The Topology of Continua that admit Mixing Homemorphisms

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A *continuum* is a compact connected metric space.

A continuum is *1-dimensional* if for every $\epsilon > 0$ there exists a finite open cover $\mathcal{U}$ of X with mesh less than ϵ such that every $x \in X$ is contained in at most 2 elements of $\mathcal{U}$.

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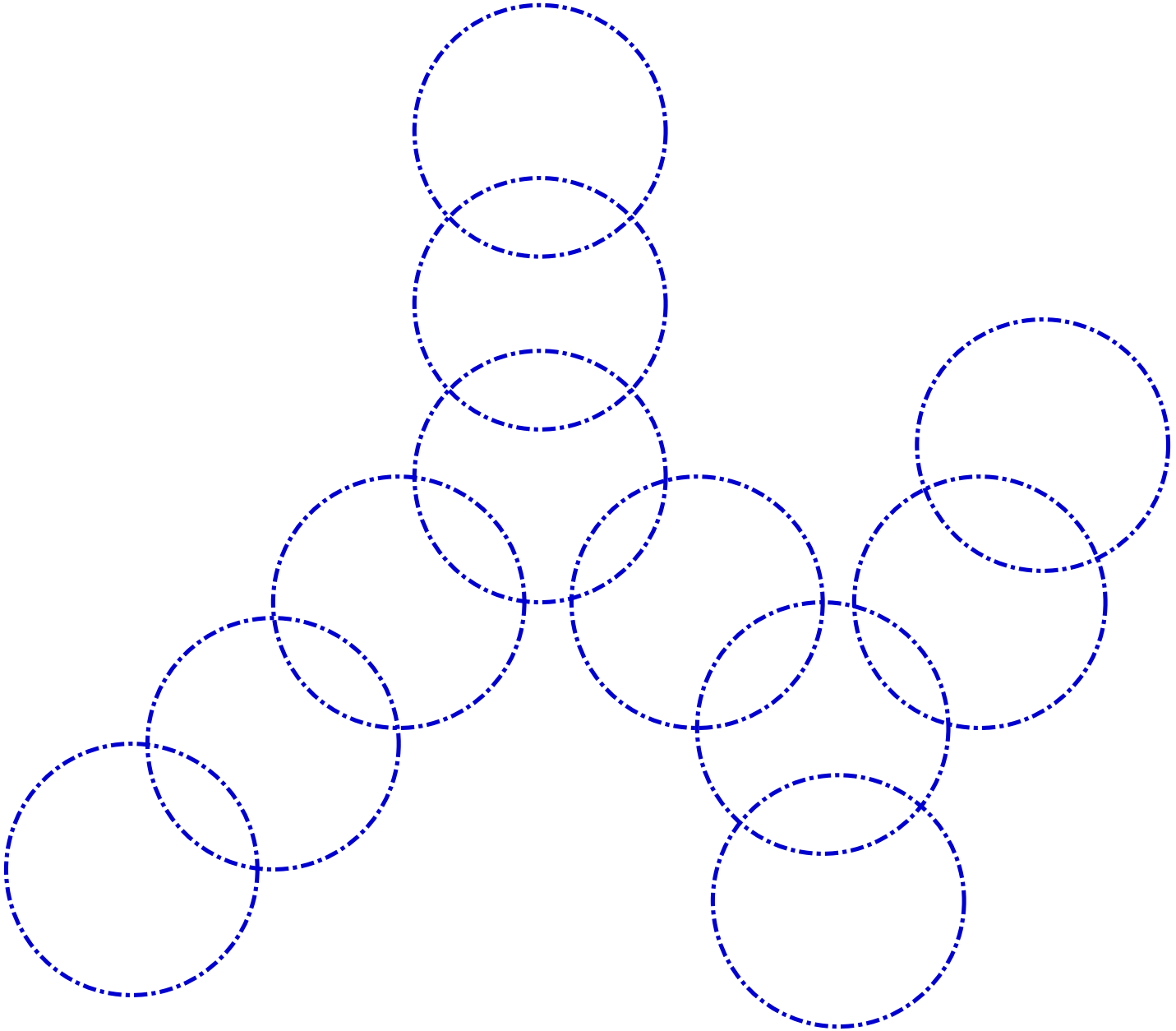


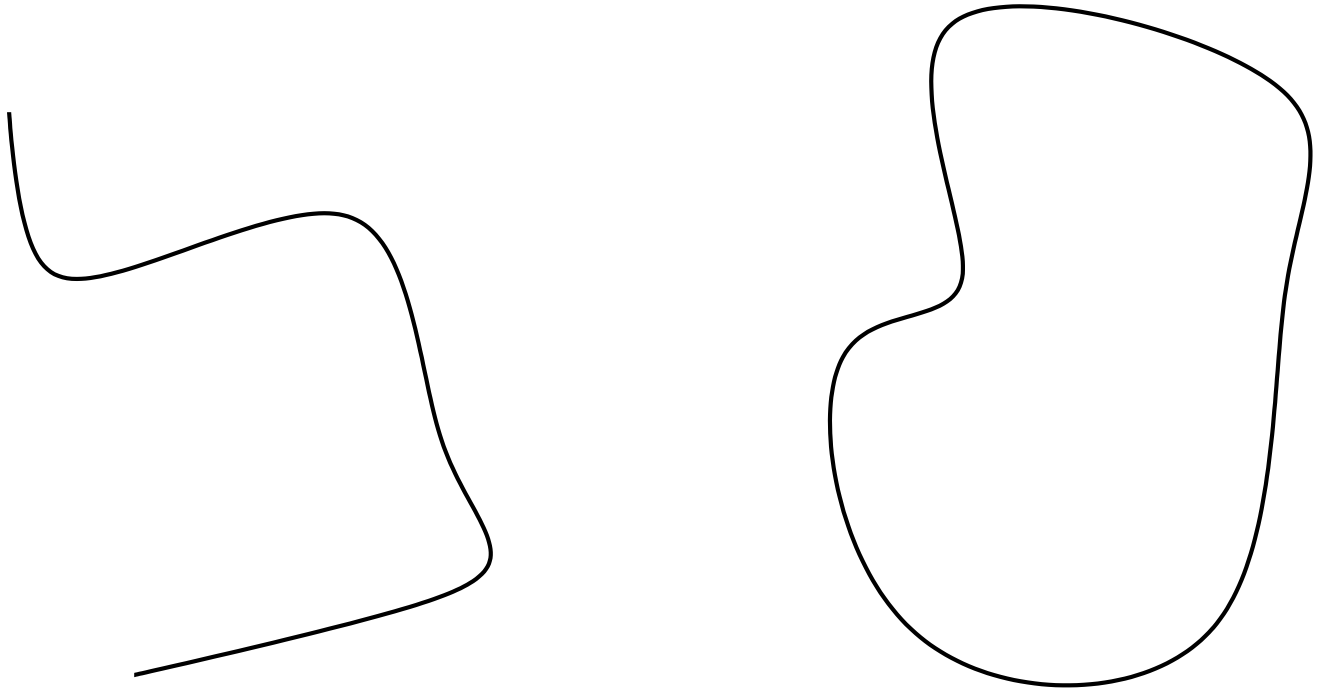


Figure: arc and simple closed curve.

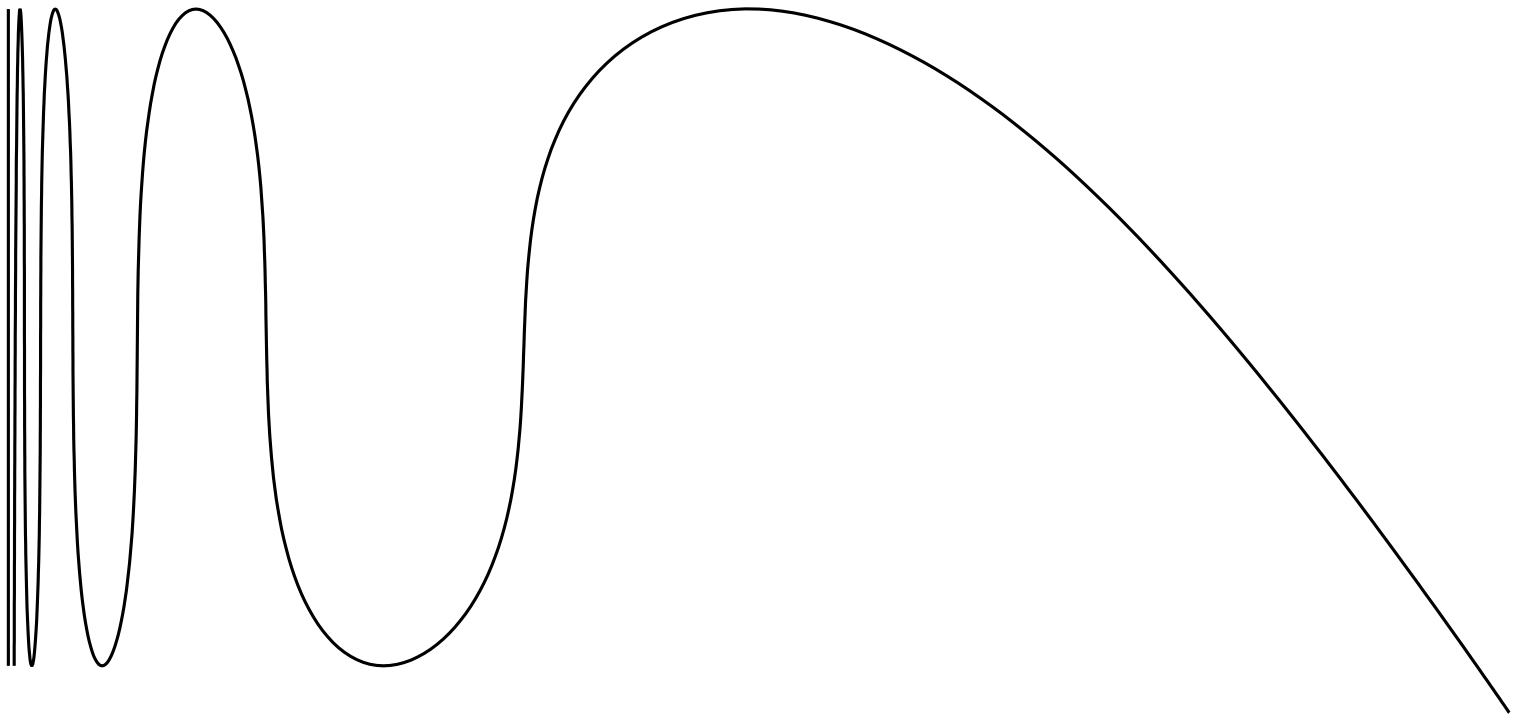


Figure: $\sin(1/x)$ curve

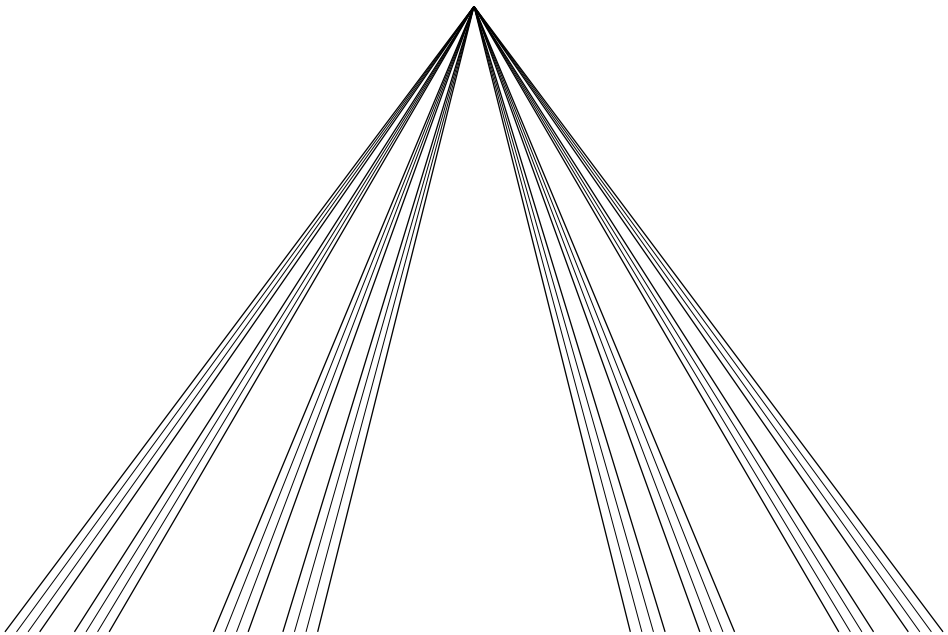


Figure: Cantor fan

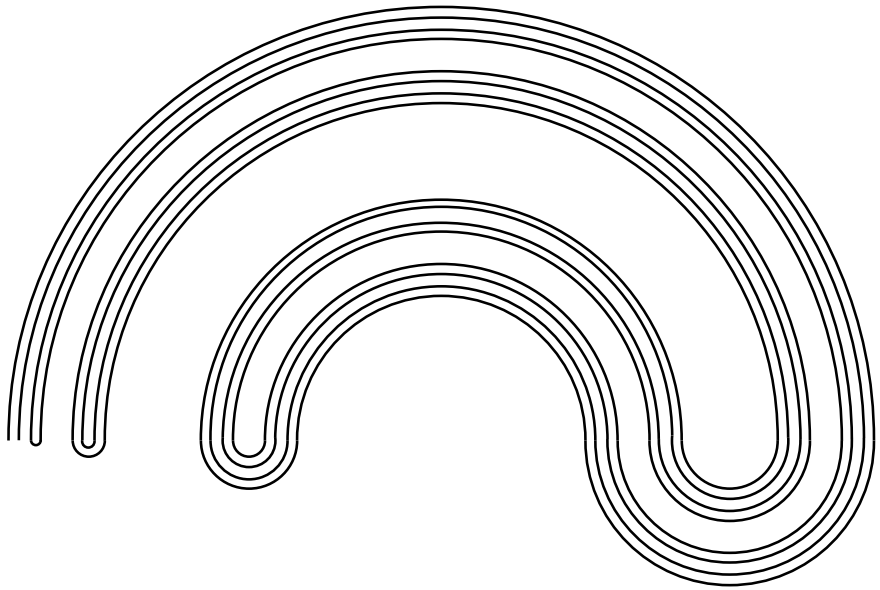


Figure: Buckethandle continuum

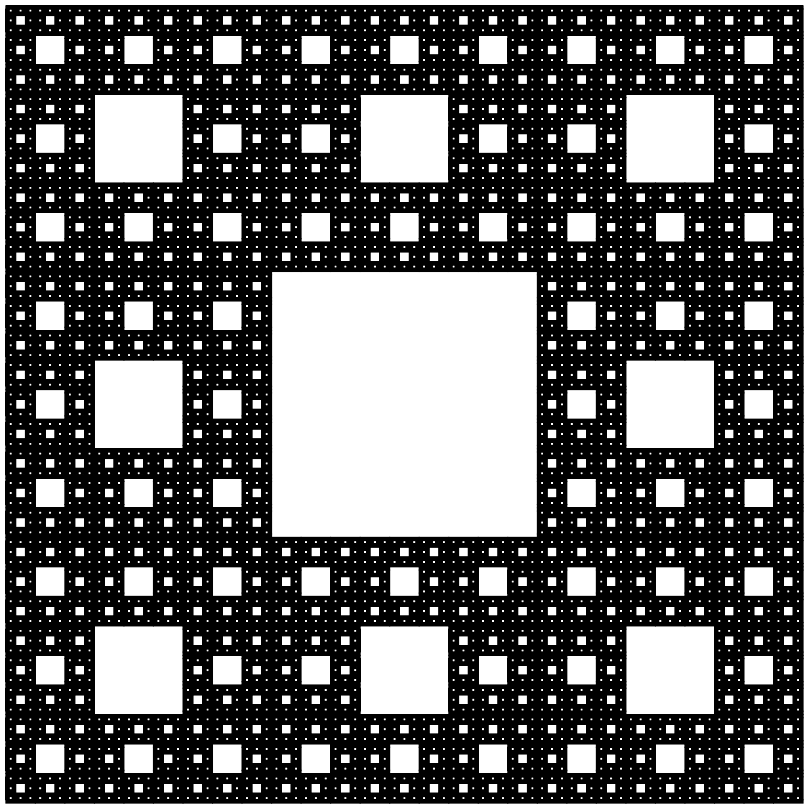


Figure: Sierpinski plane curve

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of topological spaces. For each i, let $f_i : X_{i+1} \longrightarrow X_i$ be a continuous function called a *bonding map*. The collection $\{X_i, f_i\}_{i=1}^{\infty}$ is called an *inverse system*. Each space X_i is called a *factor space* of the inverse system.

