Vietoris-type Topologies on Hyperspaces

Elza Ivanova-Dimova

Faculty of Mathematics and Informatics,
University of Sofia, Bulgaria

14th Nipissing Topology Workshop,
May 14-19, 2017, North Bay, Ontario, Canada

The author of this talk was partially supported by the contract DN02/15/19.12.2016 with the National Scientific Research Fund.

Introduction

In 1975, M. M. Choban [4] introduced a new topology on the set of all closed subsets of a topological space for obtaining a generalization of the famous Kolmogoroff Theorem on operations on sets. This new topology is similar to the upper Vietoris topology but is weaker than it. In 1997, M. M. Clementino and W. Tholen [5] introduced, for any families $\mathcal{F}$ and $\mathcal{G}$ of subsets of a set X, three new topologies on the power set $\mathcal{P}(X)$ (and its subsets), called by them, respectively, *upper* $\mathcal{F}$*-topology, lower* $\mathcal{F}$*-topology* and $(\mathcal{F}, \mathcal{G})$*-hit-and-miss topology*. The upper $\mathcal{F}$-topology turned out to be a generalized version of the Choban topology.

In 1998, G. Dimov and D. Vakarelov [10] introduced independently the notion of upper $\mathcal{F}$-topology, which was called by them *Tychonoff-type hypertopology* (later on, in [12], it was called also *upper-Vietoris-type hypertopology* because "Tychonoff hypertopology" and "upper Vietoris hypertopology" are two names of one and same notion); they used it for proving an isomorphism theorem for the category of all Tarski consequence systems. The Tychonoff-type hypertopology was studied in details in [9]. Further, in [12], the notion of lower $\mathcal{F}$-topology was introduced independently and was called *lower-Vietoris-type hypertopology*. Finally, in [13], the notion of $(\mathcal{F}, \mathcal{G})$-hit-and-miss topology was introduced independently and was called *Vietoris-type hypertopology*.

In this talk we will present our results from [12] and [13]. They are about lower-Vietoris-type hypertopologies and Vietoris-type hypertopologies. These results are different from those from [5]. So, we will study lower-Vietoris-type hypertopologies and Vietoris-type hypertopologies and, in particular, we will generalize many results of E. Cuchillo-Ibáñez, M. A. Morón and F. R. Ruiz del Portal [6], some of the results of E. Michael [14] about the hyperspaces with Vietoris topology and some of the results of H.-J. Schmidt in [15]. As a corollary, we will get that for every continuous map $f : X \longrightarrow X$, where X is a continuum, there exist a subcontinuum K of X such that $f(K) = K$.

Preliminaries

We denote by $\mathbb{N}$ the set of all natural numbers (hence, $0 \notin \mathbb{N}$), by $\mathbb{R}$ the real line (with its natural topology) and by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{-\infty, \infty\}$.
Let X be a set. We denote by $|X|$ the cardinality of X and by $\mathcal{P}(X)$ (resp., by $\mathcal{P}'(X)$) the set of all (non-empty) subsets of X.
Let $\mathcal{M}, \mathcal{A} \subseteq \mathcal{P}(X)$ and $A \subseteq X$. We set:

- $A^+_{\mathcal{M}} := \{M \in \mathcal{M} \mid M \subseteq A\}$;
- $\mathcal{A}^+_{\mathcal{M}} := \{A^+_{\mathcal{M}} \mid A \in \mathcal{A}\}$;
- $A^-_{\mathcal{M}} := \{M \in \mathcal{M} \mid M \cap A \neq \emptyset\}$
- $\mathcal{A}^-_{\mathcal{M}} := \{A^-_{\mathcal{M}} \mid A \in \mathcal{A}\}$;
- $-\mathcal{M} := \{X \setminus M \mid M \in \mathcal{M}\}$;

- $Fin(X) := \{M \subseteq X \mid 0 < |M| < \aleph_0\}$;
- $Fin_n(X) := \{M \subseteq X \mid 0 < |M| \leq n\}$, where $n \in \mathbb{N}$;
- $\mathcal{A}^\cap := \{\bigcap_{i=1}^{k} A_i \mid k \in \mathbb{N}, A_i \in \mathcal{A}\}$.
- $\mathcal{A}^\cup := \{\bigcup_{i=1}^{k} A_i \mid k \in \mathbb{N}, A_i \in \mathcal{A}\}$.

Let $(X, \mathcal{T})$ be a topological space. We put

- $CL(X) := \{M \subseteq X \mid M \text{ is closed in } X,\ M \neq \emptyset\}$.
- $Comp(X) := \{M \in CL(X) \mid M \text{ is compact}\}$.

When $\mathcal{M} = CL(X)$, we will simply write A^+ and A^- instead of $A^+_{\mathcal{M}}$ and $A^-_{\mathcal{M}}$; the same for subfamilies $\mathcal{A}$ of $\mathcal{P}(X)$. The closure of a subset A of X in $(X, \mathcal{T})$ will be denoted by $cl_{(X,\mathcal{T})}A$ or $\overline{A}^{(X,\mathcal{T})}$ (we will also write, for short, $cl_X A$ or $\overline{A}^X$ and even $\overline{A}^{\mathcal{T}}$). By "neighborhood" we will mean an "open neighborhood". The regular spaces are not assumed to be T_1-spaces; by a T_3-space we mean a regular T_1-space.
If X and Y are sets and $f : X \longrightarrow Y$ is a function then, as usual, we denote by $f \restriction X$ the function between X and $f(X)$ which is a restriction of f. If $(X, \mathcal{T})$ and $(Y, \mathcal{O})$ are topological spaces and $f : X \longrightarrow Y$ is an injection, then we say that f is an *inversely continuous function* if the function $(f \restriction X)^{-1} : f(X) \longrightarrow X$ is continuous.

Let X be a topological space. Recall that the *upper Vietoris topology* Υ_{+X} on $CL(X)$ (called also *Tychonoff topology on* $CL(X)$) has as a base the family of all sets of the form

$$U^+ = \{F \in CL(X) \mid F \subseteq U\},$$

where U is open in X, and *the lower Vietoris topology* Υ_{-X} on $CL(X)$ has as a subbase all sets of the form

$$U^- = \{F \in CL(X) \mid F \cap U \neq \emptyset\},$$

where U is open in X. The *Vietoris topology* Υ_X on $CL(X)$ is defined as the supremum of Υ_{+X} and Υ_{-X}, i.e., $\Upsilon_{+X} \cup \Upsilon_{-X}$ is a subbase for Υ_X.

