze:
$\mathrm{d}(F_i \times Z, X \times Z) \leq \mathrm{d}(X \times Z, X \times Z) = \dim X \times Z$,
$\mathrm{d}(X \setminus O, X) \leq \dim X \leq \dim X \times Z$.

Definition [M. Charalambous 2004]

A subspace X of Y is called *countably accessible*, if for any cozero-set G of X there are functions $f_{i,s} : X \to [0,1]$, $i \in \mathbb{N}$, $s \in S$, such that each $f_{i,s}|\mathrm{coz} f_{i,s}$ has a continuous extension from Y to $[0,1]$; $\{\mathrm{coz} f_{i,s} : s \in S\}$ is locally finite in X for each i; and G is open in the topology on X, generated by $\{f_{i,s} : i \in \mathbb{N}, s \in S\}$.

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Theorem [M. Charalambous 2004]

Let X be a countably accessible subspace of Y. Then

$$\dim X \le \dim Y.$$

Dimension-like invariants of type Ind (big inductive dimension)

Definition. Dimension $\mathrm{I}(X, \mathcal{F})$ of X by the normal base $\mathcal{F}$ [S. Iliadis 2005]

Let $\mathcal{F}$ be a normal base of X. *Big inductive dimension* I *of* X *by the normal base* $\mathcal{F}$ is the integer greater or equal to -1 defined by induction:

(a) $\mathrm{I}(X, \mathcal{F}) = -1$ iff $X = \emptyset$;

(b) $\mathrm{I}(X, \mathcal{F}) \leq n$, where n is not negative integer iff for any two disjoint closed sets $F_1, F_2 \in \mathcal{F}$ there exists their partition $L \in \mathcal{F}$ (i.e. $X \setminus L = O_1 \cup O_2$, $F_i \subset O_i$, $X \setminus O_i \in \mathcal{F}$ is open in X) such that $\mathrm{Ind}\, L \leq n - 1$.

Recall that for a uniform space (X, μ) the preimage of closed set under the uniformly continuous function $f : X \to [0, 1]$ is called uniformly closed.

Big inductive uniform dimension of M. Charalambous, big inductive relative dimension of A. Chigogidze

Recall that for a uniform space (X, μ) the preimage of closed set under the uniformly continuous function $f : X \to [0, 1]$ is called uniformly closed.

Definition. Big inductive uniform dimension [M. Charalambous 1973]

Uniform dimension $\mu-$Ind of a uniform space (X, μ) corresponds to the case $\mathrm{I}(X, \mathcal{F})$, where $\mathcal{F}$ is the normal base on X of uniformly closed sets.

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Recall that for a space Y by $Z(Y)$ we denote the set of all zero-sets of Y and for $X \subset Y$ put $Z(X, Y) = X \cap Z(Y)$.

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Definition. Big inductive relative dimension [A. Chigogidze 1977]

Relative dimension $\mathrm{I}(X, Y)$ of subset $X \subset Y$ corresponds to the case $\mathrm{I}(X, \mathcal{F})$, where $\mathcal{F}$ is the normal base on X of sets $Z(X, Y)$.

Big inductive dimension Ind_0 of V. Filippov, M. Charalambous, dimension Ind

Definition of Ind_0 [V. Filippov, M. Charalambous 1973]

Dimension Ind_0 of X correspond to the case $\mathrm{I}(X, \mathcal{F})$, where the normal base $\mathcal{F}$ on X is $Z(X)$.

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Definition of Ind [E.Čech 1931]

Big inductive dimension Ind of a normal space X correspond to the case $\mathrm{I}(X, \mathcal{F})$, where $\mathcal{F}$ is the normal base of closed sets of X.

Application of the big inductive dimension I by the normal base.