If each factor space is the same space X then the inverse system can be written as $\{X, f_i\}_{i=1}^{\infty}$. Likewise, if each bonding map is the same map f, then the inverse system can be written as $\{X, f\}_{i=1}^{\infty}$. Also, the inverse system $\{X_i, f_i\}_{i=1}^{\infty}$ is sometimes written as

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \ \ldots \ X_{i-1} \xleftarrow{f_{i-1}} X_i \xleftarrow{f_i} \ \ldots .$$

Every inverse system $\{X_i, f_i\}_{i=1}^{\infty}$ determines a topological space $\widehat{X}$ called the **inverse limit** of the system and is written $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$. The space $\widehat{X}$ is the subspace of the Cartesian product $\prod_{i=1}^{\infty} X_i$ given by

$$\widehat{X} = \varprojlim\{X_i, f_i\}_{i=1}^{\infty} = \{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i | f_i(x_{i+1}) = x_i\}.$$

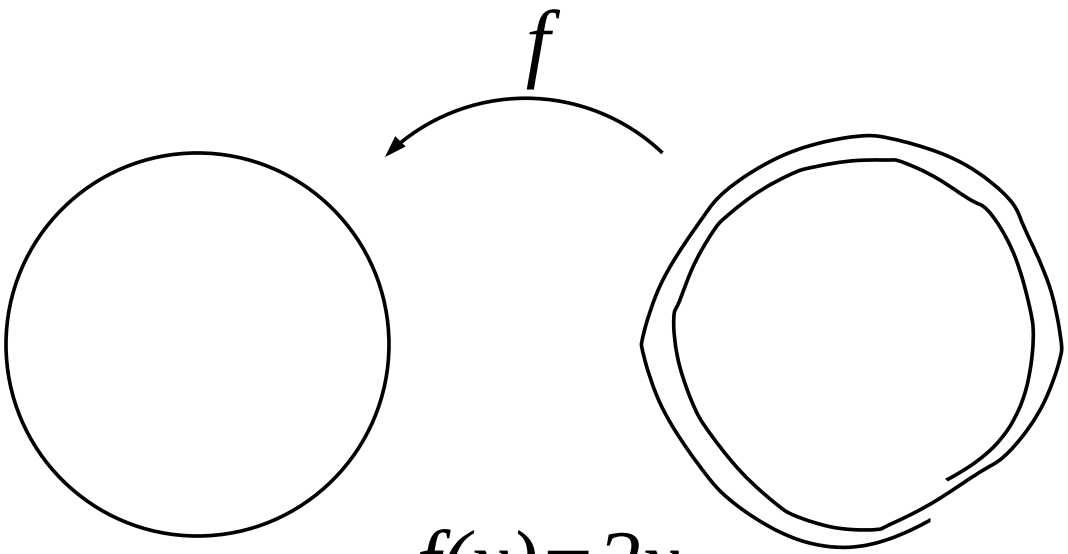
$\widehat{X}$ has the subspace topology induced on it by $\prod_{i=1}^{\infty} X_i$. If $\mathbf{x} = (x_i)_{i=1}^{\infty}$ and $\mathbf{y} = (y_i)_{i=1}^{\infty}$ are two points of the inverse limit, we define distance to be

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i},$$

where d_i is the metric on X_i.
$\mathbf{x} = (x_i)_{i=1}^{\infty}$ is often called a **thread** of the inverse system.

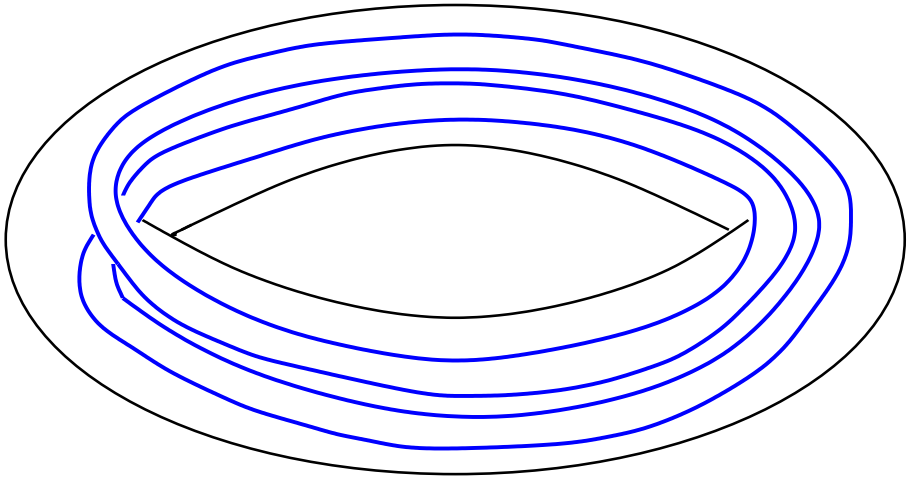
For example, let $f : S \longrightarrow S$ by $f(x) = 2x(\text{mod}1)$ Then the dyadic solenoid Σ_2 is given by

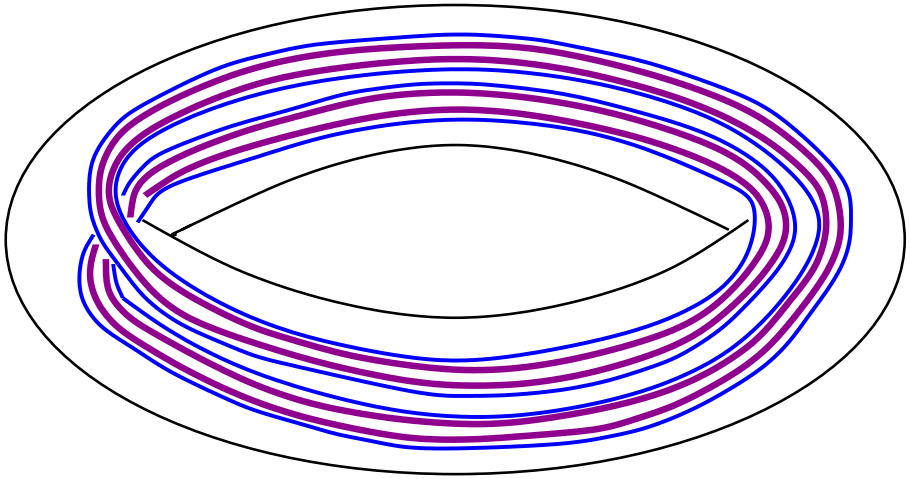
$$\Sigma_2 = \varprojlim\{S, f\}_{i=1}^{\infty} = \{(x_1, x_2, x_3, ...) \mid f(x_{i+1}) = x_i\}.$$



$$f(x)=2x$$

$$X=\{(x_1,x_2,x_3,\dots)\mid f_i(x_{i+1})=x_i\}$$





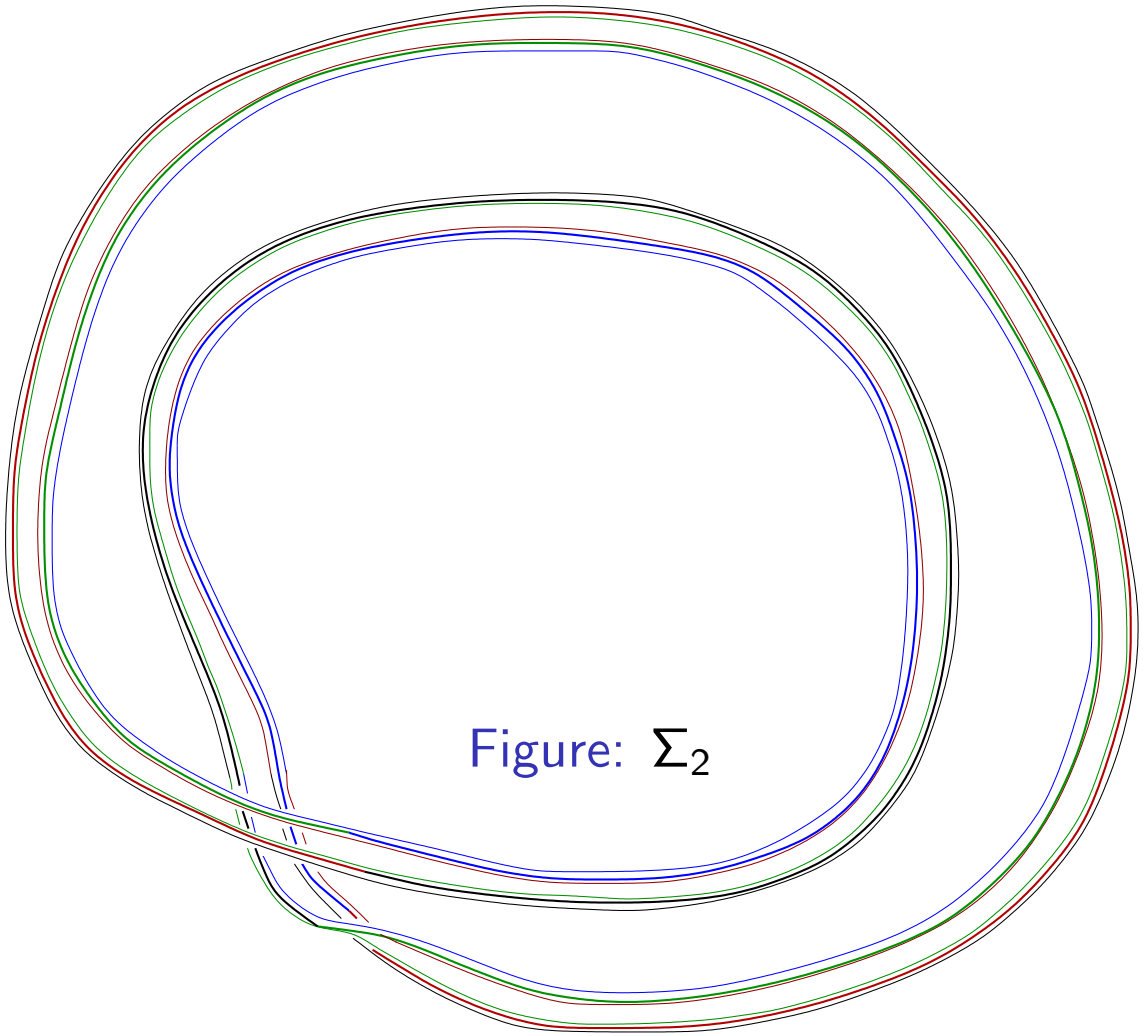
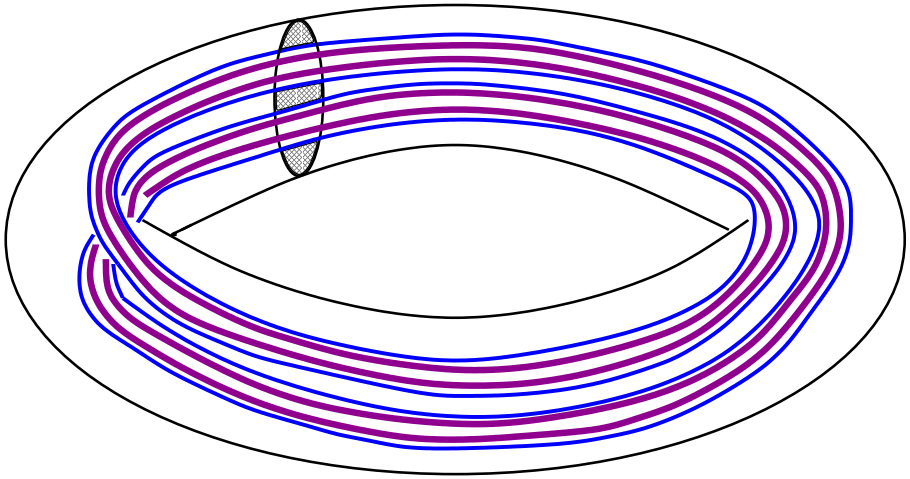
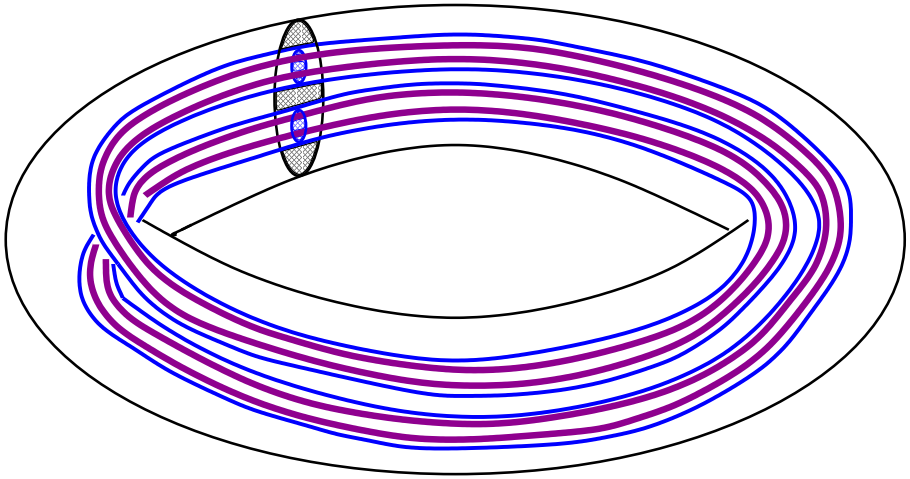


Figure: Σ_2





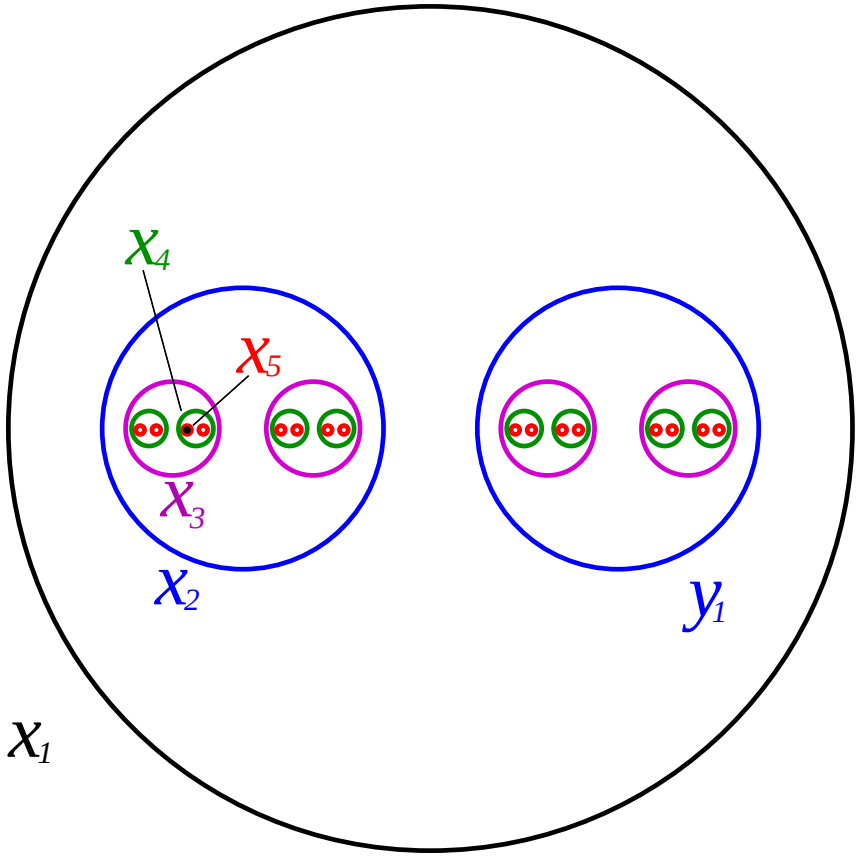


Figure: $X = \{(x_1, x_2, x_3, ...) \mid f_i(x_{i+1}) = x_i\}$.

I will only be focusing on one-dimensional continua.

Fact: If each factor is one-dimensional then the inverse limit is one-dimensional.

Fact: Every one-dimensional continuum can be expressed as the inverse limits of graphs

Shape Considerations

A connected graph G is *0-cyclic* if it contains no cycles, i.e., is a tree.

A connected graph G is *k-cyclic* if k is the smallest number edges $\{e_1, ..., e_k\}$ that must be removed from G so that there are no longer any cycles.

That is $G - \{e_1, ..., e_k\}$ is a *spanning tree* for G.

Figure: 2-cyclic graphs.

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A continuum is

1. *chainable* (also known as *arc-like*)
2. *tree-like*
3. *G-like*
4. *k-cyclic*

if it is the inverse limit of a(n)

1. arc(s)
2. tree(s)
3. topological graph(s) homeomorphic to the same graph G
4. k-cyclic topological graph(s)

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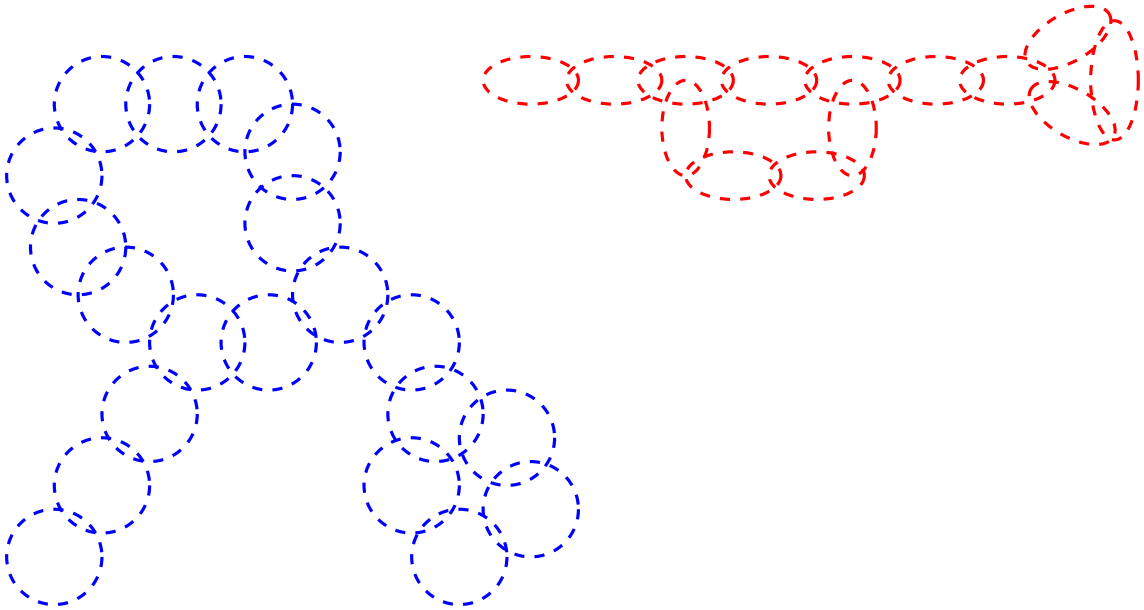


Figure: Open covers.

