Counting Weakly Bi-colored Trees

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Outline

1. Coloring
2. Connections
3. Counts

Outline

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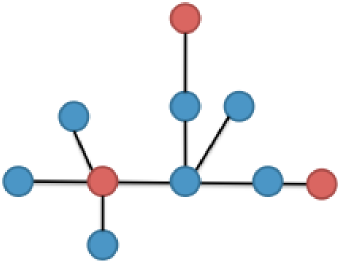
Weakly Bi-colored Trees

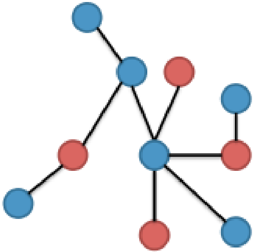
Definition

A tree is said to be *weakly bi-colored* provided it satisfies the following conditions:

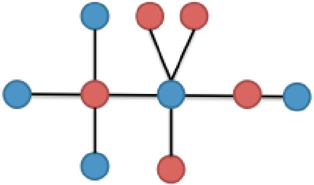
1. Two colors (say, red and blue) that both must be used at least once.
2. Adjacent vertices can both be blue.
3. Adjacent vertices cannot both be red.

Examples of Weakly Bi-colored Trees





Examples of Weakly Bi-colored Trees



- A strongly bi-colored tree, which is considered weakly bi-colored

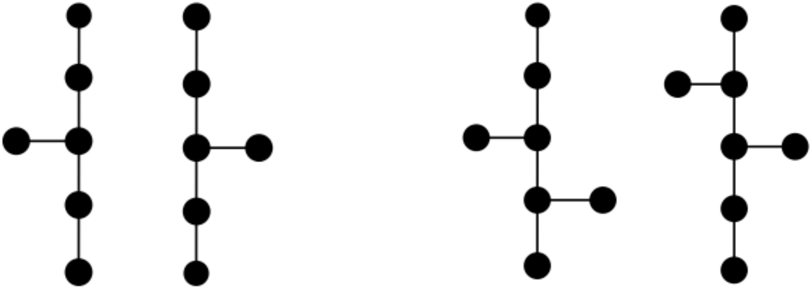
Planar Equivalent Trees

Definition

Two trees are said to be *planar equivalent* (or *planar isomorphic*) if and only if there is an orientation-preserving homeomorphism of the plane between them.

Problem: This condition complicates the count.

Examples of Planar Equivalent Trees



Color Equivalent Trees

Definition

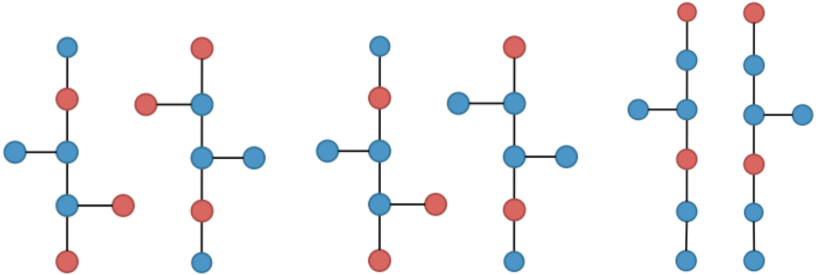
Two trees are said to be *color equivalent* (or *color isomorphic*) if and only if the mapping between the trees respects the coloring.

Definition

If two trees are both planar equivalent and color equivalent, we will say that the trees are *equivalent* or *isomorphic*.

- From now on, we will use the terms equivalent and isomorphic interchangably

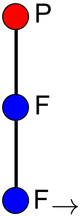
Examples of Color Planar Equivalent Trees



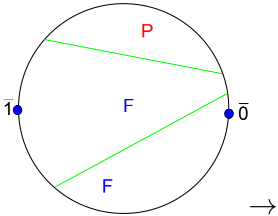
Color, Planar   Non-color, Planar   Color, Non-planar