Definition([5, 10])

(a) Let $(X, \mathcal{T})$ be a topological space and $\mathcal{M} \subseteq \mathcal{P}'(X)$. The topology $\Upsilon_{+\mathcal{M}}$ on $\mathcal{M}$ having as a base the family $\mathcal{T}^+_{\mathcal{M}}$ is called a *Tychonoff topology on* $\mathcal{M}$ (or, *upper Vietoris topology on* $\mathcal{M}$) generated by $(X, \mathcal{T})$. When $\mathcal{M} = CL(X)$, then $\Upsilon_{+\mathcal{M}}$ is just the classical upper Vietoris topology Υ_{+X} on $CL(X)$ (= Tychonoff topology $\Upsilon_{+\mathcal{M}}$ on $CL(X)$).

(b) Let X be a set and $\mathcal{M} \subseteq \mathcal{P}'(X)$. A topology $\mathcal{O}$ on the set $\mathcal{M}$ is called a *Tychonoff-type topology on* $\mathcal{M}$ (or, *upper-Vietoris-type topology on* $\mathcal{M}$) if the family $\mathcal{O} \cap \mathcal{P}(X)^+_{\mathcal{M}}$ is a base for $\mathcal{O}$.

A Tychonoff topology on $\mathcal{M}$ is always a Tychonoff-type topology on $\mathcal{M}$, but not viceversa (see [9]).

Fact ([10])

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. Then the family

$$\mathcal{B}_{\mathcal{O}} := \{A \subseteq X \mid A^{+}_{\mathcal{M}} \in \mathcal{O}\}$$

contains X and is closed under finite intersections; hence, it can serve as a base for a topology

$$\mathcal{T}_{+\mathcal{O}}$$

on X. When $\mathcal{O}$ is a Tychonoff-type topology, the family $(\mathcal{B}_{\mathcal{O}})^{+}_{\mathcal{M}}$ is a base for $\mathcal{O}$.

The lower-Vietoris-type topologies on hyperspaces

Definition

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{M} \subseteq \mathcal{P}'(X)$. The topology $\mathcal{O}_{\mathcal{T}}$ on $\mathcal{M}$ having as a subbase the family $\mathcal{T}^{-}_{\mathcal{M}}$ will be called a *lower Vietoris topology on* $\mathcal{M}$ generated by $(X, \mathcal{T})$. When $\mathcal{M} = CL(X)$, then $\mathcal{O}_{\mathcal{T}}$ is just the classical lower Vietoris topology on $CL(X)$.

Definition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{O}$ be a topology on $\mathcal{M}$. We say that $\mathcal{O}$ is a *lower-Vietoris-type topology on* $\mathcal{M}$, if $\mathcal{O} \cap \{A^{-}_{\mathcal{M}} \mid A \subseteq X\}$ is a subbase for $\mathcal{O}$.

Clearly, a lower Vietoris topology on $\mathcal{M}$ is always a lower-Vietoris-type topology on $\mathcal{M}$, but not viceversa (see Example 1 below).

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$. Then the family

$$\mathcal{P}_{\mathcal{O}} := \{A \subseteq X \mid A^{-}_{\mathcal{M}} \in \mathcal{O}\}$$

contains X, and can serve as a subbase for a topology

$$\mathcal{T}_{-\mathcal{O}}$$

on X. The family $(\mathcal{P}_{\mathcal{O}})^{-}_{\mathcal{M}}$ is a subbase of $\mathcal{O}$. The family $\mathcal{P}_{\mathcal{O}}$ is closed under arbitrary unions.

Proposition

Let X be a set and $\mathcal{M} \subseteq \mathcal{P}'(X)$. Then a topology $\mathcal{O}$ on $\mathcal{M}$ is a lower-Vietoris-type topology on $\mathcal{M}$ iff there exists a topology $\mathcal{T}$ on X and a subbase $\mathcal{S}$ for $\mathcal{T}$, such that $\mathcal{S}^-_{\mathcal{M}} = \{A^-_{\mathcal{M}} \mid A \in \mathcal{S}\}$ is a subbase for $\mathcal{O}$.

Proposition 1

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{S} \subset \mathcal{P}'(X)$. Then $\mathcal{S}^-_{\mathcal{M}}$ is a subbase for a (lower-Vietoris-type) topology on $\mathcal{M}$ if and only if $M \cap \bigcup \mathcal{S} \neq \emptyset$ for any $M \in \mathcal{M}$.

Definition

Let X be a set, $\mathcal{M}, \mathcal{S} \subseteq \mathcal{P}'(X)$ and $M \cap \bigcup \mathcal{S} \neq \emptyset$ for any $M \in \mathcal{M}$. Then the lower-Vietoris-type topology $\mathcal{O}_{\mathcal{S}}^{\mathcal{M}}$ on $\mathcal{M}$ for which $\mathcal{S}_{\mathcal{M}}^{-}$ is a subbase (see Proposition 1) will be called a *lower-Vietoris-type topology on* $\mathcal{M}$ *generated by the family* $\mathcal{S}$. When there is no ambiguity, we will simply write $\mathcal{O}_{\mathcal{S}}$ instead of $\mathcal{O}_{\mathcal{S}}^{\mathcal{M}}$.

Corollary

Let X be a set, $\mathcal{M}, \mathcal{S} \subseteq \mathcal{P}'(X)$ and $\bigcup \mathcal{S} = X$. Then $\mathcal{S}_{\mathcal{M}}^{-}$ is a subbase for a lower-Vietoris-type topology $\mathcal{O}_{\mathcal{S}}$ on $\mathcal{M}$.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{U}_i \subseteq \mathcal{P}'(X)$, $i = 1, 2$, be such that $X \in \mathcal{U}_1 \cap \mathcal{U}_2$. Let $\mathcal{O}_i$ be the topologies on $\mathcal{M}$ generated by $\mathcal{U}_i$, $i = 1, 2$. Then $\mathcal{O}_1 \equiv \mathcal{O}_2$ if and only if the next two conditions are fulfilled:

(a) for any $M \in \mathcal{M}$ and for every $U_1, \ldots, U_k \in \mathcal{U}_1$ for which $M \cap U_i \neq \emptyset$ for any $i = 1, \ldots, k$, there exist $V_1, \ldots, V_n \in \mathcal{U}_2$ such that: (1) $M \cap V_j \neq \emptyset$, for any $j = 1, \ldots, n$, and (2) if $M' \in \mathcal{M}$ and $M' \cap V_j \neq \emptyset$ for any $j = 1, \ldots, n$, then $M' \cap U_i \neq \emptyset$ for any $i = 1, \ldots, k$;

(b) for every $M \in \mathcal{M}$ and for any $V_1, \ldots, V_k \in \mathcal{U}_2$ for which $M \cap V_i \neq \emptyset$ for any $i = 1, \ldots, k$, there exist $U_1, \ldots, U_n \in \mathcal{U}_1$ such that: (1) $M \cap U_j \neq \emptyset$ for any $j = 1, \ldots, n$, and (2) if $M' \in \mathcal{M}$ and $M' \cap U_j \neq \emptyset$ for any $j = 1, \ldots, n$, then $M' \cap V_i \neq \emptyset$ for any $i = 1, \ldots, k$.