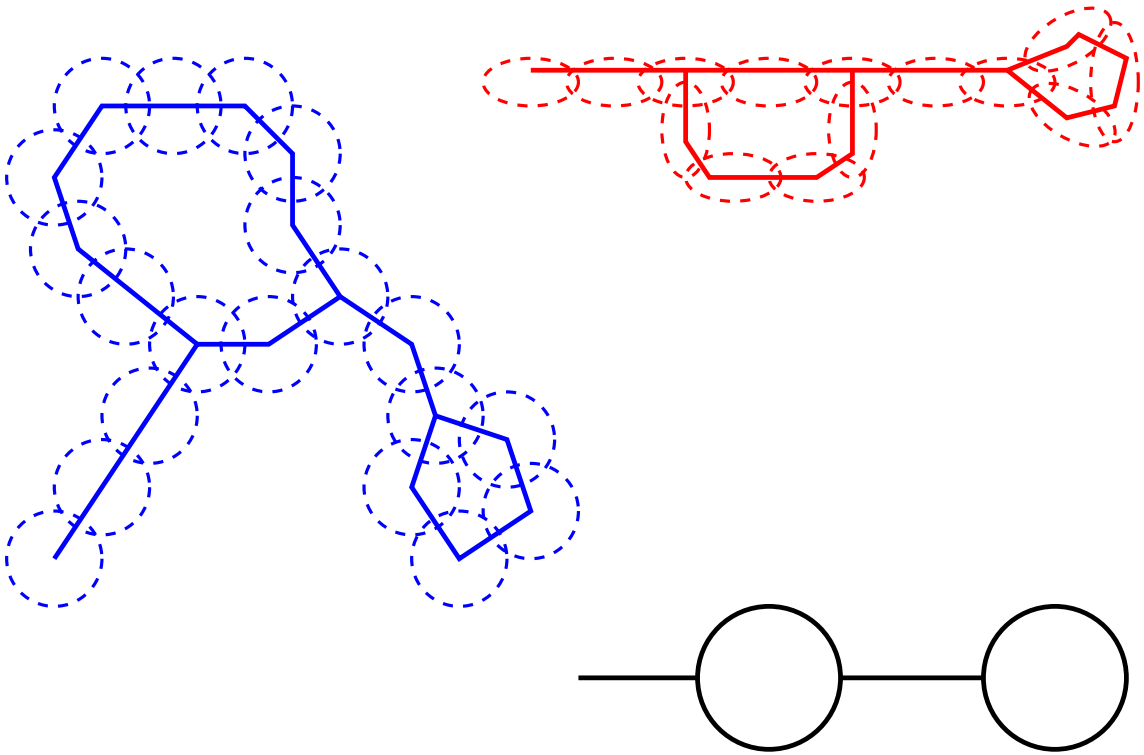


Figure: Covers with their nerves.

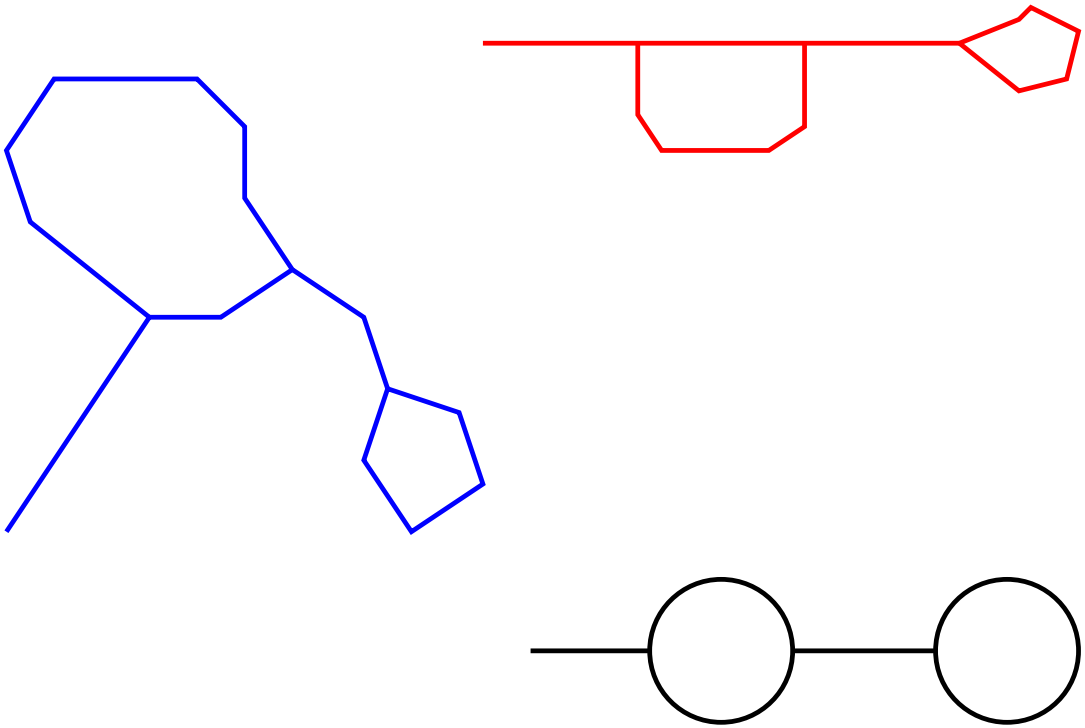
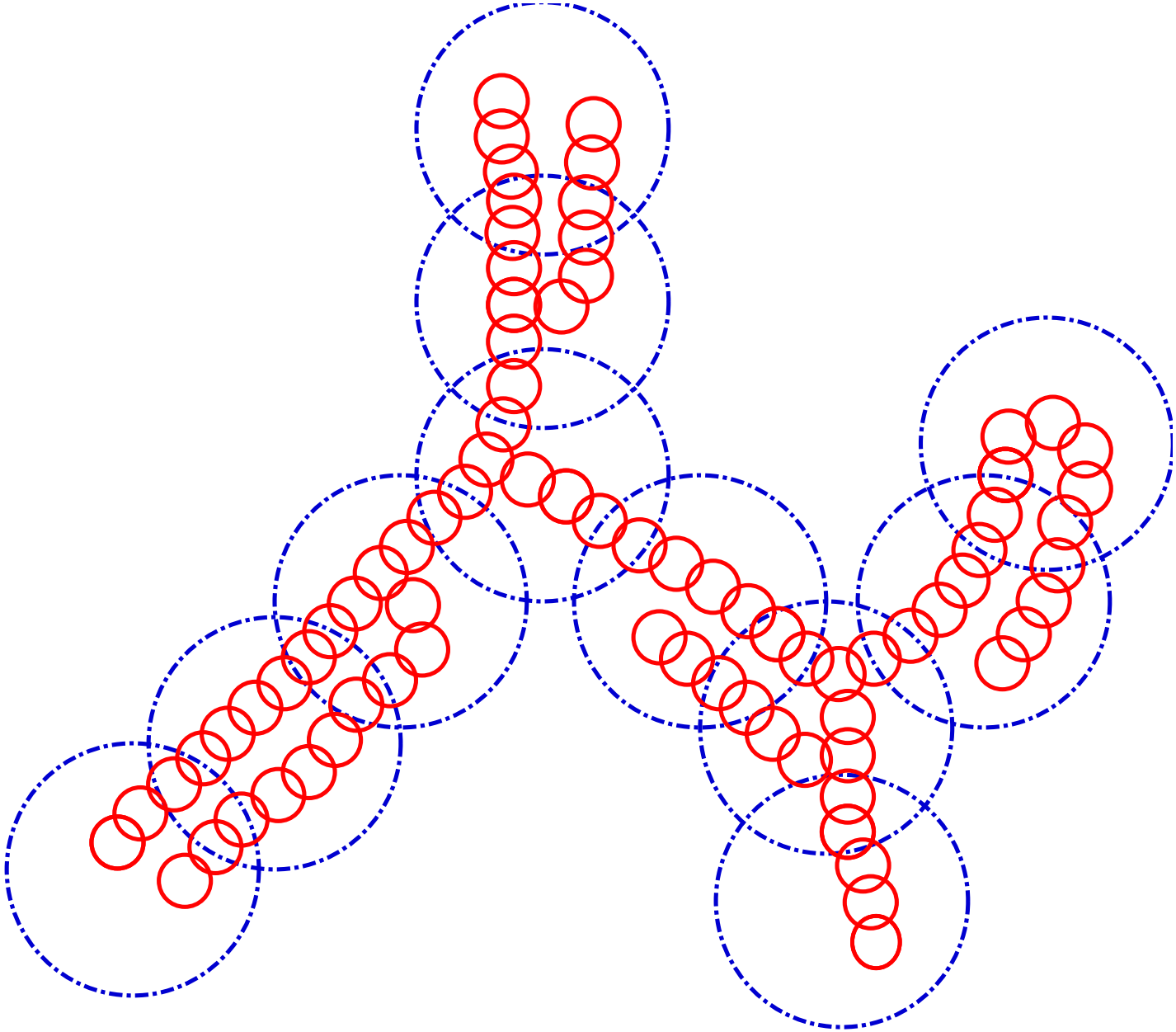


Figure: The nerves are homeomorphic to the same graph G.