Levels of Correspondence

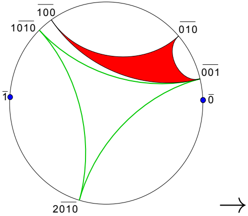
Bi-colored Tree

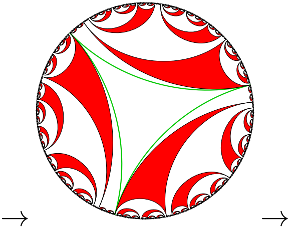


Critical Portrait

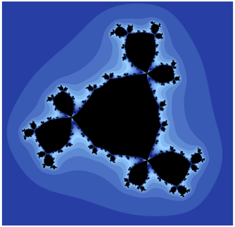


Lamination Data





Lamination



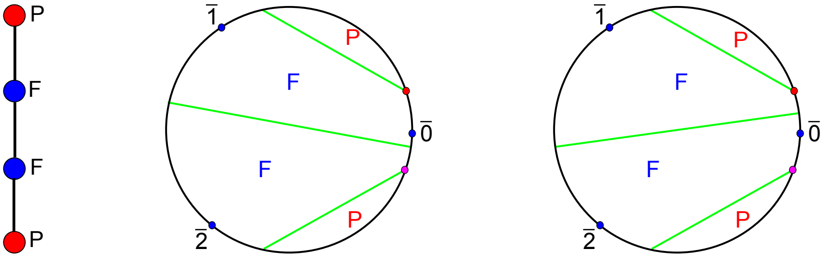
Julia Set

Correspondence Between Bi-colored trees and Crtitcal Protraits

- The correspondence between weakly bi-colored trees and critical portraits raises some interesting questions
- Going from a critical portrait to a tree is unambigous, there is only one weakly bi-colored tree for a given critical portrait.
- However, going from a weakly bi-colored tree to critical portrait there is ambiguity. A tree may correspond to multiple portraits.

Different Critical Portraits with Same Bi-colored Tree

The two critical portraits below correspond to the same weakly bicolored tree:



Strongly Bi-colored Case

Definition

A tree is said to be *strongly bi-colored* provided it satisfies the following conditions:

1. Two colors (say, red and blue) that both must be used at least once.
2. Adjacent vertices cannot both be the same color.

Strongly Bi-colored Case

- The number of strongly bi-colored planar trees with d egdes is:

$$N(d) = \frac{1}{d}\left(\frac{1}{d+1}\binom{2d}{d} + \sum_{n|d, n<d} \phi(\frac{d}{n})\binom{2n}{n}\right)_{[1]} \qquad (1)$$

- Where $\phi(x)$ is Euler's totient function, the number of positive integers less than, and relatively prime to, x.

[1]-Cosper et al, 2015

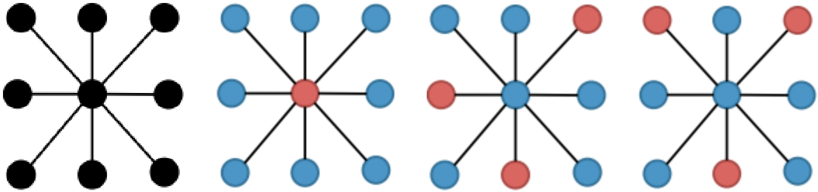
Fans

Definition

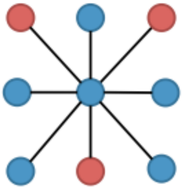
To be classified as a *fan* a tree with $n + 1$ vertices must have the following qualities:

- One central vertex of degree n
- n vertices with degree 1

Examples of Fans



Representing Fans



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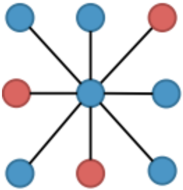

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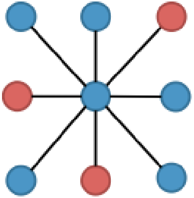
Representing Fans



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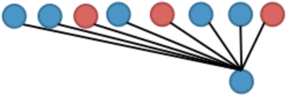

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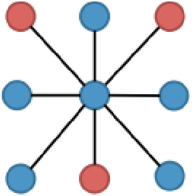
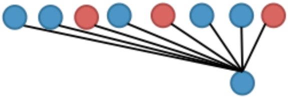


Representing Fans

- We can represent Fans as a binary sequence with 1's representing reds and 0's representing blues

$\equiv$ 01001001



$\equiv$ 00101001

Representing Fans

- We can take all rotations to be equivalent.

$$01001001 \equiv 00101001 \equiv 10010100 \equiv 01001010 \equiv 00100101 \equiv 10010010 \equiv 01001001$$

- Taking all rotations to be a equivalent means that we can reduce *Fans* to the combinatorial objects, *Necklaces*

Representing Fans

Definition

A *k*-ary *necklace* is an array of length *n* with *k* distinct characters taking all rotations to be equivalent.[2]

- The number of necklaces with a given *n* and *k* is:

$$N_k(n) = \frac{1}{n} \sum_{d|n} \phi(d) k^{\frac{n}{d}} \quad [2] \tag{2}$$

- Where $\phi(x)$ is Euler's totient function, the number of positive integers less than, and relatively prime to, x.