Definition

Let X be a set and $\mathcal{M} \subseteq \mathcal{P}(X)$. We say that $\mathcal{M}$ is a *natural family in* X if $\{x\} \in \mathcal{M}$ for any $x \in X$.

Proposition

If $(X, \mathcal{T})$ is a T_1-space, $\mathcal{M} = CL(X)$ (resp., $\mathcal{M} \subseteq \mathcal{P}'(X)$ is a natural family and a closed base for X), $\mathcal{O}$ is a lower-Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T} = \mathcal{T}_{-\mathcal{O}}$ (see Proposition 1 for $\mathcal{T}_{-\mathcal{O}}$), then $\mathcal{O} \equiv \Upsilon_{-X}$ (resp., $\mathcal{O} \equiv \mathcal{O}_{\mathcal{T}}^{\mathcal{M}}$) if and only if for every $F \in \mathcal{M}$ we have that $\overline{F}^{\mathcal{O}} = F_X^+$.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$. Then $(\mathcal{M}, \mathcal{O})$ is a T_0-space if and only if for every $M_1, M_2 \in \mathcal{M}$ with $M_1 \neq M_2$, there exist $U_1, \ldots, U_n \in \mathcal{P}_{\mathcal{O}}$ such that either ($M_1 \cap U_i \neq \emptyset$ for any $i = 1, \ldots, n$ and there exists an $i_0 \in \{1, \ldots, n\}$ for which $M_2 \cap U_{i_0} = \emptyset$) or ($M_2 \cap U_i \neq \emptyset$ for any $i = 1, \ldots, n$ and there exists an $i_0 \in \{1, \ldots, n\}$ for which $M_1 \cap U_{i_0} = \emptyset$).

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ be a natural family, $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$. Then $(\mathcal{M}, \mathcal{O})$ is a T_1-space if and only if $\mathcal{M} = \{\{x\} \mid x \in X\}$ and $(X, \mathcal{T}_{-\mathcal{O}})$ is a T_1-space.

Definition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ be a natural family in X and $\mathcal{U} \subseteq \mathcal{P}'(X)$. Then:

(a) If $\mathcal{U}' = \{U_{\alpha,i} \mid \alpha \in A, i = 1, \ldots, n_\alpha, n_\alpha \in \omega\}$ is a subfamily of $\mathcal{U}$, $U = \bigcup_{\alpha \in A} \bigcap_{i=1}^{n_\alpha} U_{\alpha,i}$ and from $M \in \mathcal{M}$ and $M \cap U_{\alpha,i} \neq \emptyset$ for some $\alpha \in A$ and for any $i = 1, \ldots, n_\alpha$, it follows that $M \cap U \neq \emptyset$, then we will say that *the set U is $\mathcal{M}^-$-covered by the family $\mathcal{U}'$;*

(b) The family $\mathcal{U}$ is said to be an $\mathcal{M}^-$-*closed family* if it contains any subset U of X which is $\mathcal{M}^-$-covered by some subfamily $\mathcal{U}'$ of $\mathcal{U}$.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$, A be an index set, $\mathcal{U}_\alpha \subseteq \mathcal{P}'(X)$ and $\mathcal{U}_\alpha$ are $\mathcal{M}^-$-closed families for any $\alpha \in A$. Then $\mathcal{U} = \bigcap_{\alpha \in A} \mathcal{U}_\alpha$ is an $\mathcal{M}^-$-closed family.

Corollary

Let X be a set and $\mathcal{M} \subseteq \mathcal{P}'(X)$. Then every family $\mathcal{N} \subseteq \mathcal{P}'(X)$ is contained in a minimal $\mathcal{M}^-$-closed family, denoted by $\mathcal{M}^-(\mathcal{N})$.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ be a natural family and $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$. Then $\mathcal{P}_{\mathcal{O}}$ (where $\mathcal{P}_{\mathcal{O}} = \{A \subseteq X \mid A^-_{\mathcal{M}} \in \mathcal{O}\}$) is an $\mathcal{M}^-$-closed family. If $\mathcal{U}, \mathcal{V} \subseteq \mathcal{P}'(X)$, $\mathcal{U}$ covers X and generates $\mathcal{O}$, and $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{P}_{\mathcal{O}}$, then $\mathcal{P}_{\mathcal{O}} = \mathcal{M}^-(\mathcal{V})$.

Corollary

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ be a natural family and $\mathcal{U}, \mathcal{V} \subseteq \mathcal{P}'(X)$ are covers of X. Then $\mathcal{O}^{\mathcal{M}}_{\mathcal{U}} \equiv \mathcal{O}^{\mathcal{M}}_{\mathcal{V}}$ if and only if $\mathcal{M}^-(\mathcal{U}) = \mathcal{M}^-(\mathcal{V})$.

Corollary

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq \mathcal{P}'(X)$ be a natural family and $\mathcal{B}$ be a base for $\mathcal{T}$. Then $\mathcal{O}^{\mathcal{M}}_{\mathcal{B}} \equiv \mathcal{O}^{\mathcal{M}}_{\mathcal{T}}$.

Example 1

Let $\mathcal{M} \subseteq \mathcal{P}'(\mathbb{R})$ and $\mathcal{M} \supseteq \mathit{Fin}_2(\mathbb{R})$. Then

$$\mathcal{U} = \{(-\infty, \beta), (\alpha, \infty), (-\infty, \beta) \cup (\alpha, \infty) \mid \alpha, \beta \in \overline{\mathbb{R}}\}$$

is an $\mathcal{M}^{-}$-closed family. Hence it generates a topology $\mathcal{O}_{\mathcal{U}}^{\mathcal{M}}$ on $\mathcal{M}$ different from the lower Vietoris topology $\mathcal{O}_{\mathcal{T}}^{\mathcal{M}}$ (= $\mathcal{O}_{\mathcal{T}}$) on $\mathcal{M}$, where $\mathcal{T}$ is the natural topology on $\mathbb{R}$.