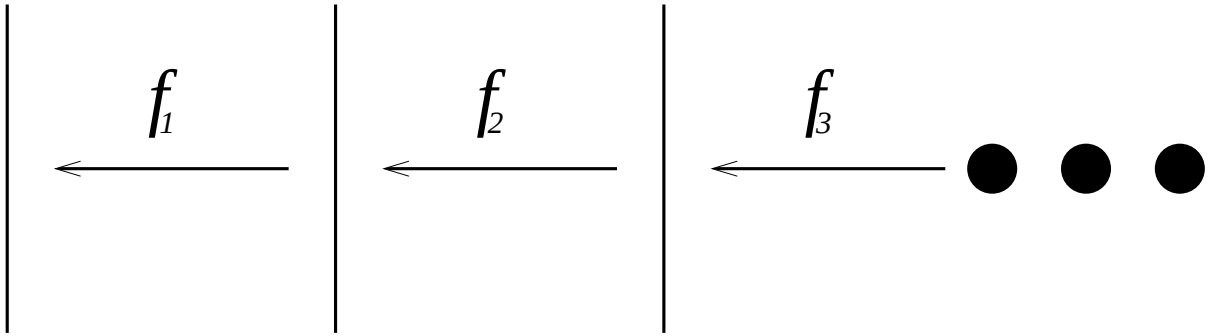


Figure: Arc-like

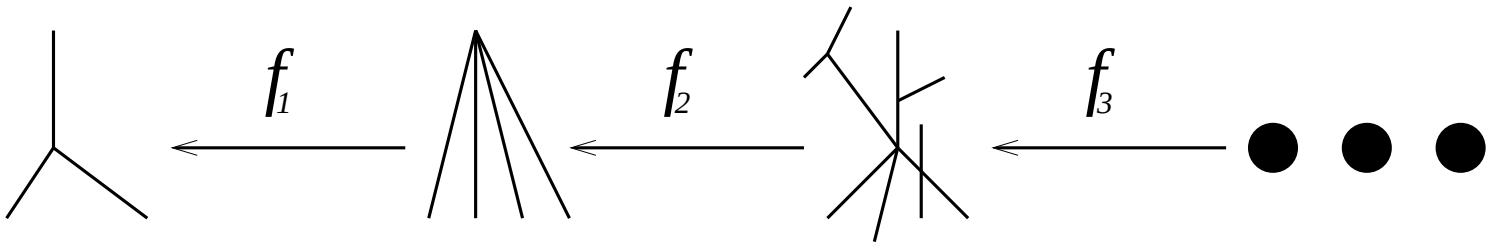


Figure: Tree-like

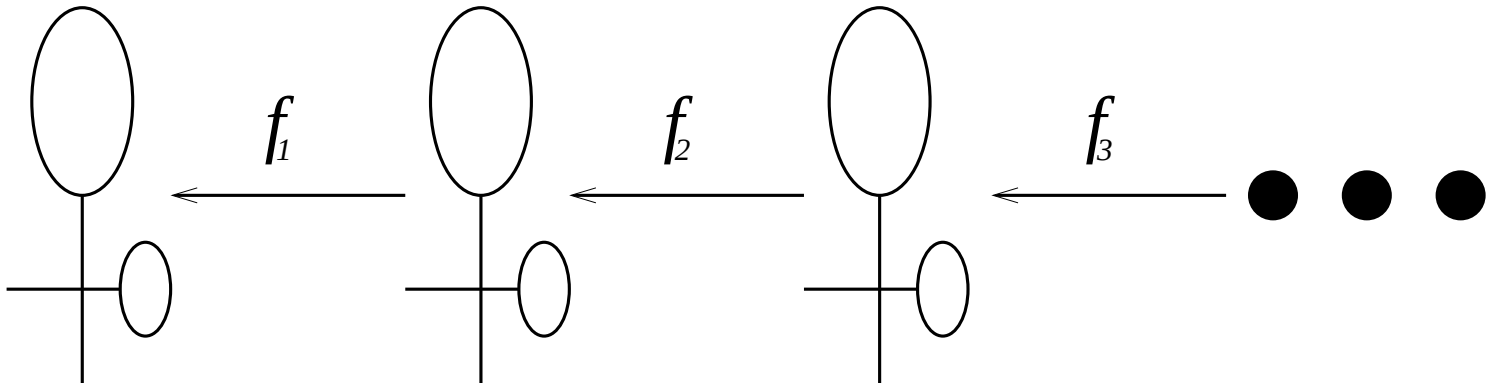


Figure: G-like

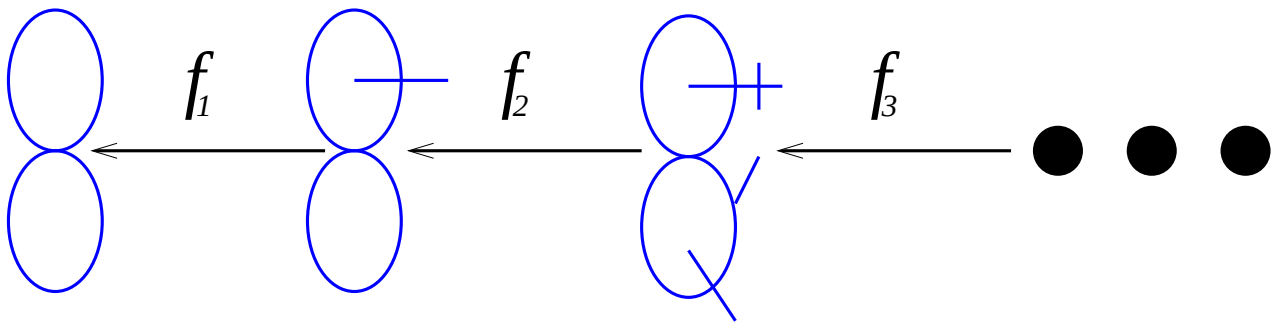


Figure: k-cyclic

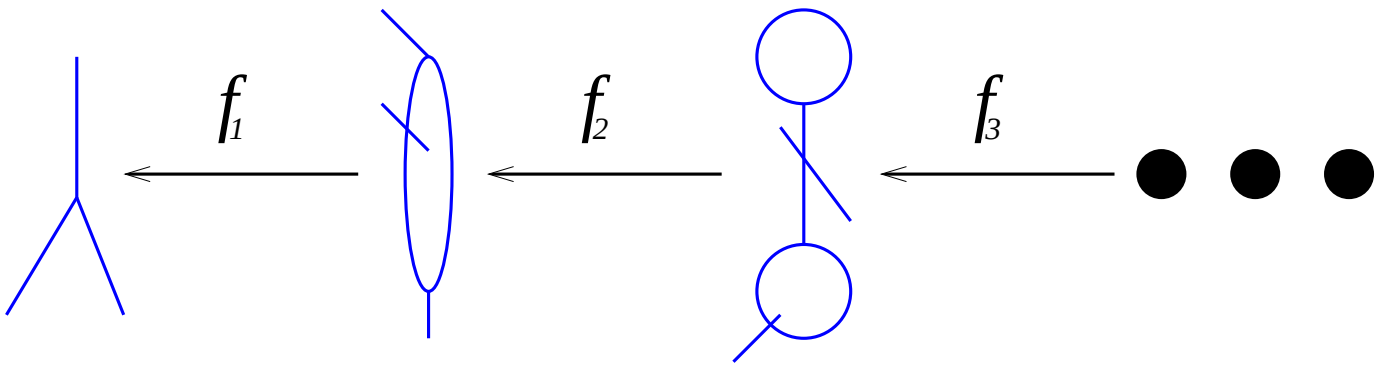
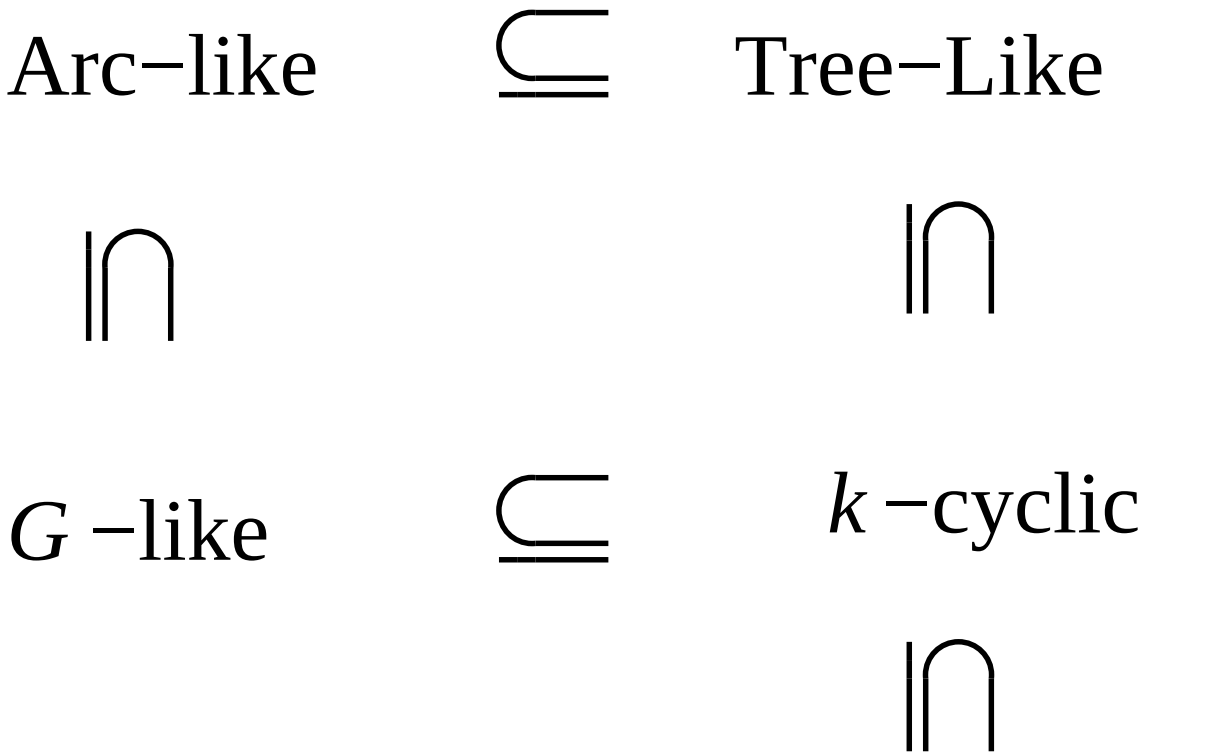


Figure: not k-cyclic


Arc−like
⊆
Tree−Like
∩
∩
G −like
⊆
k −cyclic
∩
All 1−dimensional
continua

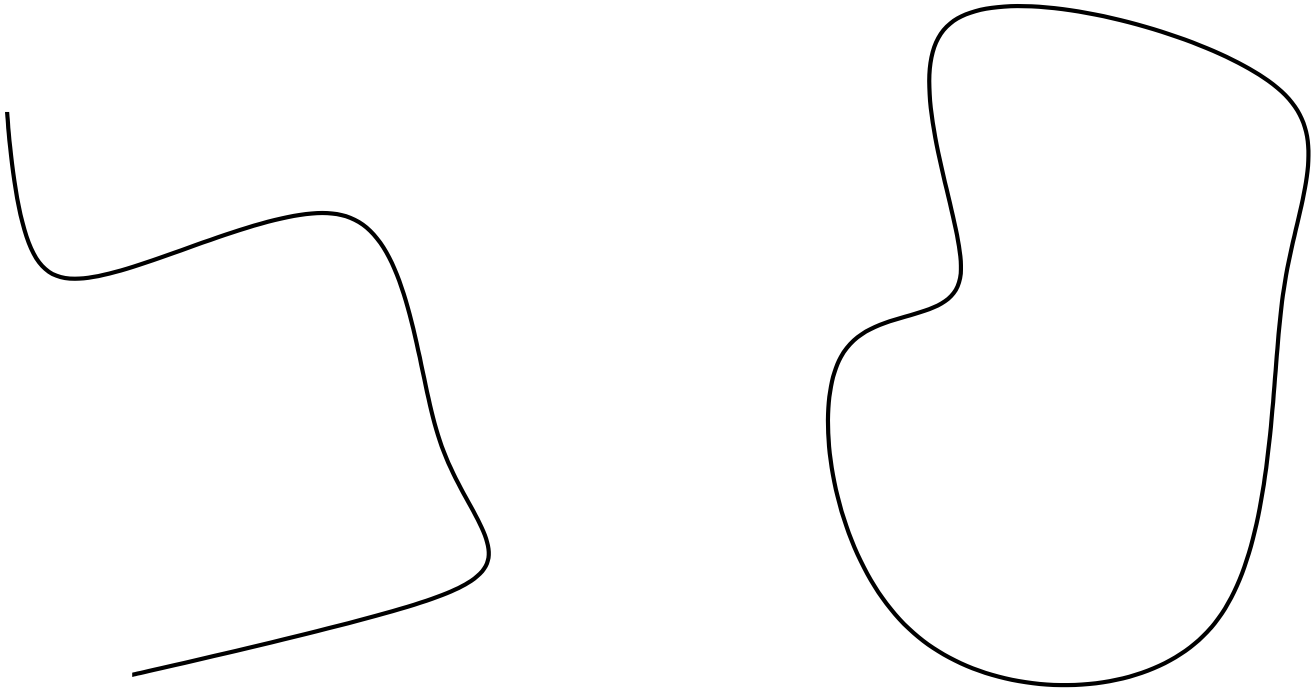


Figure: arc-like and G-like (circle-like).

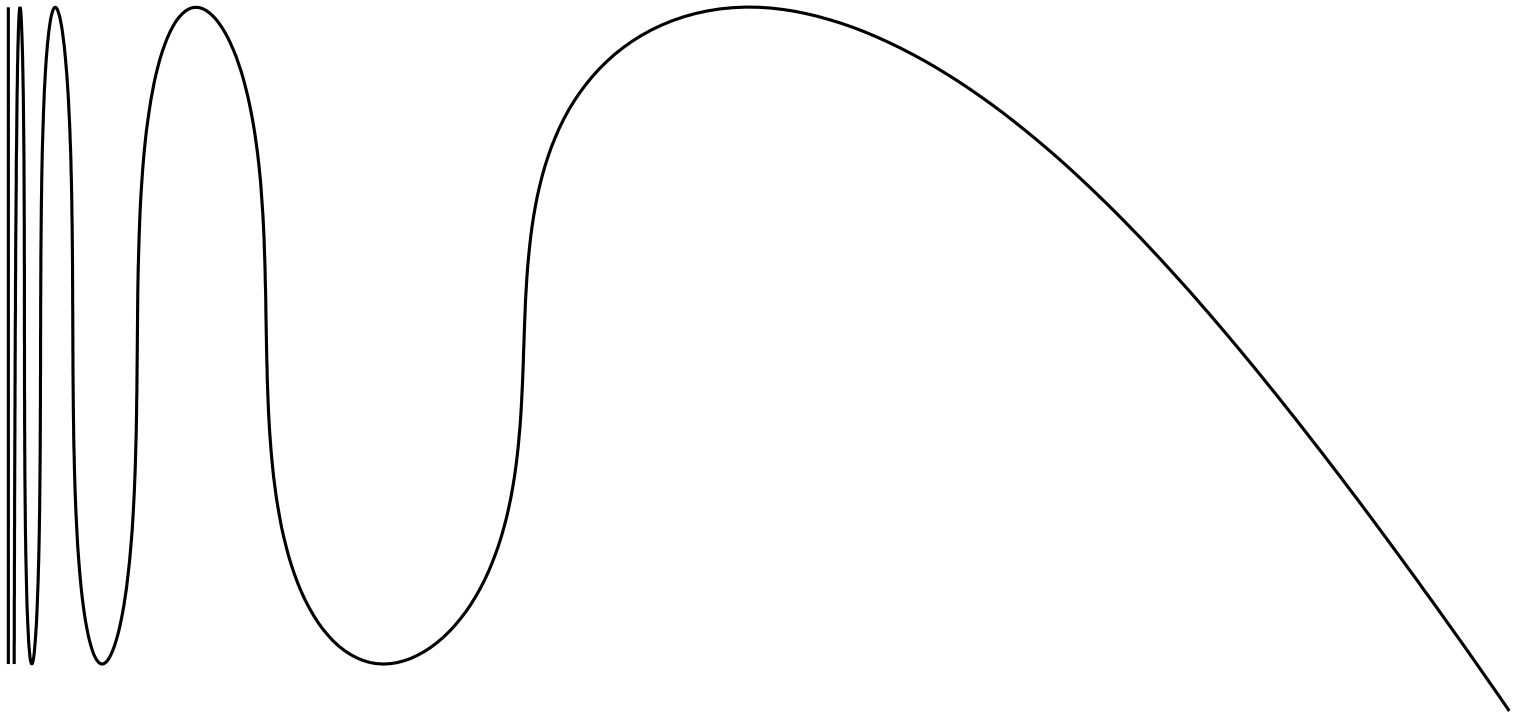


Figure: arc-like

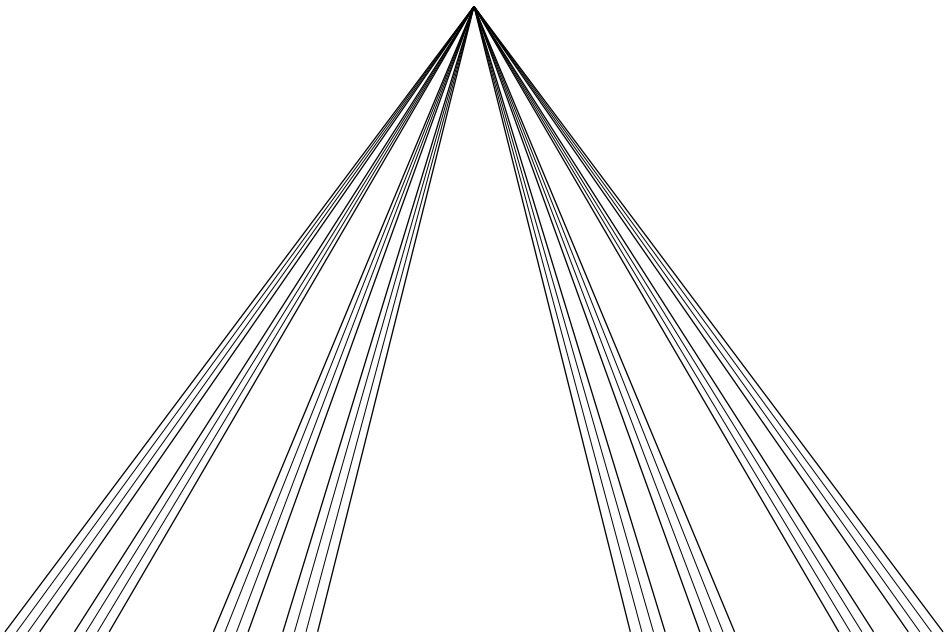


Figure: tree-like but not G-like

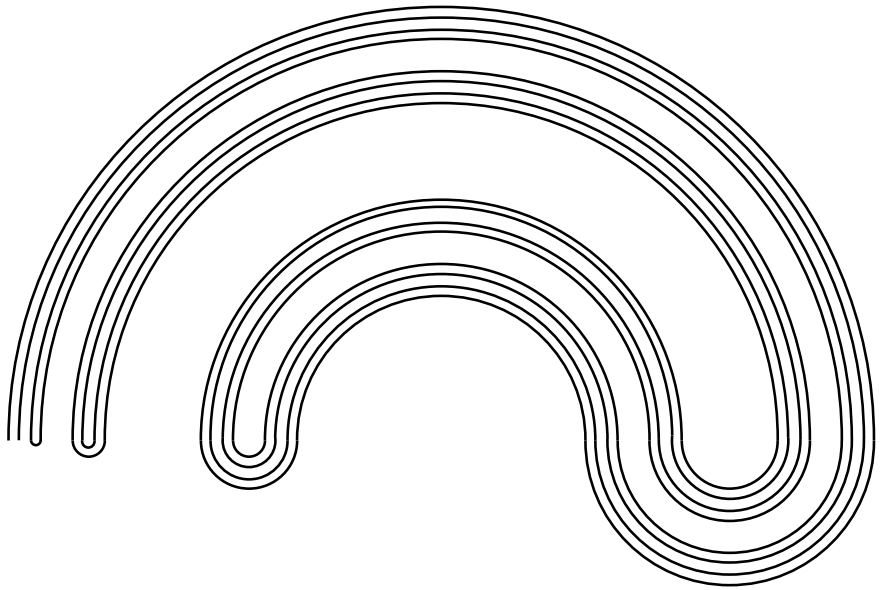


Figure: arc-like

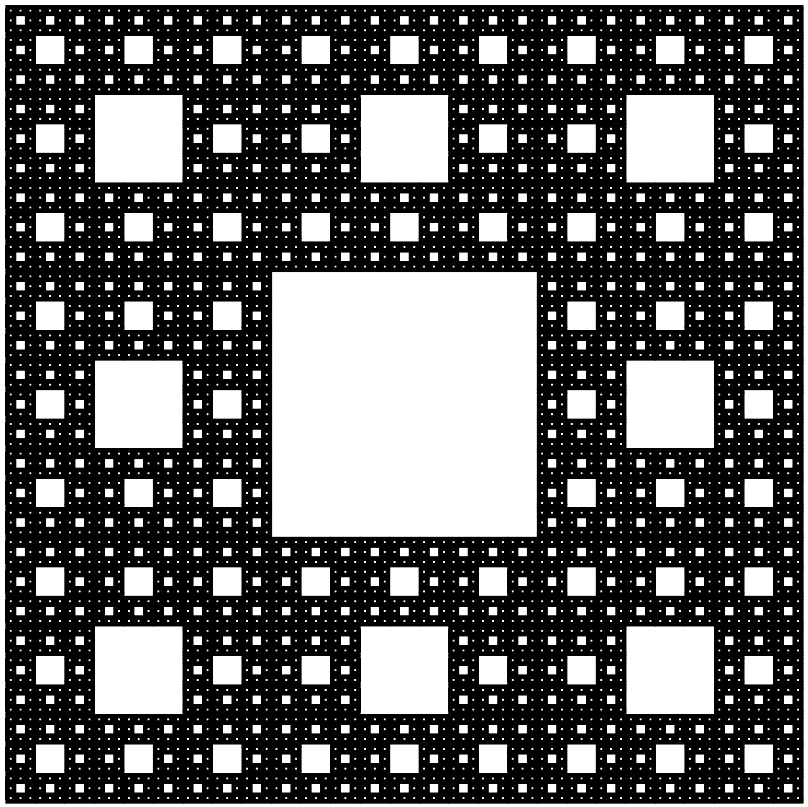


Figure: not k-cyclic

Dynamics

The primary motivation to study the necessary topology of continua that admit homeomorphisms with a particular "chaotic" properties.

In particular, I am interested in what conditions for the continuum to be

1. indecomposable
2. contain an indecomposable subcontinuum
3. $1/n$ indecomposable
4. if hereditarily decomposable, then what can be said about the topology

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A continuum X is *decomposable* if it is the union of 2 of its **proper** subcontinua.

A continuum is *indecomposable* if it is not decomposable.

Equivalently, X is indecomposable if every **proper** subcontinuum is nowhere dense.