On the inequality $\operatorname{Ind} X \times Y > \operatorname{Ind} X + \operatorname{Ind} Y$

Theorem [V. Filippov 1972]

There exist compacta X, Y such that

$$\mathrm{Ind} X = 1, \mathrm{Ind} Y = 2 \;\text{ и }\; \mathrm{Ind} X \times Y \geq \mathrm{ind} X \times Y \geq 4.$$

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Estimation [B. Pasynkov 1969]

$4 \leq \mathrm{Ind}X \times Y \leq 7.$

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Theorem [A. Karassev, K. Kozlov 2015]

$\mathrm{Ind} X \times Y = 4.$

Let $\mathcal{F}_i$ be the normal base on X_i, $i = 1, 2$. Then the set of all finite unions of elements of the form $F_1 \times F_2$ where $F_i \in \mathcal{F}_i$, $i = 1, 2$, is called *the product of normal bases* $\mathcal{F}_i$ and denoted by $\mathcal{F}_1 \otimes \mathcal{F}_2$.

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$\mathcal{F}_1 \otimes \mathcal{F}_2$ is a normal base on the product space $X_1 \times X_2$.

Theorem [2009]

Let $\mathcal{F}_i$ be the normal base on X_i, $i = 1, 2$. If the finite sum theorem is fulfilled for I in factors X_i by normal bases $\mathcal{F}_i$, $i = 1, 2$, then

$$\mathrm{I}(X_1 \times X_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \leq \mathrm{I}(X_1, \mathcal{F}_1) + \mathrm{I}(X_2, \mathcal{F}_2).$$

How to compute dimension Ind of products

Fact

For any normal base $\mathcal{F}$ on a compact space X we have

$$\mathrm{Ind}X \leq \mathrm{I}(X, \mathcal{F}).$$

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If $\mathrm{I}(X, \mathcal{F}) = 1$ for a normal base $\mathcal{F}$ on a space X then the finite sum theorem for base dimension I by $\mathcal{F}$ is fulfilled.

How to compute dimension Ind of products

Fact

For any normal base $\mathcal{F}$ on a compact space X we have

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Method

We must construct a normal base $\mathcal{F}$ on two-dimensional factor in Filippov's example for the base-dimension by which the (weak) finite sum theorem is fulfilled.

Results about the big inductive dimension I by the normal base.

Let $f\colon Y \to X$ be an irreducible map of an extremally disconnected space Y onto a compactum X.
Let us fix point z_0 in compactum Z and define an equivalence relation E on $Z \times Y$. For $(z, y), (z', y') \in Z \times Y$, put $(z, y)\ E\ (z', y')$ if $z = z'$ and either $y = y'$, or $z = z_0, f(y) = f(y')$. The quotient space is denoted as $\Psi = \Psi(Z, Y, X, f, \{z_0\}) = (Z \times Y)/E$, the quotient map as q. The continuous open map is defined $p\colon \Psi \to Z$ such that $\mathrm{pr} = p \circ q$, where pr is the projection of product.

Lemma [2015]

Let $\mathcal{F}(Z)$ be the normal base on Z, $\mathcal{F}(X)$ be the normal base on X. Let $j\colon \mathcal{F}(X) \to \mathcal{F}_{clop}(Y)$ be the correspondence:

$$j(F) = \mathrm{cl}(f^{-1}(\mathrm{int} F)),\ \ F \in \mathcal{F}(X).$$

The family $\mathcal{F}(\Psi)$ is the finite union of elements

- $q(B \times F) \cong B \times F$, where $B \in \mathcal{F}(Z)$, $F \in \mathcal{F}_{clop}(Y)$, $\{z_0\} \notin B$;
- $q((B \setminus \{z_0\}) \times F') \cup (\{z_0\} \times F)$, where $B \in \mathcal{F}(Z)$, $\{z_0\} \in B$, $F' \in \mathcal{F}_{clop}(Y)$, $F \in \mathcal{F}(X)$, $q((B \setminus \{z_0\}) \times F') \cong (B \setminus \{z_0\}) \times F'$, $j(F) = F'$