Proposition

Let X be a set, $\mathcal{M} \subset \mathcal{P}'(X)$, $X \in \mathcal{M}$ and $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$. Then $(\mathcal{M}, \mathcal{O})$ is a pseudocompact, connected, separable Baire space. Moreover, the intersection of any family of open dense subsets of $(\mathcal{M}, \mathcal{O})$ is a dense subset of $(\mathcal{M}, \mathcal{O})$.

Proposition

Let X be a set, $\mathcal{M} \subset \mathcal{P}'(X)$, $\mathcal{M}$ be a natural family, $X \in \mathcal{M}$, $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$ and $(X, \mathcal{T}_{-\mathcal{O}})$ be a T_2-space. Then $\Phi : (X, \mathcal{T}_{-\mathcal{O}}) \longrightarrow (\mathcal{M}, \mathcal{O})$, where $\Phi(x) = \{x\}$ for any $x \in X$, is a homeomorphic embedding, $\Phi(X)$ is a closed subset of $(\mathcal{M}, \mathcal{O})$, and if $|X| > 1$ then $\Phi(X)$ is nowhere dense in $(\mathcal{M}, \mathcal{O})$.

Proposition

Let X be a set, $\mathcal{M} \subset \mathcal{P}'(X)$, $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$ and $\mathcal{M} \supseteq \mathit{Fin}(X)$. Then $\mathit{Fin}(X)$ is dense in $(\mathcal{M}, \mathcal{O})$.

Proposition 2

Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be two topological spaces, $f : X \longrightarrow X'$, $\mathcal{P}$ be a subbase for $\mathcal{T}$, $\mathcal{P}'$ be a subbase for $\mathcal{T}'$, $f^{-1}(\mathcal{P}') \subseteq \mathcal{P}$, $\mathcal{M} = CL(X)$, $\mathcal{M}' = CL(X')$. Let $\mathcal{O}$ be generated by $\mathcal{P}$ and $\mathcal{O}'$ be generated by $\mathcal{P}'$. Then the map $2^f : (\mathcal{M}, \mathcal{O}) \longrightarrow (\mathcal{M}', \mathcal{O}')$, where $2^f(C) = \overline{f(C)}^{X'}$ for every $C \in \mathcal{M}$, is continuous.

Proposition

In the notation and hypothesis of Proposition 2, we have that:

(a) $2^{id_X} = id_{CL(X)}$;

(b) $2^{g \circ f} = 2^g \circ 2^f$.

Proposition

Let X, Y be sets, $\mathcal{P} \subseteq \mathcal{P}'(X)$, $\mathcal{S} \subseteq \mathcal{P}'(Y)$, $\bigcup \mathcal{P} = X$, $\bigcup \mathcal{S} = Y$, $\mathcal{T}$ (resp., $\mathcal{T}'$) be the topology on X (resp., Y) generated by the subbase $\mathcal{P}$ (resp., $\mathcal{S}$), $f : X \longrightarrow Y$, $f^{-1}(\mathcal{S}) \subseteq \mathcal{P}$, $\mathcal{M} \subseteq CL(X, \mathcal{T})$, $\mathcal{M}' = CL(Y, \mathcal{T}')$, $\mathcal{O} = \mathcal{O}_{\mathcal{P}}^{\mathcal{M}}$ and $\mathcal{O}' = \mathcal{O}_{\mathcal{S}}^{\mathcal{M}'}$. Let $(X, \mathcal{T})$ be a T_2-space, $(Y, \mathcal{T}')$ be a T_1-space, $\mathcal{M}$ be a natural family and $2^f : (\mathcal{M}, \mathcal{O}) \longrightarrow (\mathcal{M}', \mathcal{O}')$, where $2^f(F) = \overline{f(F)}^Y$ for every $F \in \mathcal{M}$, be a closed map. Then the map f is closed.

With our next result we generalize a theorem of H.-J. Schmidt [15, Theorem 11(1)] about *commutability between hyperspaces and subspaces* (see also [7] for similar results).

Proposition 3

Let $(X, \mathcal{T})$ be a space, $\mathcal{P}$ be a subbase for $\mathcal{T}$, $X \in \mathcal{P}$, $\mathcal{O} = \mathcal{O}_{\mathcal{P}}^{CL(X)}$. For any $A \subset X$, set $\mathcal{P}_A = \{U \cap A \mid U \in \mathcal{P}\}$ and $\mathcal{O}_A = \mathcal{O}_{\mathcal{P}_A}^{CL(A)}$. Then $i_{A,X} : (CL(A), \mathcal{O}_A) \longrightarrow (CL(X), \mathcal{O})$, where $i_{A,X}(F) = \overline{F}^X$, is a homeomorphic embedding.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{M}$ be a natural family, $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$ and $(X, \mathcal{T}_{-\mathcal{O}})$ be a T_2-space. Then $(X, \mathcal{T}_{-\mathcal{O}})$ is compact if and only if $(\mathcal{M}, \mathcal{O})$ is compact.

Proposition

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq \mathcal{P}'(X)$ be a natural family and $\mathcal{O} = \mathcal{O}_{\mathcal{T}}^{\mathcal{M}}$. Then $w(X, \mathcal{T}) \leq \tau (\geq \aleph_0)$ if and only if $w(\mathcal{M}, \mathcal{O}) \leq \tau$.

Homotopy, extensions of maps and fixed point properties in lower-Vietoris-type hyperspaces

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ be a natural family, $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$ and $X \in \mathcal{M}$. Then $(\mathcal{M}, \mathcal{O})$ has a trivial homotopy type.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ be a natural family, $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$ and $X \in \mathcal{M}$. Then $(\mathcal{M}, \mathcal{O})$ is an absolute extensor for the class of all topological spaces (i.e., every continuous function f from a closed subspace of a space Z to $(\mathcal{M}, \mathcal{O})$ can be continuously extended to Z).

Theorem

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{P}$ be a subbase for $\mathcal{T}$, $\emptyset \neq \mathcal{M} \subseteq \{X \setminus U \mid U \in \mathcal{P}\}$, $\emptyset \notin \mathcal{M}$ and $X \in \mathcal{M}$. If $\mathcal{O}$ is the topology on $\mathcal{M}$ generated by $\mathcal{P}$ and $\mathcal{M}$ is closed under intersections of decreasing subfamilies, then $(\mathcal{M}, \mathcal{O})$ has the fixed point property.