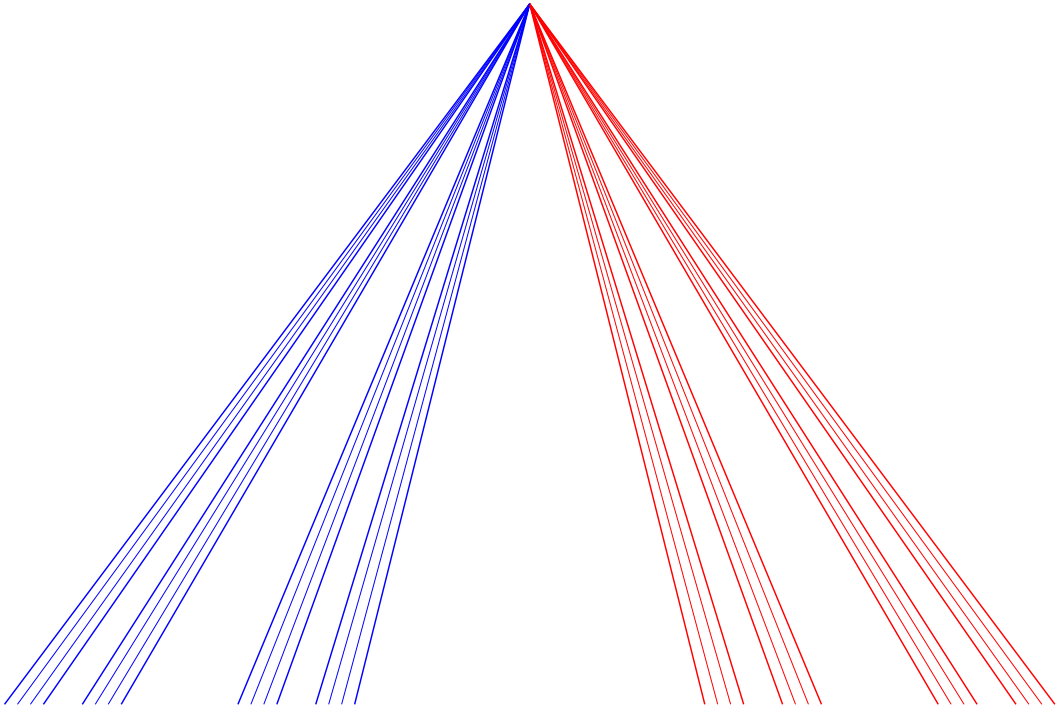
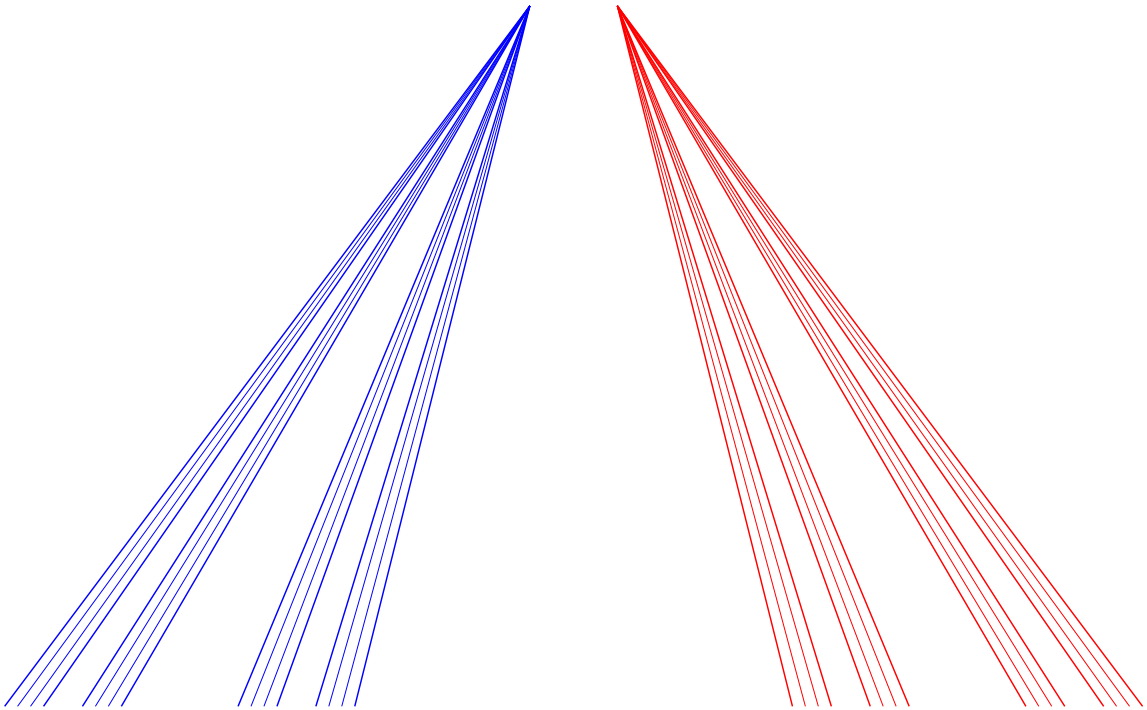


Figure: Decomposable



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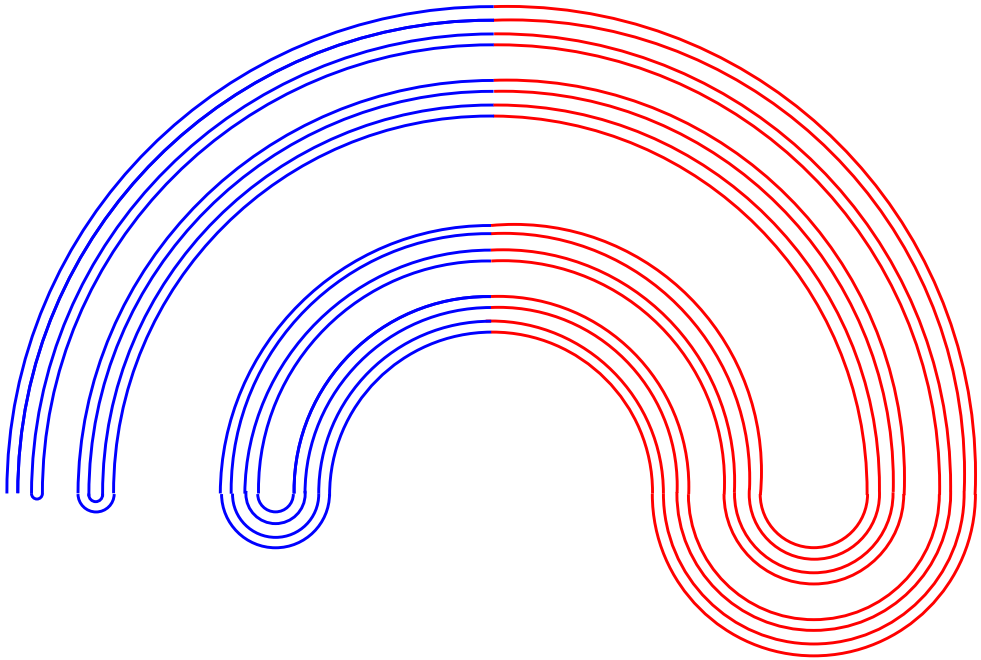
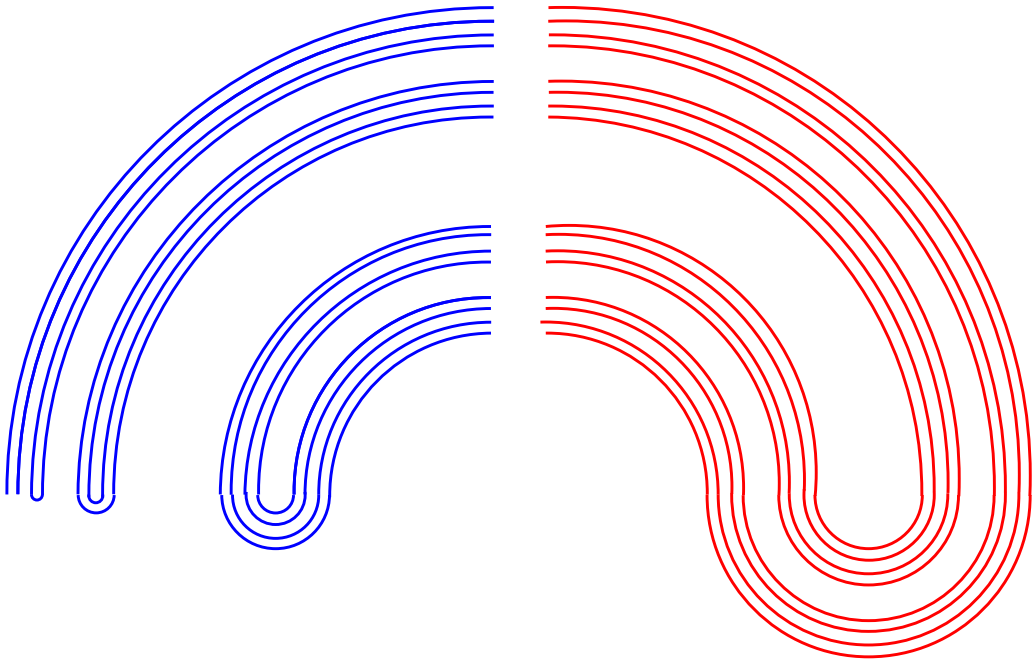


Figure: Indecomposable



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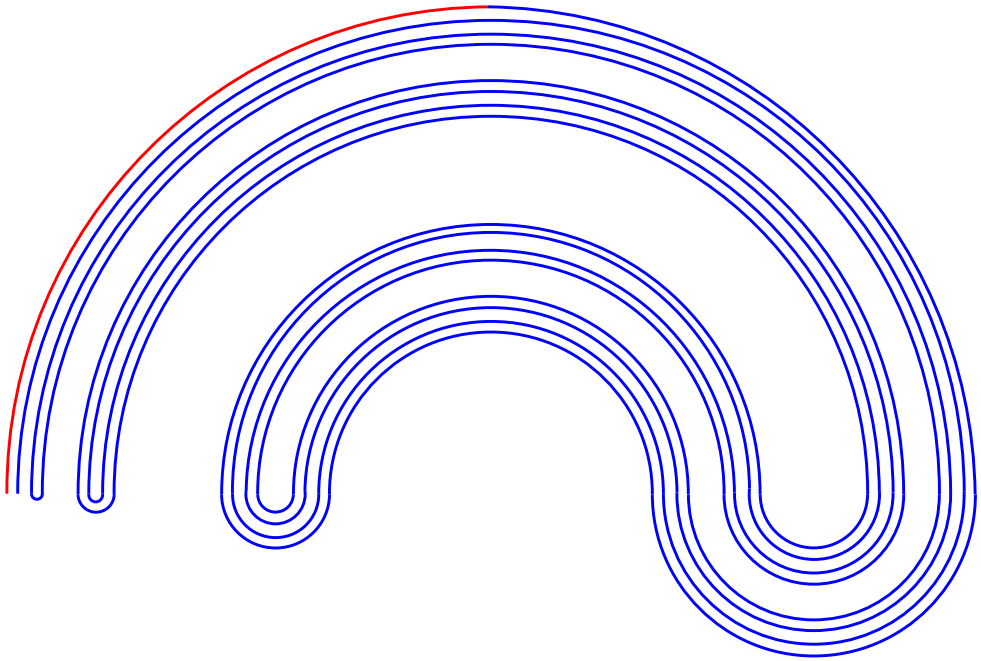

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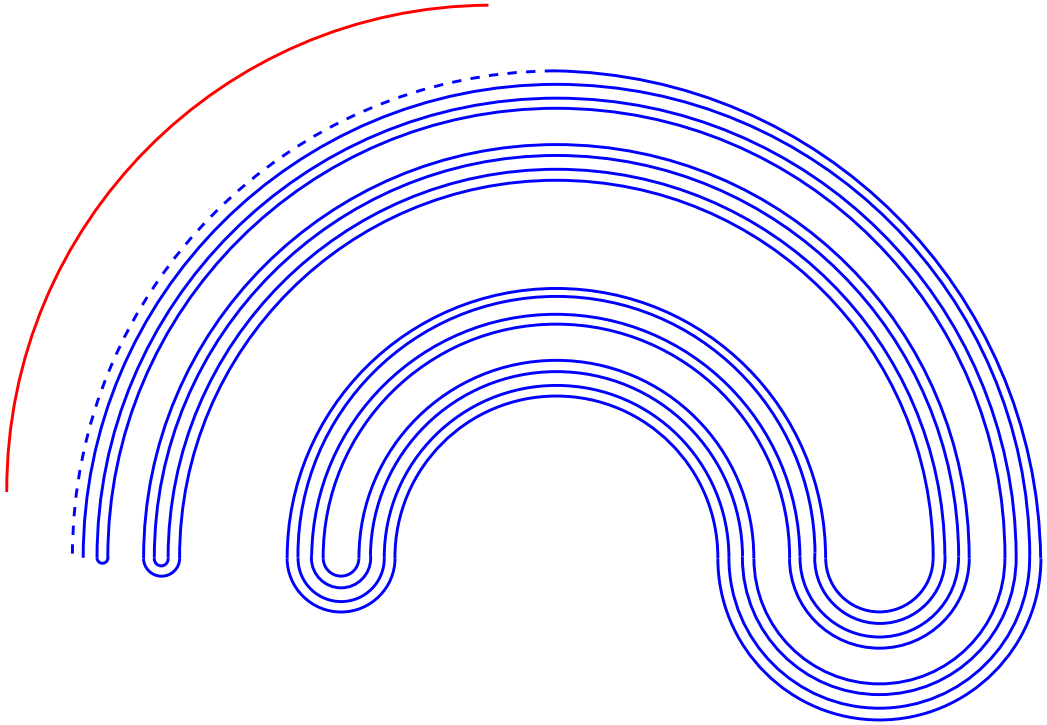

Figure: Indecomposable

A map f is *positively continuum-wise fully expansive* if for every $\epsilon > 0$ and nondegenerate subcontinuum Y, there exists $N = N(\epsilon, Y)$ such that $\mathrm{d}_H(f^n(Y), X) < \epsilon$ for all $n \geq N$.

A map f is *weakly positively continuum-wise fully expansive* if there exist a dense subset Z of the hyperspace of X such that for every $\epsilon > 0$ and subcontinuum $Y \in Z$, there exists $N = N(\epsilon, Y)$ such that $\mathrm{d}_H(f^n(Y), X) < \epsilon$ for all $n \geq N$.

Map f is *continuum-wise expansive* if there exists a $c > 0$ such that for every nondegenerate subcontinuum $Y \subset X$, there is an $n \in \mathbb{N}$ such that $\operatorname{diam}(f^n(Y)) \geq c$.

A map f is *mixing* if for every open sets U, V of X^n, there exists an M such that $f^m(U) \cap V \neq \emptyset$ for all $m \geq M$.

That is, if U is open, then $\mathrm{d}_H(f^n(U), X) \to 0$ as $n \to \infty$.

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Theorem

If $f : X \longrightarrow X$ is a map of a continuum then:

$$f \text{ is fully continuum-wise expansive.}$$
$$\Downarrow$$
$$f \text{ is weakly fully continuum-wise expansive.}$$
$$\Downarrow$$
$$f \text{ is mixing.}$$

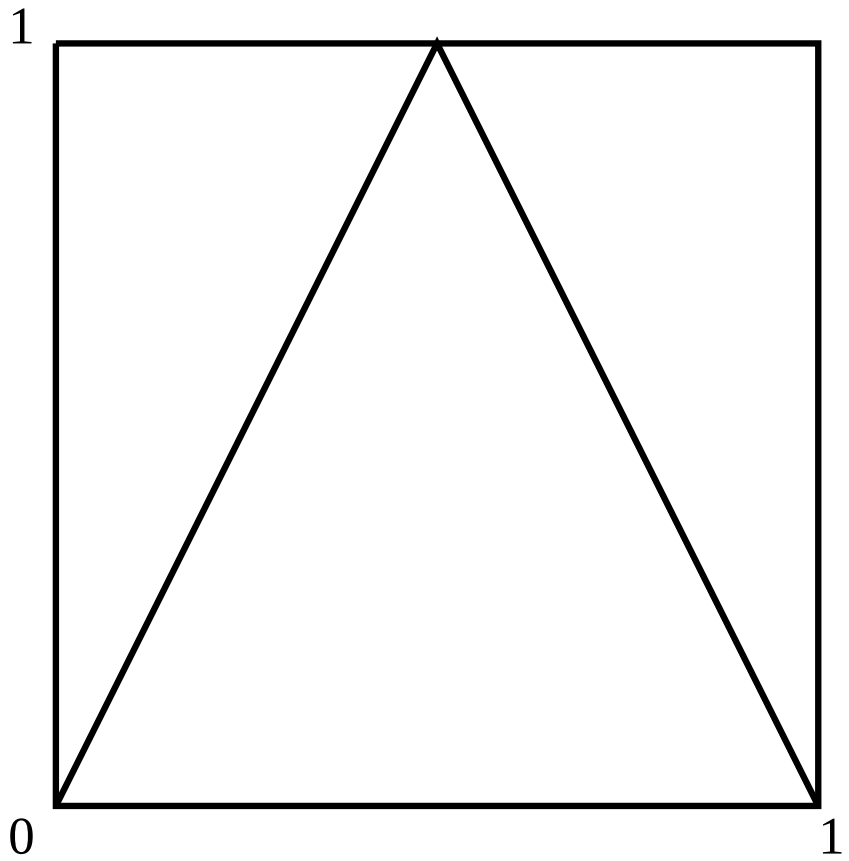
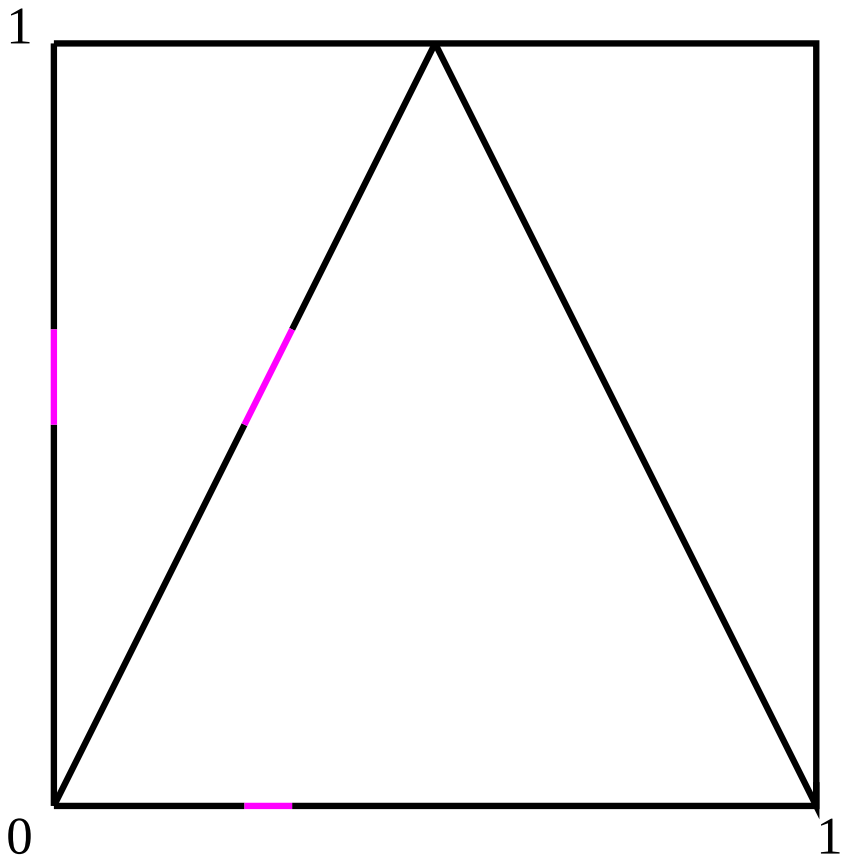


Figure: The tent map, $T : [0, 1] \longrightarrow [0, 1]$, is a positively continuum-wise fully expansive function.