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Theorem [2015]

$$\mathrm{I}(\Psi, \mathcal{F}(\Psi)) \leq \max\{\mathrm{I}(X, \mathcal{F}(X)), \mathrm{I}(Z, \mathcal{F}(Z))\}.$$

Theorem [A. Karassev 2015]

For any normal base $\mathcal{F}$ on a linear ordered compactum J we have

(i) $\mathsf{I}(J, \mathcal{F}) = 0$ iff J is hereditarily disconnected;

(ii) $\mathsf{I}(J, \mathcal{F}) = 1$ or $\mathsf{I}(J, \mathcal{F}) = \infty$ iff J is not hereditarily disconnected.

In particular, for any normal base $\mathcal{F}$ on the segment $[0, 1]$ either $\mathsf{I}([0, 1], \mathcal{F}) = 1$, or $\mathsf{I}([0, 1], \mathcal{F}) = \infty$.

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Theorem [2015]

For any $n \in \mathbb{N}$, $n \geq 2$, the set of all inductive dimensions I by the normal bases of I^n is $\{k \in \mathbb{N} : k \geq n\} \cup \{\infty\}$.

Lemma [A. Karassev 2015]

Suppose that for the normal base $\mathcal{F}$ on a linearly ordered compactum J we have $\mathrm{I}(J, \mathcal{F}) < \infty$. Then for any two closed disjoint subsets of J there exists their $\mathcal{F}$-partition which is hereditarily disconnected.

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Theorem [V. Chatyrko 2010]

Let X be a compact space and $\mathcal{F}$ a normal base on X such that $\mathrm{I}(X, \mathcal{F}) = 0$. Then $\mathrm{I}(X, \mathcal{F}') = 0$ for every normal base $\mathcal{F}'$ on X.

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Example [V. Chatyrko 2010]

Let X be the closed interval $[a, b]$, $a < b$, of the real line and Q_1, Q_2 (countable) dense subsets of X such that $a \in Q_1$ and $b \in Q_2$. Then the family $\mathcal{F}_{Q_1,Q_2}$ of all finite unions of elements of the family $\{[x, y] : x \in Q_1, y \in Q_2, x \leq y\}$ is a normal base on X; $\mathrm{I}(X, \mathcal{F}_{Q_1,Q_2}) = \infty$.

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Any dense in itself compactum is a *resolvable* space [E. Hewitt 1943].

Theorem [2015]

Let A be the closed subset of a hereditarily normal space X, $\mathcal{F}(A)$ be the normal base on A. Then the minimal ring $\mathcal{F}(X)$ of closed sets containing $\mathcal{F}(A)$ and the family $\{F \in \mathcal{Z}(X) : F \cap A \in \mathcal{F}(A)\}$ is the normal base on X.
If X is a perfectly normal space and the normal base $\mathcal{F}(X)$ is separating then $\max\{\mathrm{I}(A, \mathcal{F}(A)),\ \mathrm{Ind}\ \ F : F \in \mathcal{F}(X),\ F \cap A = \emptyset\} \leq \mathrm{I}(X, \mathcal{F}(X)) \leq \max\{\mathrm{I}(A, \mathcal{F}(A)),\ \mathrm{Ind}\ \ X\}$.

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the big inductive dimension I by it of X is finite
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For which $n, m \in \mathbb{N}$, $n \leq m$, are there metrizable compactum such that all its possible base normal inductive dimensions I form the set:

(a) $\{n, \ldots, m\}$;

(b) $\{k \in \mathbb{N} : k \geq n\}$?

For what $n, m \in \mathbb{N}$, $n \leq m$, are there metrizable compacta such that all possible base normal inductive dimensions I form the set $\{n, \ldots, m\} \cup \{\infty\}$?

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Find all possible base normal inductive dimensions of the dendrite "comb".

I should like to thank the Organizing Committee of the Workshop for the opportunity to present a talk.