Corollary

Let X be a compact space, $\mathcal{P}$ be a subbase for X, $\emptyset \in \mathcal{P}$, $\mathcal{M} = \{X \setminus U \mid U \in \mathcal{U}(\mathcal{P})\} \setminus \{\emptyset\}$, where $\mathcal{U}(\mathcal{P}) = \{U = \bigcup \mathcal{P}' \mid \mathcal{P}' \subseteq \mathcal{P}\}$, and $\mathcal{O}$ be the topology on $\mathcal{M}$ generated by $\mathcal{P}$. Then $(\mathcal{M}, \mathcal{O})$ has the fixed point property.

Corollary ([6])

Let $(X, \mathcal{T})$ be a compact Hausdorff space. Then $(CL(X), \Upsilon_{-X})$ has the fixed point property.

Corollary ([6])

Let $(X, \mathcal{T})$ be a compact Hausdorff space and $f : X \longrightarrow X$ be a continuous map. Then there exists a compact subspace K of X such that $f(K) = K$.

Corollary

Let $(X, \mathcal{T})$ be a compact Hausdorff space, $\mathcal{M} \subseteq CL(X)$ be closed under any intersections of decreasing subfamilies and $X \in \mathcal{M}$. Then $(\mathcal{M}, \mathcal{O}_{\mathcal{T}}^{\mathcal{M}})$ has the fixed point property.

Corollary

Let $(X, \mathcal{T})$ be a continuum,

$$\mathcal{M} = \{K \subseteq X \mid K \text{ is a non-empty continuum}\},$$

$\mathcal{P} \supseteq \{X \setminus K \mid K \in \mathcal{M}\}$ and $\mathcal{O}$ be the topology on $\mathcal{M}$ generated by $\mathcal{P}$. Then $(\mathcal{M}, \mathcal{O})$ has the fixed point property.

Corollary

Let $(X, \mathcal{T})$ be a continuum and $f : X \longrightarrow X$ be a continuous map. Then there exists a continuum $K \subseteq X$, such that $f(K) = K$.

The Vietoris-type topologies on hyperspaces

Definition

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{M} \subseteq \mathcal{P}'(X)$. The topology $\Upsilon_{\mathcal{M}}$ on $\mathcal{M}$ having as a subbase the family $\mathcal{T}^{+}_{\mathcal{M}} \cup \mathcal{T}^{-}_{\mathcal{M}}$ is called *the Vietoris topology on* $\mathcal{M}$. When $\mathcal{M} = CL(X)$ then $\Upsilon_{\mathcal{M}} \equiv \Upsilon_X$.

Definition([5, 13])

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. The topology $\mathcal{O}$ is called a *Vietoris-type topology on* $\mathcal{M}$ if the family $(\mathcal{B}_{\mathcal{O}})^{+}_{\mathcal{M}} \cup (\mathcal{P}_{\mathcal{O}})^{-}_{\mathcal{M}}$ is a subbase for $\mathcal{O}$.

Remark

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{M} \subseteq \mathcal{P}'(X)$. Then, clearly, $\Upsilon_{\mathcal{M}}$ is a Vietoris-type topology on $\mathcal{M}$. As it is noted in [5], the converse is not always true even when $\mathcal{M} = CL(X)$. We will also give such an example below. Note that when $\mathcal{M} \subseteq CL(X)$, then $\Upsilon_{\mathcal{M}} \equiv (\Upsilon_X)|_{\mathcal{M}}$.

Notation

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. We denote by

$$\mathcal{T}_{\mathcal{O}}$$

the topology on X having $\mathcal{B}_{\mathcal{O}} \cup \mathcal{P}_{\mathcal{O}}$ as a subbase and we say that $\mathcal{T}_{\mathcal{O}}$ is the *V-topology on* X *induced by the topological space* $(\mathcal{M}, \mathcal{O})$. We denote by

$$\mathcal{O}_u$$

the topology on $\mathcal{M}$ having $(\mathcal{B}_{\mathcal{O}})^+_{\mathcal{M}}$ as a base, and by

$$\mathcal{O}_l$$

the topology on $\mathcal{M}$ having $(\mathcal{P}_{\mathcal{O}})^-_{\mathcal{M}}$ as a subbase.

The following assertion can be easily proved.

Fact

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. Then $\mathcal{T}_{\mathcal{O}} = \mathcal{T}_{+\mathcal{O}} \vee \mathcal{T}_{-\mathcal{O}}$, $\mathcal{O}_u$ is a Tychonoff-type topology on $\mathcal{M}$ and $\mathcal{O}_l$ is a lower-Vietoris-type topology on $\mathcal{M}$. $\mathcal{O}$ is a Vietoris-type topology on $\mathcal{M}$ iff $\mathcal{O} = \mathcal{O}_u \vee \mathcal{O}_l$.

Fact

Let $(X, \mathcal{T})$ be a topological space. If $\mathcal{M} \subseteq CL(X)$, then $(\Upsilon_{+X})|_{\mathcal{M}}$ and $(\Upsilon_{-X})|_{\mathcal{M}}$ are Vietoris-type topologies on $\mathcal{M}$. Moreover, if $\mathcal{M} \subseteq \mathcal{P}'(X)$, then $\Upsilon_{+\mathcal{M}}$ and $\Upsilon_{-\mathcal{M}}$ are Vietoris-type topologies on $\mathcal{M}$.

Definition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$. Then $\mathcal{O}$ is called a *strong Vietoris-type topology on* $\mathcal{M}$ if $\mathcal{T}_{+\mathcal{O}} \equiv \mathcal{T}_{-\mathcal{O}}$.

Proposition

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq CL(X)$ and $\mathcal{M}$ be a natural family. Let $\mathcal{O} = (\Upsilon_X)|_{\mathcal{M}}$. Then $\mathcal{O}$ is a strong Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T} \equiv \mathcal{T}_{\mathcal{O}}$. In particular, for every T_1-space X, Υ_X is a strong Vietoris-type topology on $CL(X)$.

Example

Let $X = \{0, 1, 2\}$ and $\mathcal{T} = \{\emptyset, X, \{0\}, \{0, 2\}\}$. Then $(X, \mathcal{T})$ is a T_0-space such that $\mathcal{O} = \Upsilon_X$ is not a strong Vietoris-type topology on $CL(X)$. Also, we have that $\mathcal{T} \neq \mathcal{T}_{-\mathcal{O}}$, $\mathcal{T} \neq \mathcal{T}_{+\mathcal{O}}$, $\mathcal{T} \neq \mathcal{T}_{\mathcal{O}}$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}_{-\mathcal{O}}$.