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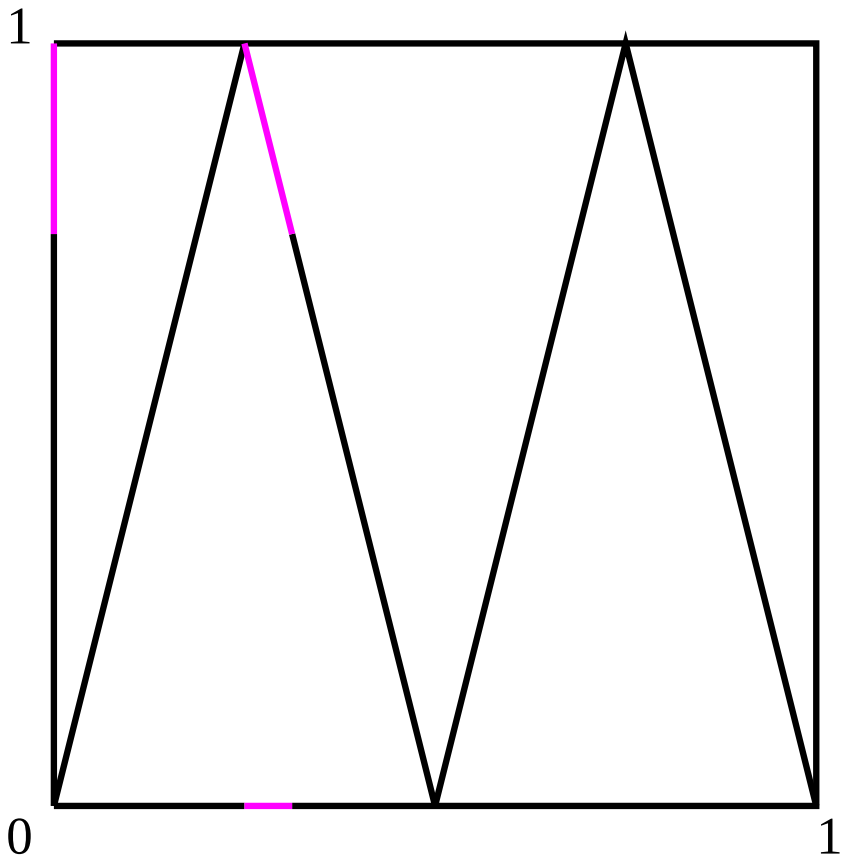

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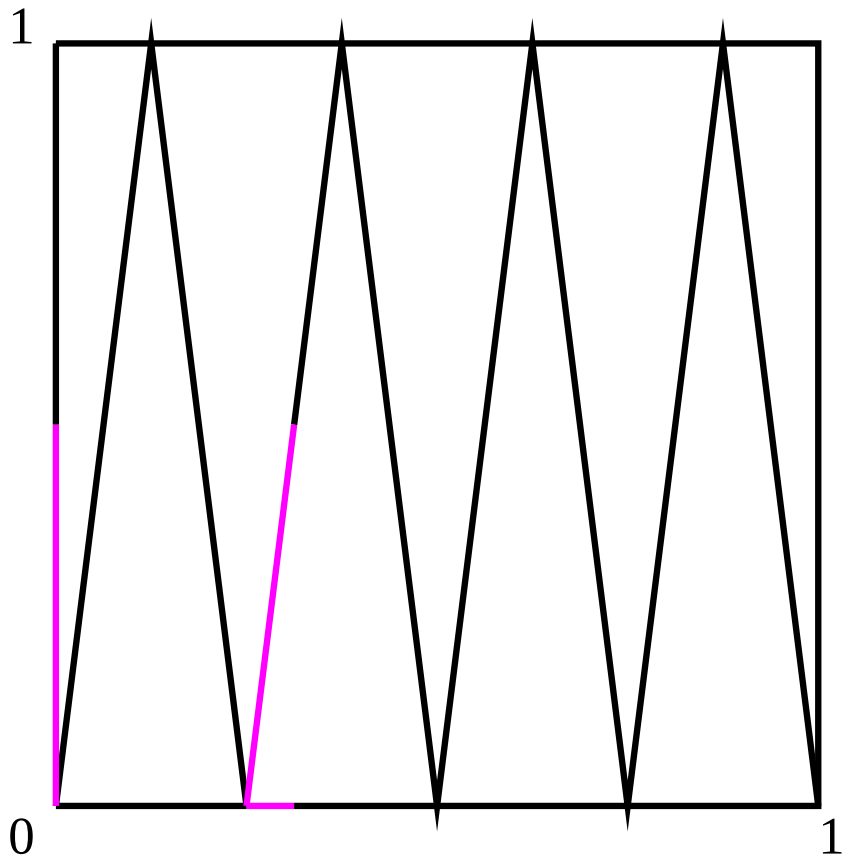

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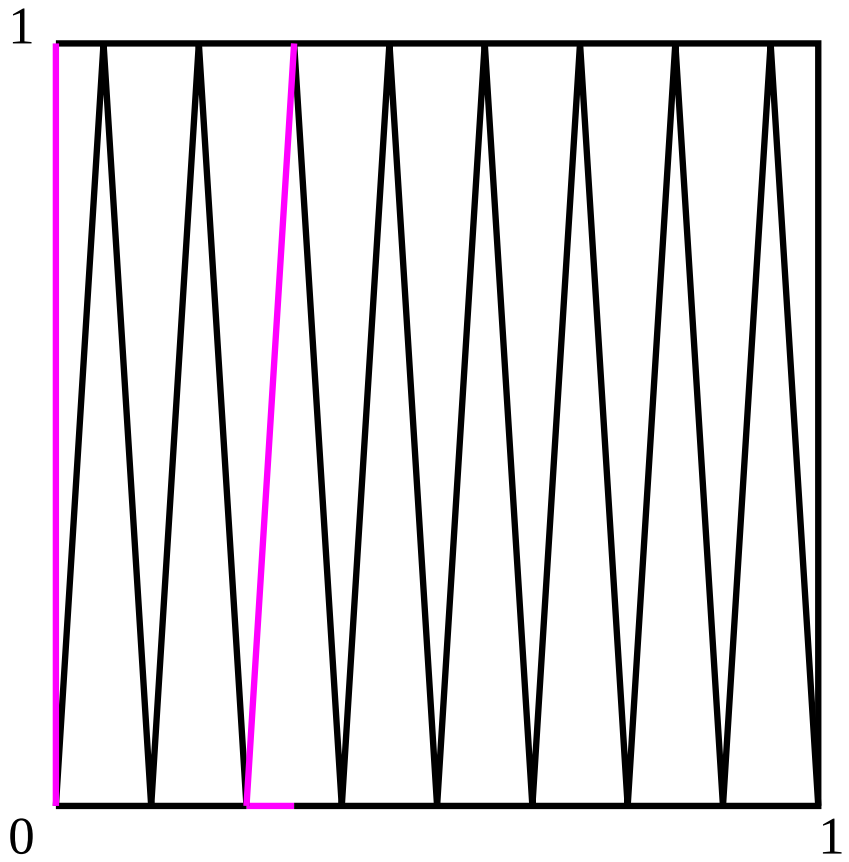

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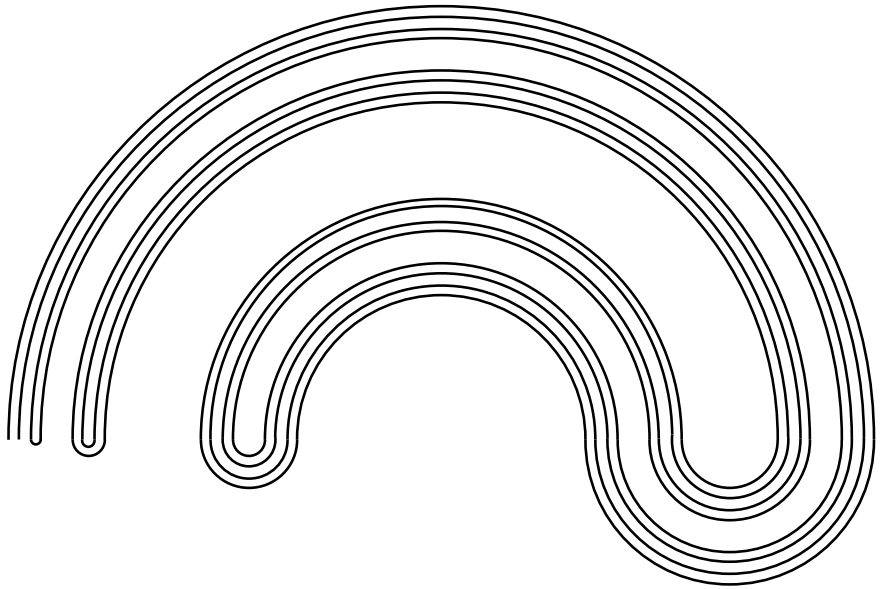
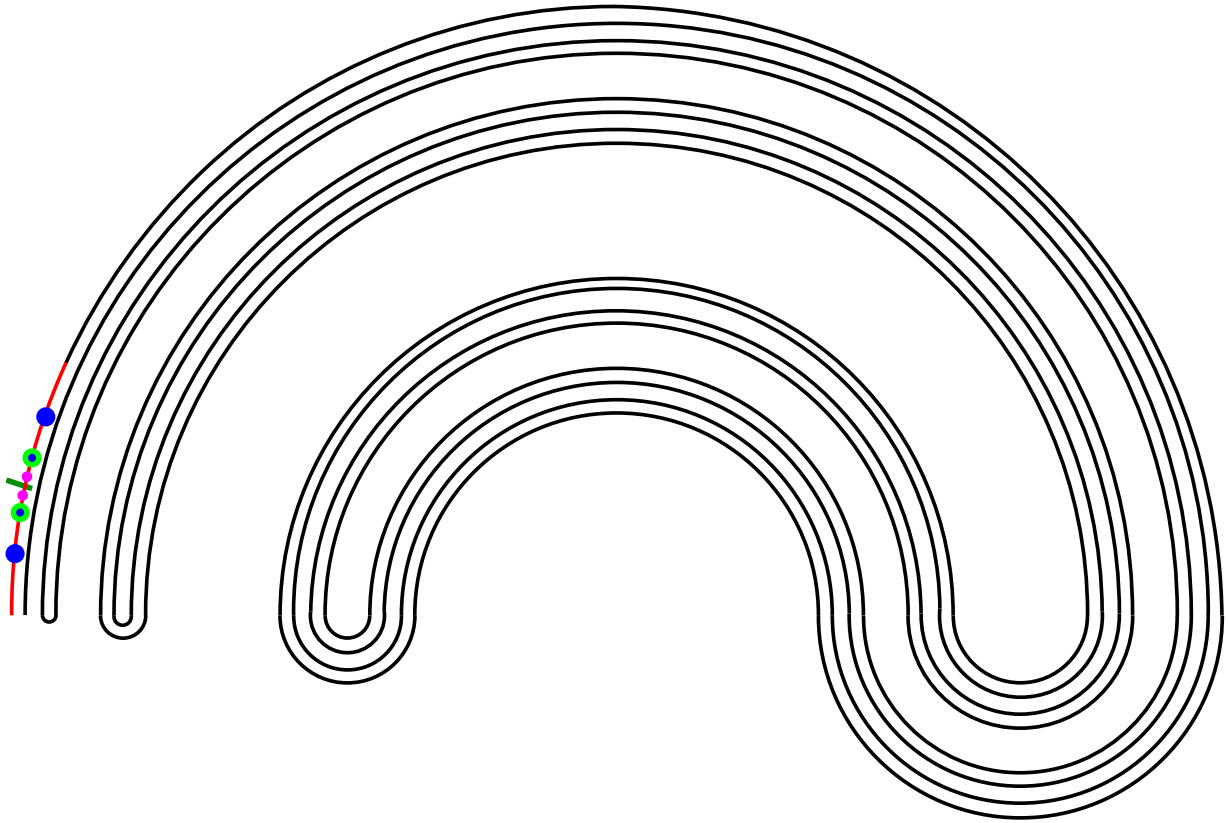


Figure: The buckhandle continuum is the inverse limit of the tent map: $\varprojlim([0,1], T)$



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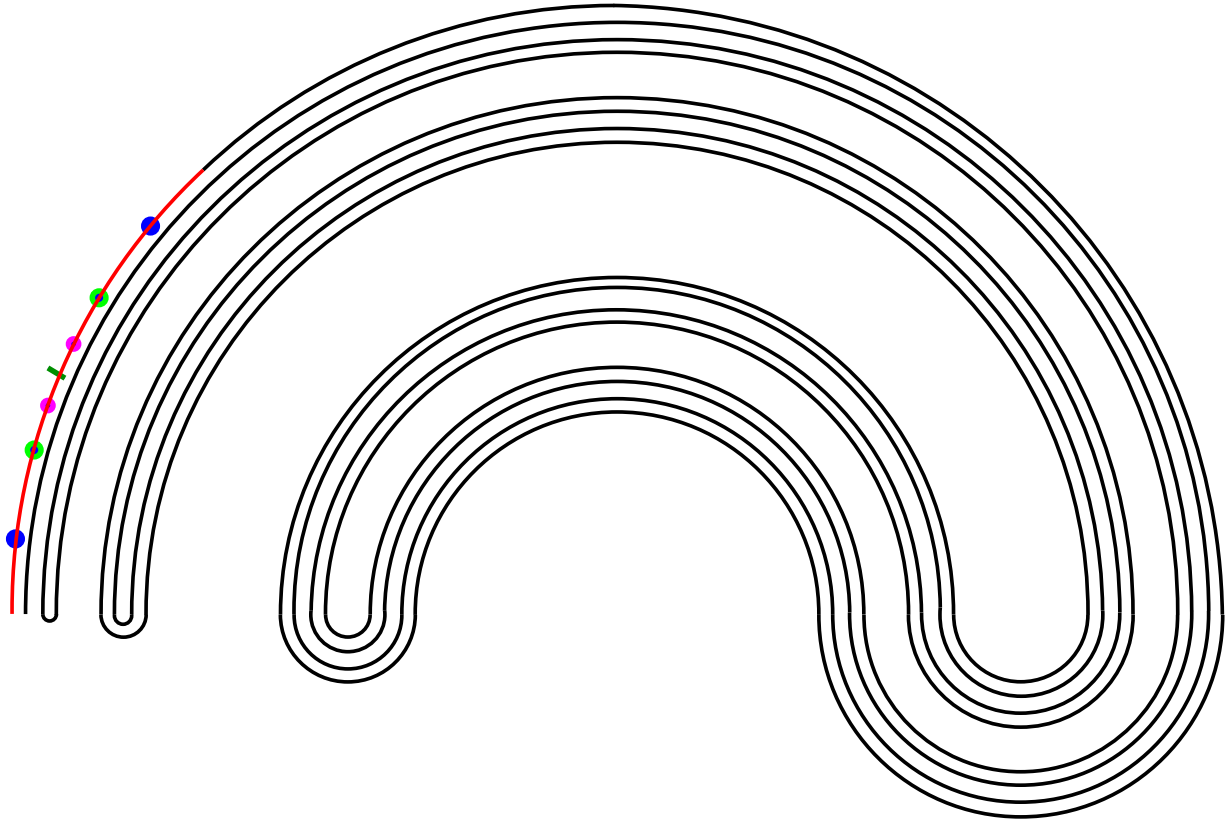