[2]-Weisstein

Representing Fans

- A fan with $n+1$ vertices can be represented as a necklace of length n with $k=2$ distinct characters(1 and 0)
- The number of weakly bi-colored fans with $n+1$ vertices (or n egdes) is:

$$N(n) = \frac{1}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}} \tag{3}$$

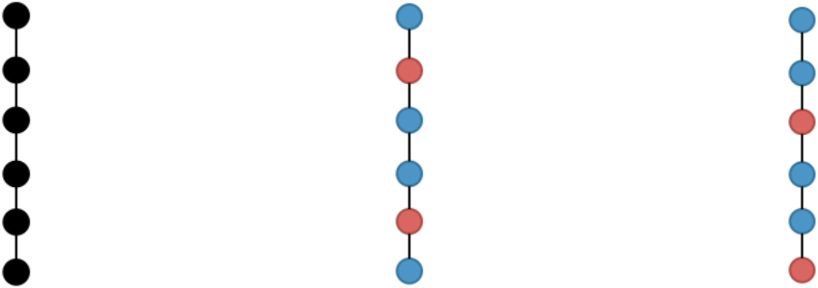
Chains

Definition

A *Chain* is a tree with the following qualities:

- two terminal vertices of degree one
- Any non-terminal vertex is of degree two

Examples of Chains



Representing Chains

- Chains can also be represented with binary sequences of 1's and 0's.
- 0's representing blues, 1's representing reds and 1's cannot be sequential

$\equiv$ 00101001

$\equiv$ 00101001

Counting Chains

- We can think of the problem of counting chains by trying to count to the number of sequences of 1's and 0's where 1's are not allowed to touch

Counting Chains

- Say we have n vertices (or slots in the sequence)
- Let i be the number of reds (or 1's) with $1 \leq i \leq \lceil n/2 \rceil$
- Let $n - i$ be the number of blues (or 0's)

Counting Chains

- Now suppose we were to put down all of the 0's (or blues): _0_0_0_..._0_0_
- This gives us $n - i + 1$ slots to place the i 1's (or reds) and once there is a 1 (or red) in a slot nothing else can be place in said slot
- Meaning we have $n - i + 1$ places and i objects to place, so there are $\binom{n-i+1}{i}$ ways to place the 1's (or reds)

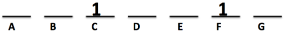
Counting Chains

- However $\binom{n-i+1}{i}$ will overcount because rotation is not considered in this number.
- Some trees will be counted twice.
- Some trees will be counted once and these will be called *palindromes*

Palindromes and Non-palindromes

- Non-palindromes are counted twice and palindromes are counted once due to the condition of planar equivalence.

Non-Palindromes



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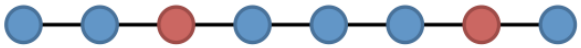

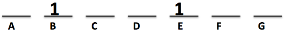

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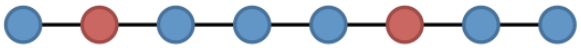
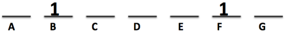
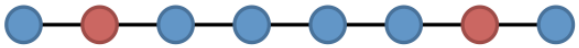


Palindrome



$\equiv$ 01000010 $\equiv$



Counting Palindromes

- In order to correct the overcounting, we need to count the palindromes.
- We can do this in a similar process as how we found the overall count.
- This time we have to consider only the first half of the chain and whether the number of vertices and the number of reds are even or odd.

Counting Palindromes

- We place down half of the blues and place half of the reds in between blues, then flip it across horizontally to create a palindrome.
- So, by counting the ways of arranging the first half, we can count the number of palindromes.

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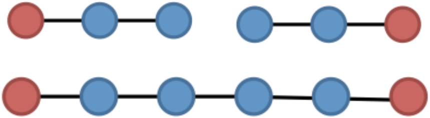


Counting Palindromes

1. If n is even and i is odd, there will be no palindromes.

- Recall $n =$ number of vertices and $i =$ number of reds

Counting Palindromes

1. If n is even and i is is odd, there will be no palindromes.
2. If n is even and i is even, there will be $\binom{\frac{n-i}{2}}{\frac{i}{2}}$ palindromes.

- Recall $n =$ number of vertices and $i =$ number of reds

Counting Palindromes

1. If n is even and i is odd, there will be no palindromes.
2. If n is even and i is even, there will be $\binom{\frac{n-i}{2}}{\frac{i}{2}}$ palindromes.
3. If n is odd and i is odd, there will be $\binom{\frac{n-i}{2}}{\frac{i-1}{2}}$ palindromes.

- Recall $n =$ number of vertices and $i =$ number of reds

Counting Palindromes

1. If n is even and i is odd, there will be no palindromes.
2. If n is even and i is even, there will be $\binom{\frac{n-i}{2}}{\frac{i}{2}}$ palindromes.
3. If n is odd and n is odd, there will be $\binom{\frac{n-i}{2}}{\frac{i-1}{2}}$ palindromes.
4. If n is odd and i is even, there will be $\binom{\frac{n-i+1}{2}}{\frac{i}{2}}$ palindromes.