Example

Let $X = \{0, 1, 2\}$ and $\mathcal{T} = \{\emptyset, X, \{0\}, \{0, 2\}, \{1, 2\}, \{2\}\}$. Then $(X, \mathcal{T})$ is a T_0-space such that $\mathcal{O} = \Upsilon_X$ is not a strong Vietoris-type topology on $CL(X)$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}_{-\mathcal{O}} \supsetneqq \mathcal{T}_{+\mathcal{O}} = \mathcal{T}$.

Example

Let $\mathcal{U} = \{(\alpha, \beta) \mid \alpha, \beta \in \overline{\mathbb{R}}\}$ and $\mathbb{R}$ be regarded with its natural topology. Then the topology $\mathcal{O}$ on $CL(\mathbb{R})$ having as a subbase the family $\mathcal{U}^- \cup \mathcal{U}^+$ is a strong Vietoris-type topology on $CL(\mathbb{R})$ different from the Vietoris topology $\Upsilon_{\mathbb{R}}$ on $CL(\mathbb{R})$. Also, $\mathcal{T}_{\mathcal{O}}$ coincides with the natural topology $\mathcal{T}$ on $\mathbb{R}$.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$, $\emptyset \notin \mathcal{M}$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. Let $\mathcal{T} = (-\mathcal{M}) \cup \{X\}$ be a T_1-topology on X. Then $\mathcal{O}$ is the upper Vietoris topology on $\mathcal{M}$ generated by $(X, \mathcal{T})$ iff $\mathcal{O}$ is the finest upper-Vietoris-type topology on $\mathcal{M}$ for which $\mathcal{B}_{\mathcal{O}} \subseteq \mathcal{T}$.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$, $\emptyset \notin \mathcal{M}$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. Let $\mathcal{T} = (-\mathcal{M}) \cup \{X\}$ be a T_1-topology on X. Then $\mathcal{O}$ is the lower Vietoris topology on $\mathcal{M}$ generated by $(X, \mathcal{T})$ iff $\mathcal{O}$ is the finest lower-Vietoris-type topology on $\mathcal{M}$ for which $\mathcal{P}_{\mathcal{O}} \subseteq \mathcal{T}$.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}(X)$, $\emptyset \notin \mathcal{M}$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. Let $\mathcal{T} = (-\mathcal{M}) \cup \{X\}$ be a T_1-topology on X. Then $\mathcal{O}$ is the Vietoris topology on $\mathcal{M}$ generated by $(X, \mathcal{T})$ iff $\mathcal{O}$ is the finest Vietoris-type topology on $\mathcal{M}$ for which $\mathcal{B}_{\mathcal{O}} \cup \mathcal{P}_{\mathcal{O}} \subseteq \mathcal{T}$.

Some properties of the hyperspaces with Vietoris-type topologies

In this section, some of the results of E. Michael [14] concerning hyperspaces with Vietoris topology will be extended to analogous results for the hyperspaces with Vietoris-type topology.

Proposition

Let $(X, \mathcal{T})$ be a space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $n \in \mathbb{N}$, $Fin_n(X) \subseteq \mathcal{M}$, $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T}_{\mathcal{O}} \subseteq \mathcal{T}$. Let $J_n(X)$ be the subspace of $(\mathcal{M}, \mathcal{O})$ consisting of all sets of cardinality $\leq n$. Then the map $j_n : X^n \longrightarrow J_n(X)$, where $j_n(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$, is continuous.

Proposition

Let $(X, \mathcal{T})$ be a space, $\mathcal{M} \subseteq \mathcal{P}'(X)$ be a natural family, $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$. Then $j_1 : (X, \mathcal{T}) \longrightarrow J_1(X)$ is a homeomorphism.

Proposition

If $(X, \mathcal{T})$ is a T_2-space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{M}$ is a natural family, $\mathcal{O}$ is a Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T}_{-\mathcal{O}} \supseteq \mathcal{T}$, then $J_1(X)$ is closed in $(\mathcal{M}, \mathcal{O})$.

Proposition

Let $(X, \mathcal{T})$ be a space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $Fin_2(X) \subseteq \mathcal{M}$, $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$, $\mathcal{T}_{\mathcal{O}} \subseteq \mathcal{T}$ and $J_1(X)$ be closed in $(\mathcal{M}, \mathcal{O})$. Then X is a T_2-space.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $Fin(X) \subseteq \mathcal{M}$ and $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$. Then $J(X) = \bigcup_{i \in \mathbb{N}} J_i(X)$ is dense in $(\mathcal{M}, \mathcal{O})$.

Definition 1

Let X be a set and $\mathcal{P} \subseteq \mathcal{P}(X)$. Set $w(X, \mathcal{P}) = \min\{|\mathcal{P}'| \mid (\mathcal{P}' \subseteq \mathcal{P}) \wedge (\forall U \in \mathcal{P} \text{ and } \forall x \in U\, \exists V \in \mathcal{P}' \text{ such that } x \in V \subseteq U)\}$.

Clearly, $w(X, \mathcal{P}) \leq |\mathcal{P}|$; also, when $\mathcal{P}$ is a topology on X, then $w(X, \mathcal{P})$ is just the weight of the topological space $(X, \mathcal{P})$.

Fact 1

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{P}$ be a subbase for $(X, \mathcal{T})$. Then:

(a) the families $\mathcal{P}'$ from Definition 1 are also subbases for $(X, \mathcal{T})$;

(b) if $w(X, \mathcal{T}) \geq \aleph_0$ then $w(X, \mathcal{P}) \geq w(X, \mathcal{T})$.

Remark

Let X be a set and $\mathcal{P} \subseteq \mathcal{P}(X)$. Then, clearly, there exists a unique topology $\mathcal{T}(\mathcal{P})$ on X for which $\mathcal{P} \cup \{X\}$ is a subbase. Obviously, if $\bigcup \mathcal{P} = X$ then $\mathcal{P}$ is a subbase for $\mathcal{T}(\mathcal{P})$. Hence, in Definition 1 we can always assume that X is a topological space and $\mathcal{P} \cup \{X\}$ is a subbase for X.