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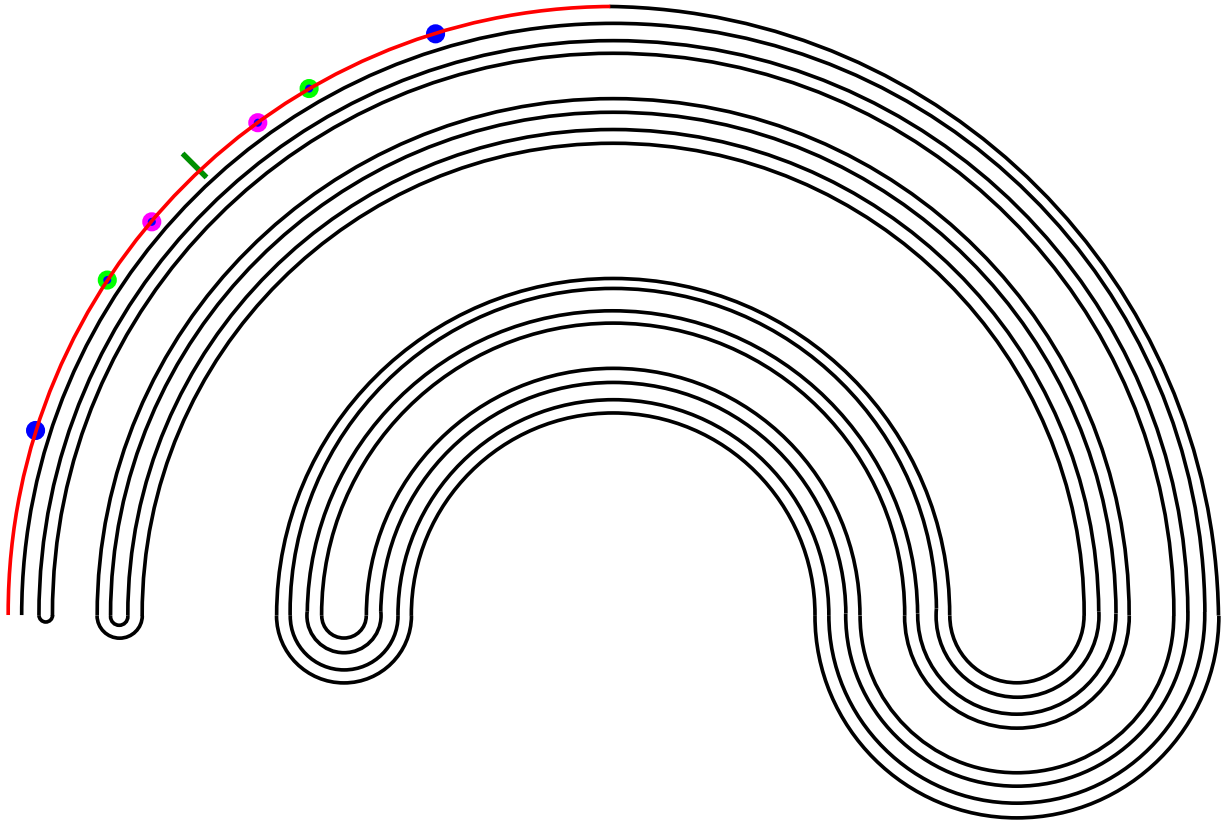

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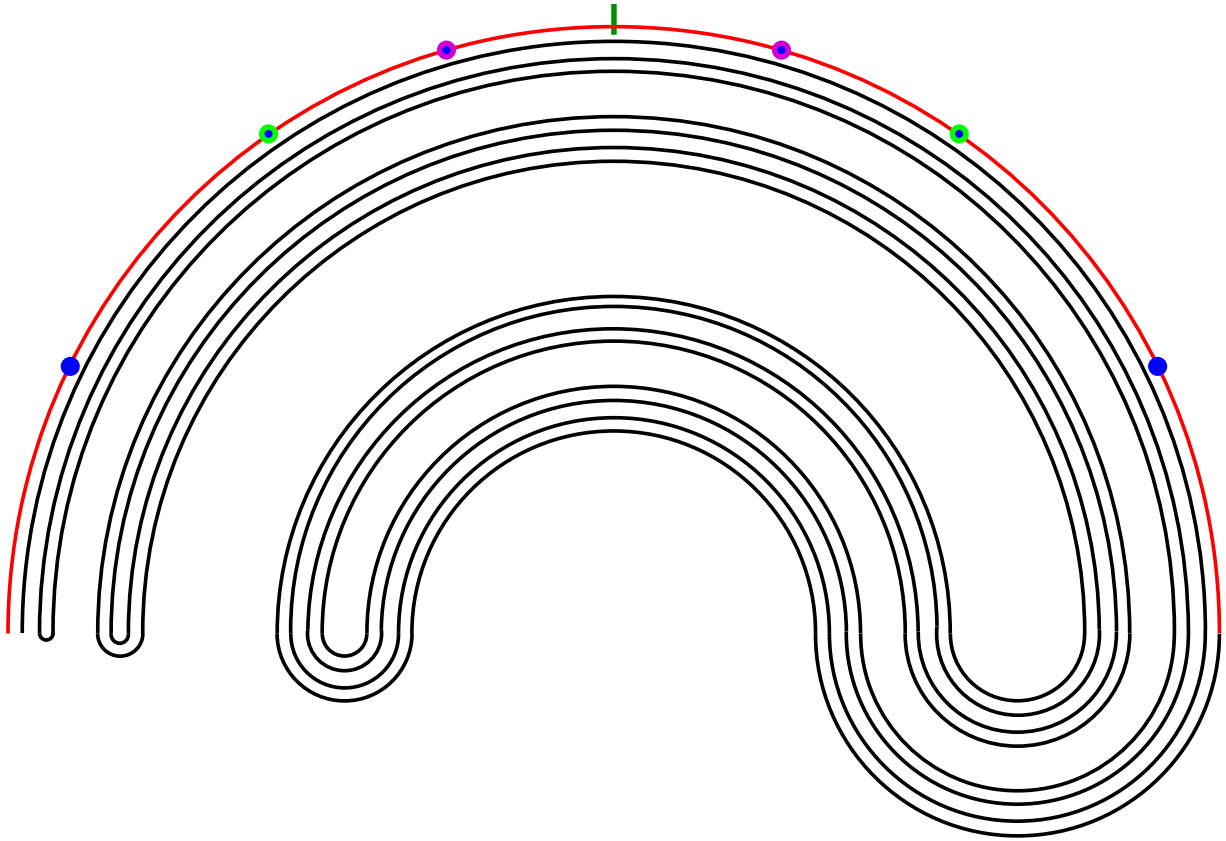

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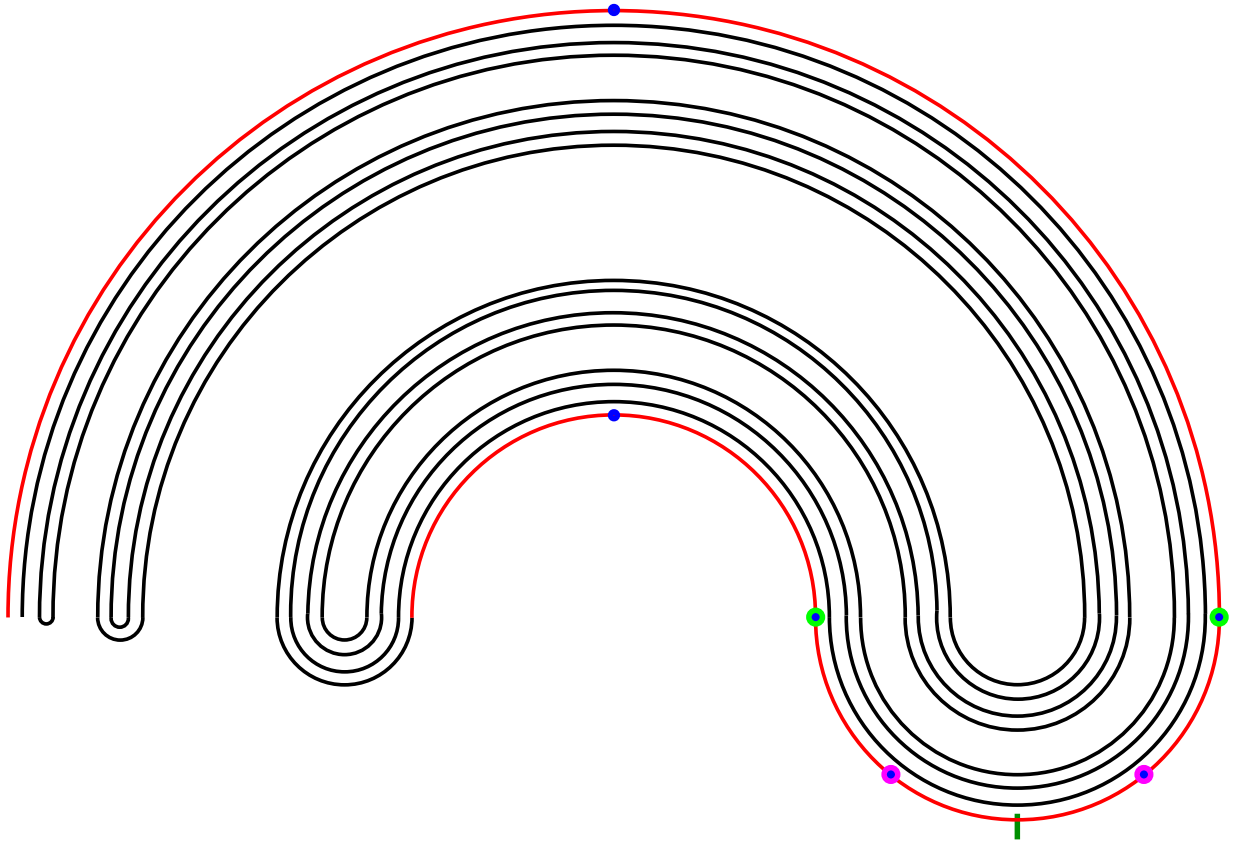

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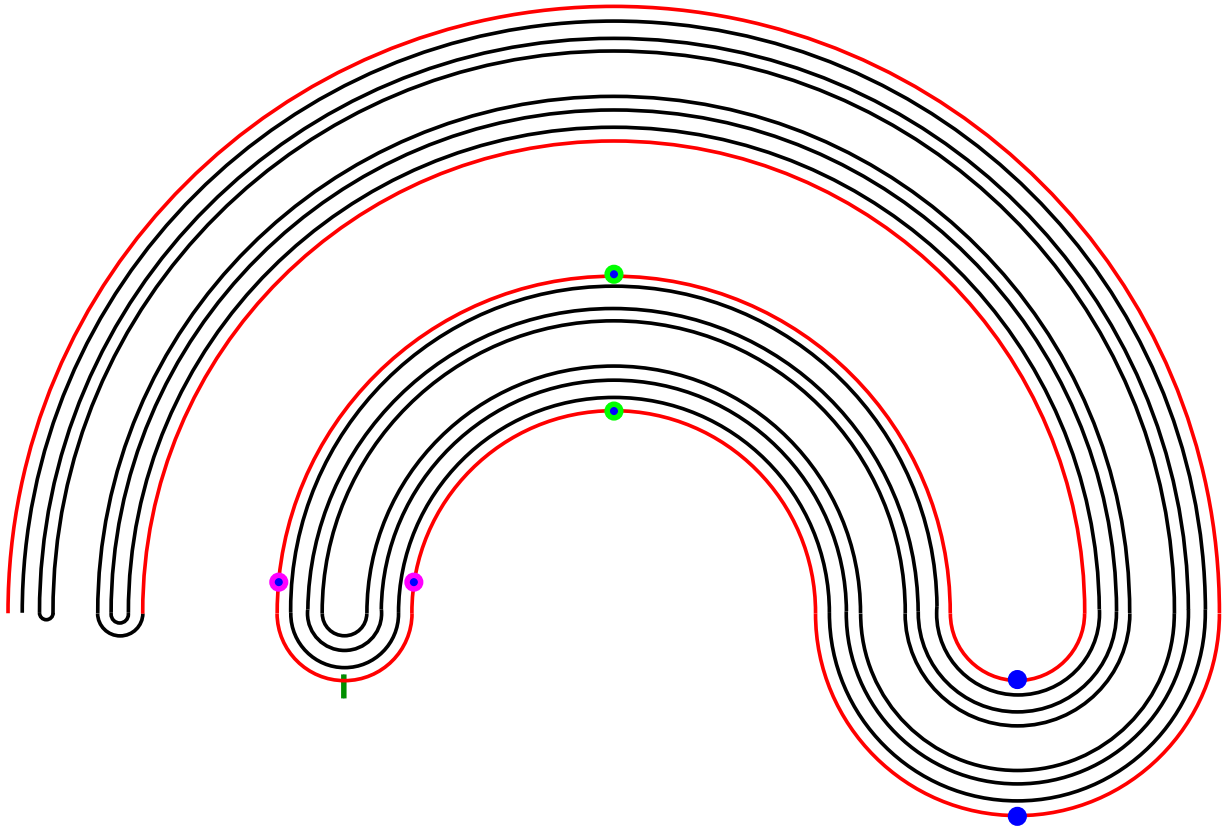

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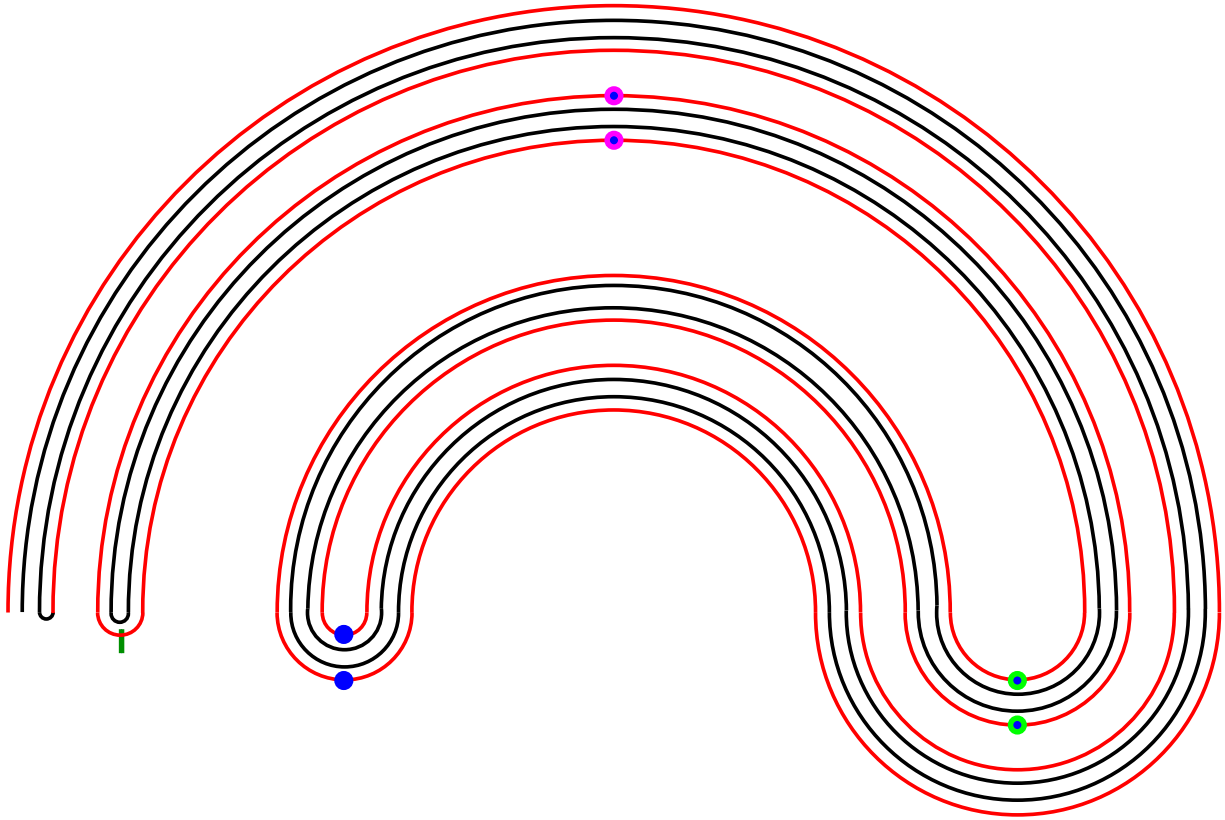