- Recall $n =$ number of vertices and $i =$ number of reds

Final Count for Chains

- By combining the two counts we found, we have that the number of weakly bi-colored planar chains with n vertices is:

$$N(n) = \frac{1}{2}\sum_{i=1}^{\lceil \frac{n}{2} \rceil}\left(\binom{n-i+1}{i} + P(n,i)\right) \tag{4}$$

Where $P(n,i) =$ number of palindromes:

$$P(n,i) = \begin{cases} 0, n \text{ even and } i \text{ odd} \\ \binom{\frac{n-i}{2}}{\frac{i}{2}}, n \text{ even and } i \text{ even} \\ \binom{\frac{n-i}{2}}{\frac{i-1}{2}}, n \text{ odd and } i \text{ odd} \\ \binom{\frac{n-i+1}{2}}{\frac{i}{2}}, n \text{ odd and } i \text{ even} \end{cases} \tag{5}$$

Caterpillars and Lobsters

Definition

A *Caterpillar* is a tree with the following qualities:

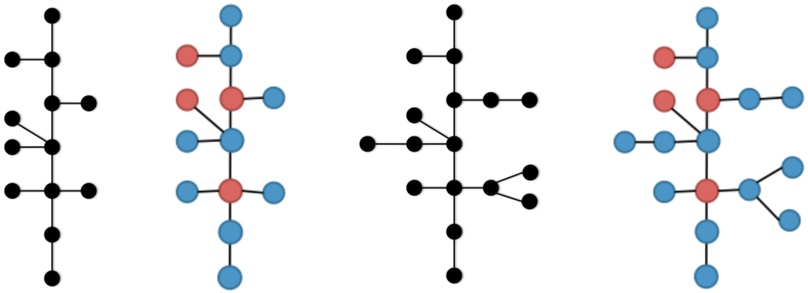
- There is some longest backbone chain of length k, $4 \leq k \leq n-1$
- Any vertex not on the backbone is a distance of one off the backbone and does not increase the backbone length.

Definition

A *Lobster* is a tree with the following qualities:

- There is some longest backbone chain of length k, $4 \leq k \leq n-1$
- Any vertex not on the backbone is a distance of at most two off the backbone and does not increase the backbone length.

Examples of Caterpillars and Lobsters

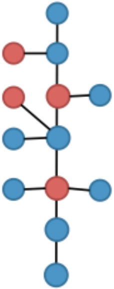


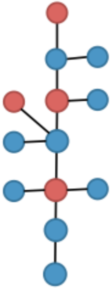
Problems arising from Caterpillars and Lobsters

- In a caterpillar or lobster, the longest backbone is not necessarily unique.
- Two trees can be equivalent and have backbones of the same length, but the backbones are different colors.
- This is something that needs to be considered when making the count.

Problems Arising from Caterpillars and Lobsters

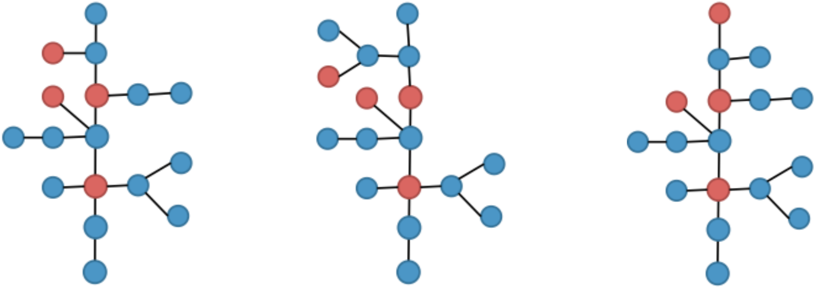
- Two caterpillars that are equivalent, but have different backbones.





Problems Arising from Caterpillars and Lobsters

- Three equivalent lobsters, each with a different backbone.



Super-Lobster

Definition

A *Super-Lobster* is a tree with the following qualities:

- There is some longest backbone chain of length k, $3 \leq k \leq n-1$
- Any vertex not on the backbone is a length of m, $1 \leq m \leq \lfloor \frac{k}{2} \rfloor$, off the backbone and does not increase the backbone length.

- Any tree is representable as a *super-lobster*
- Same problem as caterpillars and lobsters, the longest backbone may not be unique.

Caterpillars, Lobster, and Super-Lobsters

- Currently, the count for caterpillars is being explored.
- Count for caterpillars $\rightarrow$ count for lobsters $\rightarrow$ count for super-lobsters