In connection with Fact 1, note that the following assertion holds (it should be well-known):

Lemma

Let X be a space, $w(X) = \tau \geq \aleph_0$ and $\mathcal{P}$ be a subbase for X. Then there exists a $\mathcal{P}' \subseteq \mathcal{P}$, such that $|\mathcal{P}'| = \tau$ and $\mathcal{P}'$ is a subbase for X.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a lower-Vietoris-type topology on $\mathcal{M}$. Let $\tau \geq \aleph_0$, $\mathcal{P}' \subseteq \mathcal{P}(X)$ and $(\mathcal{P}')^-_{\mathcal{M}}$ be a subbase for $\mathcal{O}$. If $w(X, \mathcal{P}') \leq \tau$, then $w(\mathcal{M}, \mathcal{O}) \leq \tau$. In particular, if $\tau \geq \aleph_0$ and $w(X, \mathcal{P}_{\mathcal{O}}) \leq \tau$ then $w(\mathcal{M}, \mathcal{O}) \leq \tau$.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$. Let $\tau \geq \aleph_0$, $w(X, \mathcal{P}_{\mathcal{O}}) \leq \tau$ and there exists a family $\mathcal{B} \subseteq \mathcal{B}_{\mathcal{O}}$ such that $|\mathcal{B}| \leq \tau$ and $\forall M \in \mathcal{M}$, $\forall U \in \mathcal{B}_{\mathcal{O}}$ with $M \subseteq U$, $\exists V \in \mathcal{B}$ with $M \subseteq V \subseteq U$. Then $w(\mathcal{M}, \mathcal{O}) \leq \tau$.

The space $(Comp(X), (\Upsilon_X)|_{Comp(X)})$ will be denoted by $\mathcal{Z}(X)$.

Corollary([14])

Let $(X, \mathcal{T})$ be a T_1-space and $w(X) \geq \aleph_0$. Then $w(\mathcal{Z}(X)) = w(X)$.

Proposition

Let $(X, \mathcal{T})$ be a space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $Fin(X) \subseteq \mathcal{M}$, $\mathcal{O}$ be a Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$. If $d(X) \geq \aleph_0$ then $d(X) \geq d(\mathcal{M}, \mathcal{O})$.

Proposition

Let X be a set, $\mathcal{M} \subseteq \mathcal{P}'(X)$ and $\mathcal{O}$ be a topology on $\mathcal{M}$. If $\mathcal{M} \subseteq \{X \setminus A \mid A \in \mathcal{P}_{\mathcal{O}}\}$ or $\mathcal{M} \subseteq \mathcal{B}_{\mathcal{O}}$ then $(\mathcal{M}, \mathcal{O})$ is a T_0-space.

Corollary([14])

If $(X, \mathcal{T})$ is a topological space then $(CL(X), \Upsilon_X)$ is a T_0-space.

Proposition 4

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{P}$ be a subbase for $(X, \mathcal{T})$ and $\mathcal{M} \subseteq \{X \setminus U \mid U \in \mathcal{P}\}$. Let $\mathcal{M}$ be a natural family and $\mathcal{O}$ be the topology on $\mathcal{M}$ having as a subbase the family $\mathcal{P}^-_{\mathcal{M}} \cup \mathcal{P}^+_{\mathcal{M}}$. Then $\mathcal{O}$ is a strong Vietoris-type topology on $\mathcal{M}$, $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$ and $(\mathcal{M}, \mathcal{O})$ is a T_1-space.

Remark

The condition "$\mathcal{M} \subseteq \{X \setminus U \mid U \in \mathcal{P}\}$" in Proposition 4 can be replaced by the following one:

(*) if $M, M' \in \mathcal{M}$ and $M \setminus M' \neq \emptyset$ then $\exists U, V \in \mathcal{P}$ such that $M' \subseteq U$, $M \not\subseteq U$, $M \cap V \neq \emptyset$ and $M' \cap V = \emptyset$.

Note that condition (*) is fulfilled if the following condition holds:

(**) if $M \in \mathcal{M}$ and $x \notin M$ then $\exists U, V \in \mathcal{P}$ such that $M \subseteq U \subseteq X \setminus \{x\}$ and $x \in V \subseteq X \setminus M$.

Corollary([14])

Let $(X, \mathcal{T})$ be a T_1-space. Then $(CL(X), \Upsilon_X)$ is a T_1-space.

Definition

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{P}$ be a subbase for $(X, \mathcal{T})$. $(X, \mathcal{T})$ is said to be $\mathcal{P}$-*regular*, if for every $x \in X$ and for every $U \in \mathcal{P}$ such that $x \in U$, there exist $V, W \in \mathcal{P}$ with $x \in V \subset X \setminus W \subset U$.

Clearly, a topology space $(X, \mathcal{T})$ is $\mathcal{T}$-regular iff it is regular. Also, every $\mathcal{P}$-regular space is regular.

Example

Let $(\mathbb{R}, \mathcal{T})$ be the real line with its natural topology and $\mathcal{P} = \{(\alpha, \beta) \setminus F \mid \alpha, \beta \in \overline{\mathbb{R}},\ \alpha < \beta,\ F \subset \mathbb{R},\ |F| < \aleph_0\}$ or $\mathcal{P} = \{(\alpha, \beta) \mid \alpha, \beta \in \overline{\mathbb{R}},\ \alpha < \beta\}$. Then $\mathcal{P}$ is a base for $\mathcal{T}$, $\mathcal{P}^{\cap} = \mathcal{P}$, $(\mathbb{R}, \mathcal{T})$ is regular but it is not $\mathcal{P}$-regular.

Proposition

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{P}$ be a subbase for $(X, \mathcal{T})$ and $\mathcal{M} \subseteq \{X \setminus U \mid U \in \mathcal{P}\}$. Let $\mathcal{M}$ be a natural family, X be $\mathcal{P}$-regular and $\mathcal{O}$ be the topology on $\mathcal{M}$ having as a subbase the family $\mathcal{P}^-_{\mathcal{M}} \cup \mathcal{P}^+_{\mathcal{M}}$. Then $\mathcal{O}$ is a strong Vietoris-type topology on $\mathcal{M}$, $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$ and $(\mathcal{M}, \mathcal{O})$ is a T_2-space.

Proposition 5

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{M} \subseteq \mathcal{P}'(X)$, $\mathcal{P}$ be a base for $(X, \mathcal{T})$, $\mathcal{M} = \{X \setminus U \mid U \in \mathcal{P}\}$ and $\mathcal{P} = \mathcal{P}^{\cap}$. Let $\mathcal{M}$ be a natural family, $\mathcal{O}$ be the topology on $\mathcal{M}$ having as a subbase the family $\mathcal{P}^-_{\mathcal{M}} \cup \mathcal{P}^+_{\mathcal{M}}$ and $(\mathcal{M}, \mathcal{O})$ be a T_2-space. Then X is $\mathcal{P}$-regular.

Remark

The conditions "$\mathcal{M} = \{X \setminus U \mid U \in \mathcal{P}\}$" and "$\mathcal{M}$ is a natural family" in Proposition 5 can be replaced by "$\mathcal{M} \supseteq \{X \setminus U \mid U \in \mathcal{P}\}$, $FinX \subseteq \mathcal{M}$ and if $x \in U \in \mathcal{P}$ then $\{x\} \cup (X \setminus U) \in \mathcal{M}$".

Corollary([14])

Let $(X, \mathcal{T})$ be a T_1-space. Then $(X, \mathcal{T})$ is a T_3-space iff $(CL(X), \Upsilon_X)$ is a T_2-space.

Proposition

Let $(X, \mathcal{T})$ be a compact T_1-space, $\mathcal{O}$ be a Vietoris-type topology on $CL(X, \mathcal{T})$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$. Then $(CL(X, \mathcal{T}), \mathcal{O})$ is a compact space.

Subspaces and hyperspaces

In this section, we will regard the problem of continuity or inverse continuity of the maps of the form $i_{A,X}$ (see Proposition 3 for the definition of the maps $i_{A,X}$) for the hyperspaces with a srong Vietoris-type topology. This problem was regarded by H.-J.Schmidt [15] for the lower Vietoris topology, by G. Dimov [7, 8] for the (upper) Vietoris topology and by Barov-Dimov-Nedev [2, 3] for the upper Vietoris topology.

The next assertion is trivial.

Proposition

Let A and X be sets, and $f : A \longrightarrow X$ be a function. Let, for $i = 1, 2$, $\mathcal{T}_i$ (resp., $\mathcal{O}_i$) be a topology on A (resp., X). Let the maps $f : (A, \mathcal{T}_1) \longrightarrow (X, \mathcal{O}_1)$ and $f : (A, \mathcal{T}_2) \longrightarrow (X, \mathcal{O}_2)$ be continuous. Then $f : (A, \mathcal{T}_1 \vee \mathcal{T}_2) \longrightarrow (X, \mathcal{O}_1 \vee \mathcal{O}_2)$ is a continuous map.

Proposition

Let $(X, \mathcal{T})$ be a T_1-space, $\mathcal{M} = CL(X, \mathcal{T})$, $\mathcal{O}$ be a strong Vietoris-type topology on $\mathcal{M}$ and $\mathcal{T}_{\mathcal{O}} = \mathcal{T}$. Let A be a subspace of X, $\mathcal{M}_A = CL(A)$, $\mathcal{O}_-^A$ be the topology on $\mathcal{M}_A$ having as a subbase the family $(\mathcal{P}_{\mathcal{O}}^A)_{CL(A)}^-$, where $\mathcal{P}_{\mathcal{O}}^A = \{U \cap A \mid U \in \mathcal{P}_{\mathcal{O}}\}$, and let $\mathcal{O}_+^A$ be the topology on $\mathcal{M}_A$ having as a base the family $(\mathcal{B}_{\mathcal{O}}^A)_{CL(A)}^+$, where $\mathcal{B}_{\mathcal{O}}^A = \{U \cap A \mid U \in \mathcal{B}_{\mathcal{O}}\}$. Then:

(a) $\mathcal{O}^A = \mathcal{O}_-^A \vee \mathcal{O}_+^A$ is a strong Vietoris-type topology on $\mathcal{M}_A$, and

(b) the map $i_{A,X} : (\mathcal{M}_A, \mathcal{O}^A) \longrightarrow (\mathcal{M}, \mathcal{O})$, where $i_{A,X}(F) = \overline{F}^X$ for every $F \in \mathcal{M}_A$, is continuous (resp., inversely continuous) if and only if the map $i_{A,X,+} : (\mathcal{M}_A, \mathcal{O}_+^A) \longrightarrow (\mathcal{M}, \mathcal{O}_u)$ is continuous (resp., inversely continuous) (here $i_{A,X,+}(F) = \overline{F}^X$ for every $F \in CL(A)$).

REFERENCES

[1] A. V. Arhangel'skiĭ and V. I. Ponomarev,
Fundamentals of General Topology in Problems and Exercises,
Nauka, Moskva, 1974 (In Russian).

[2] S. Barov, G. Dimov and S. Nedev,
On a theorem of H.-J. Schmidt,
C. R. Acad. Bulgare Sci., 46(3), 1993, 9–11.

[3] S. Barov, G. Dimov and S. Nedev,
On a question of M. Paoli and E. Ripoli,
Boll. Un. Mat. Ital. A(7) 10, 1996, no.1, 127–141.

[4] M. M. Čoban,
Operations over sets,
Sibirsk. Mat. Ž., 16 (1975), no. 6, 1332–1351.

[5] M. M. Clementino and W. Tholen,
A characterization of the Vietoris topology,
Topology Proceedings, 22 (1997), 71–95.

[6] E. Cuchillo-Ibáñez, M. A. Morón and F. R. Ruiz del Portal,
Lower semifinite topology in hyperspaces,
Topology Proceedings, 17 (1992), 29–39.

[7] G. Dimov,
On the commutability between hyperspaces and subspaces, and on Schmidt's conjecture,
Rendiconti dell' Istituto di Matematica dell' Università di Trieste, 25, Fasc. I-II, 1993, 175–194.

[8] G. Dimov,
Some remarks on the commutability between hyperspaces and subspaces, and on Schmidt's conjecture,
Compt. Rend. Acad. Bulgare Sci., 47(4), 1994, 5–8.

[9] G. Dimov, F. Obersnel and G. Tironi,
On Tychonoff-type hypertopologies,
Proceedings of the Ninth Prague Topological Symposium (Prague, Czech Republic, August 19–25, 2001), Topology Atlas, Toronto, 2002, 51–70 (see also arXiv:math/0204121v1 [math.GN] 10 Apr 2002).

[10] G. Dimov and D. Vakarelov,
On Scott consequence systems,
Fundamenta Informaticae, 33 (1998), 43–70.

[11] R. Engelking,
General Topology,
PWN, Warszawa, 1977.

[12] E. Ivanova-Dimova,
Lower-Vietoris-type topologies on hyperspaces,
Topology Appl. 220 (2017), 100–110.

[13] E. Ivanova-Dimova,
Vietoris-type topologies on hyperspaces,
arXiv:1701.01181, 4 Jan 2017, 1–15.

[14] E. Michael,
Topologies on spaces of subsets,
Trans. Amer. Math. Soc. 71 (1951), 152–182.

[15] H.-J. Schmidt,
Hyperspaces of quotient and subspaces. I. Hausdorff topological spaces,
Math. Nachr. 104 (1981), 271–280.

E N D

THANK YOU