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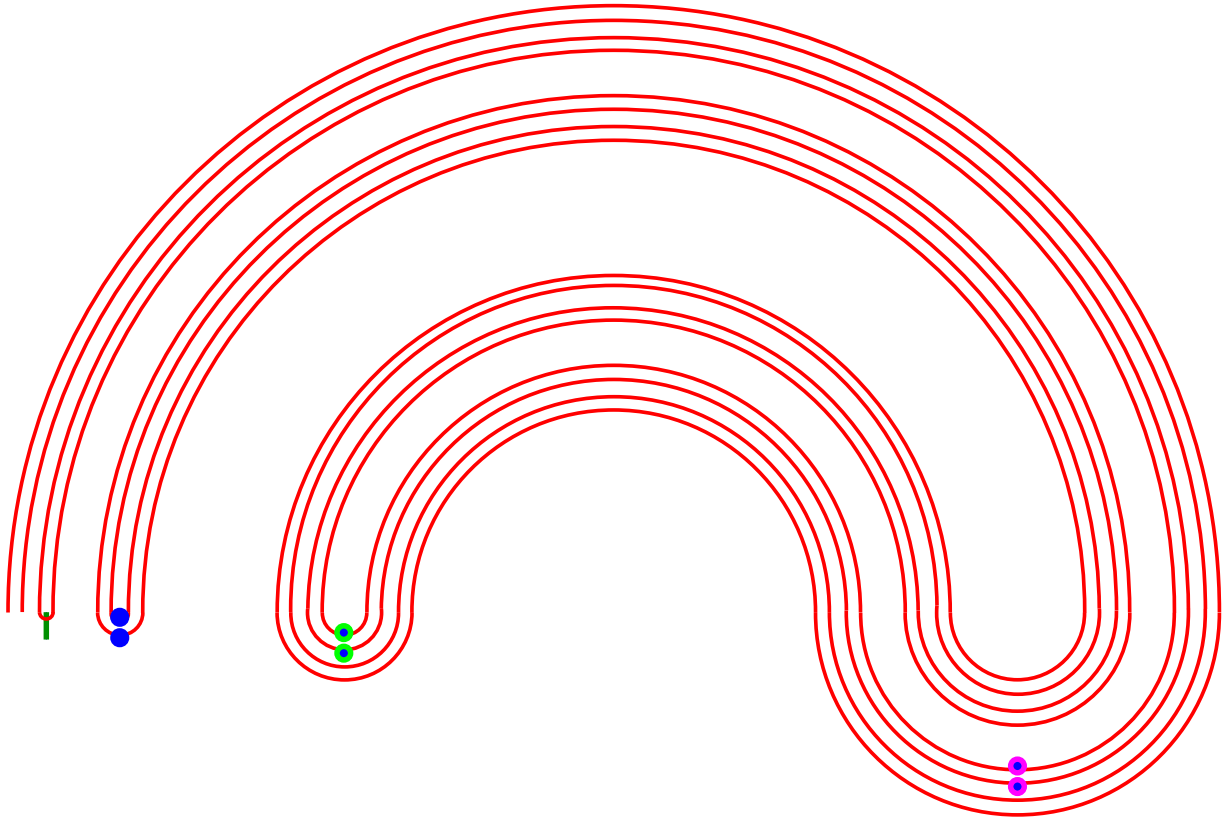

The shift of the tent map, $\widehat{T}$, is a positively continuum-wise fully expansive homeomorphism.

Theorem

(Kato) If X is a continuum that admits a positively fully continuum-wise expansive homeomorphism then X is indecomposable.

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(Mouron) If X is a finitely cyclic continuum that admits a continuum wise expansive homeomorphism, then X must contain a nondegenerate indecomposable sub continuum.

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A continuum X is $1/n$ *indecomposable* if whenever $\{A_i\}_{i=1}^n$ are a collection of pairwise disjoint subcontinua, at least one of $\{A_i\}_{i=1}^n$ has empty interior. Note: $1/2$ indecomposable is also called *semi-indecomposable*.

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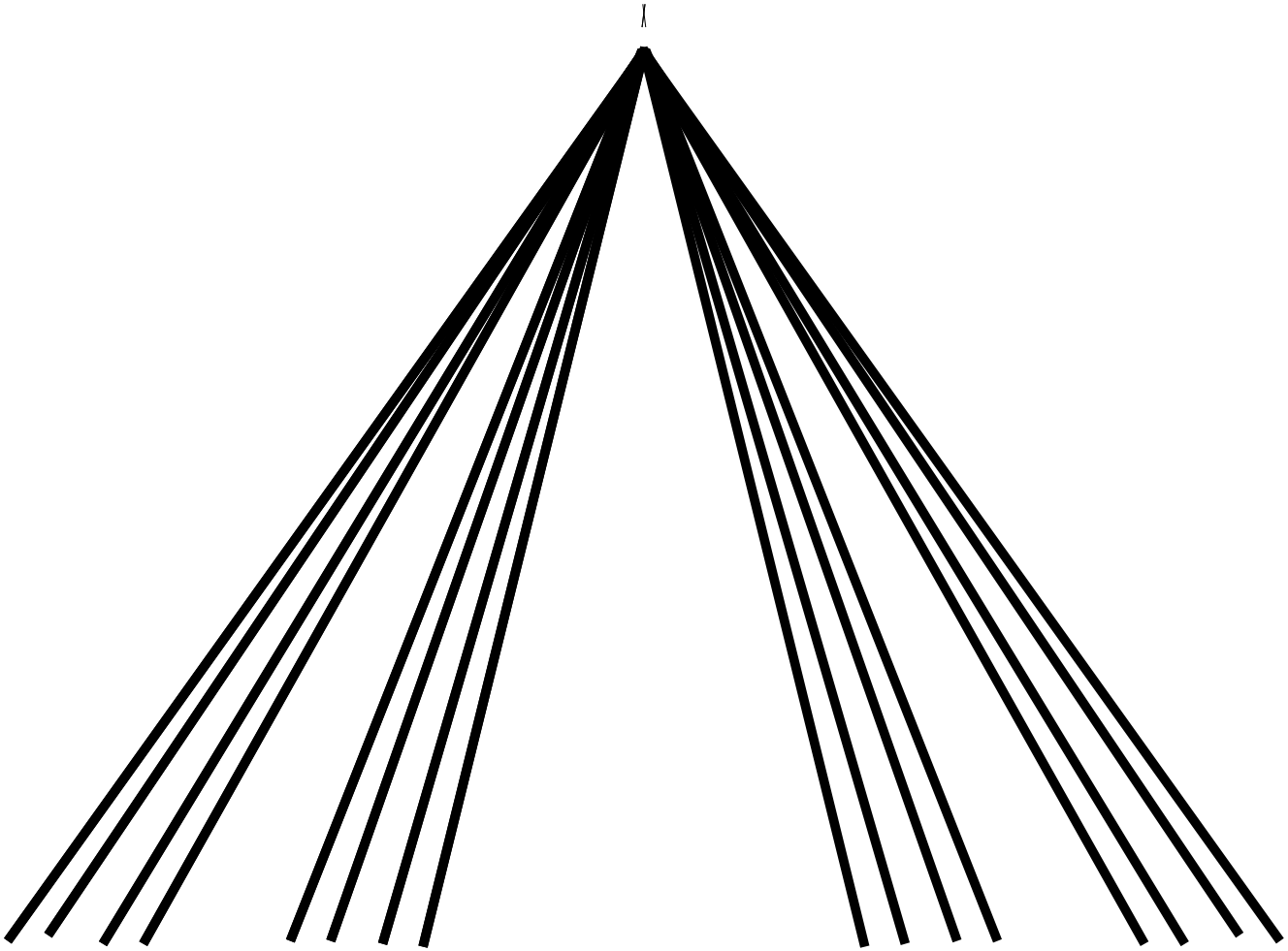
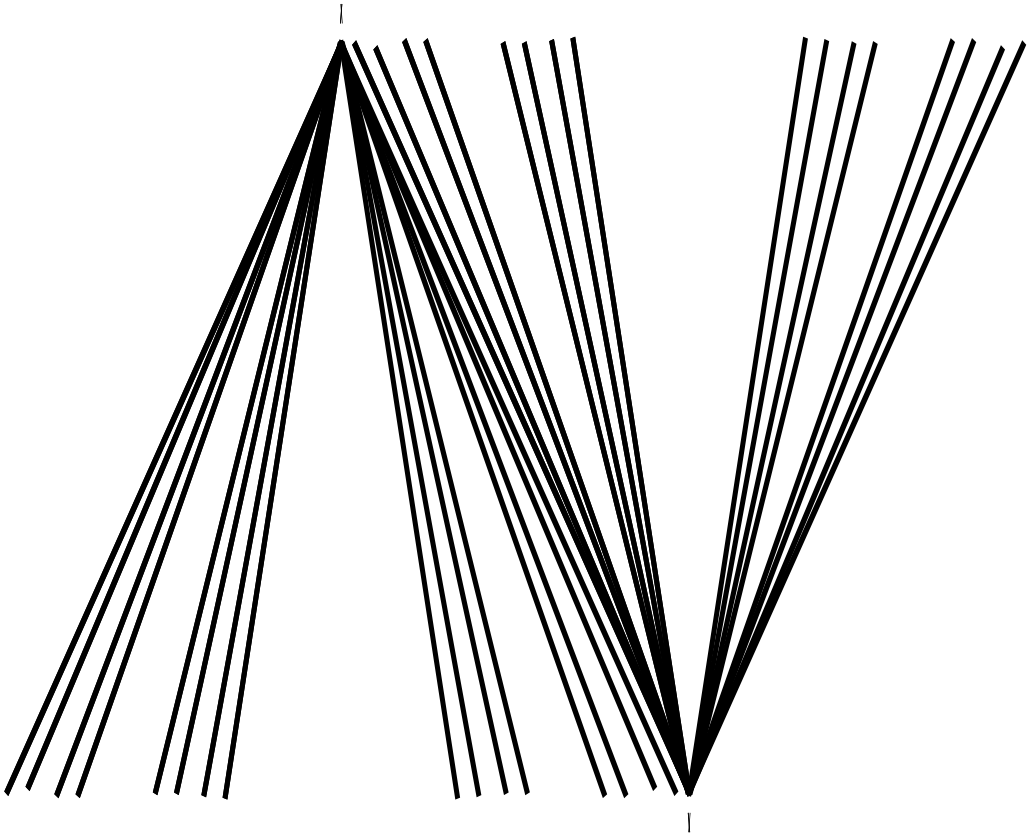
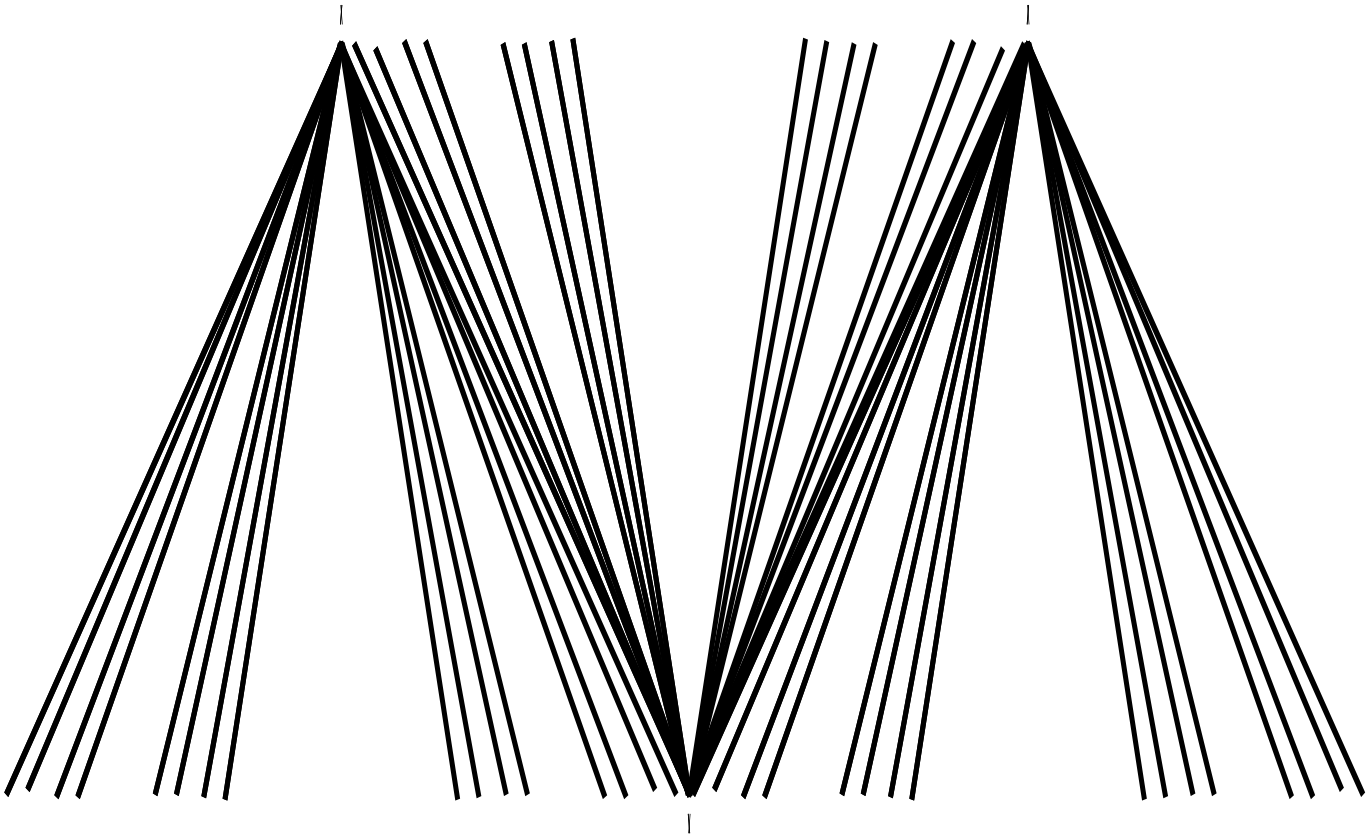


The Cantor Fan is 1/2 or *semi-* indecomposable.



This is 1/3-indecomposable.

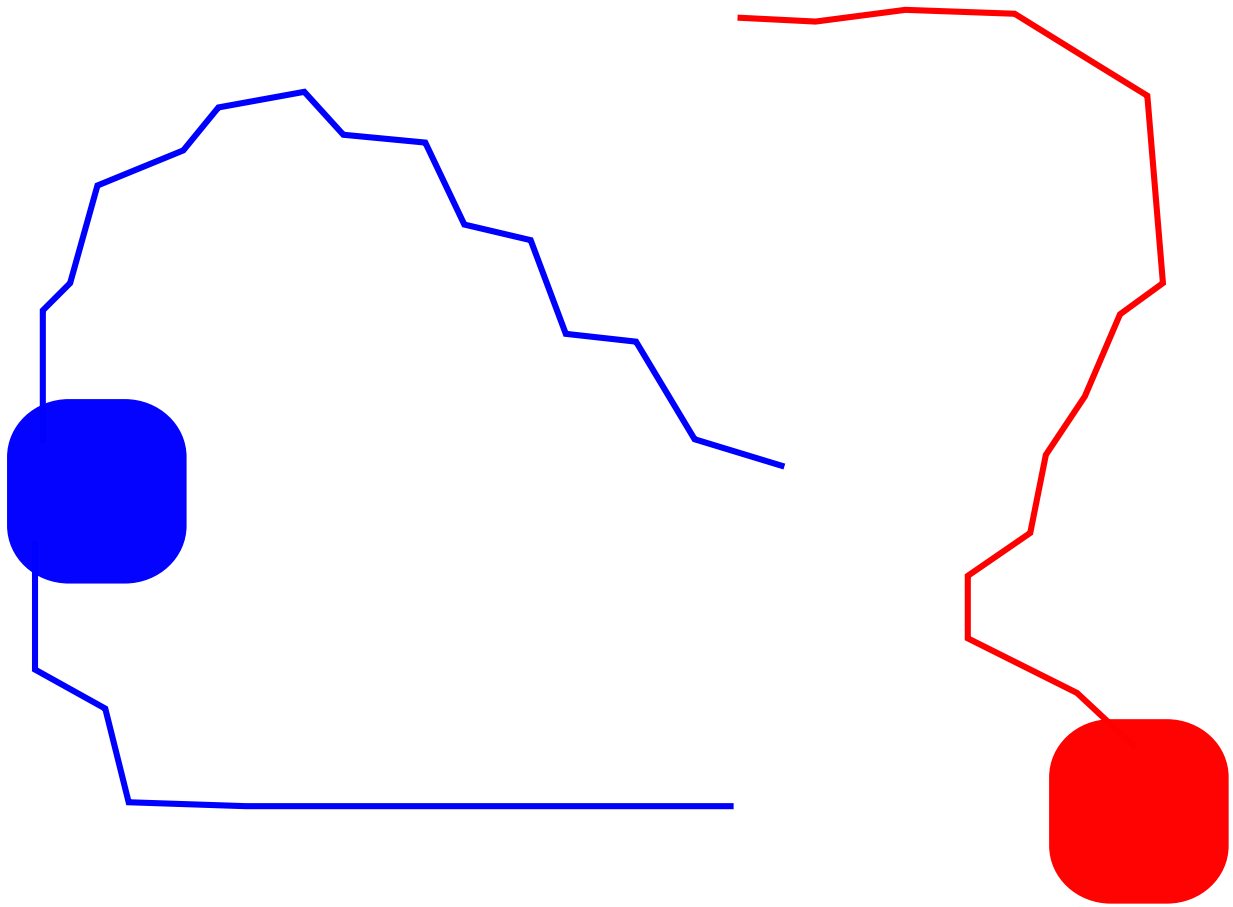


This is 1/4-indecomposable.

Theorem

Let X be a G-like continuum where G is a graph with $m \geq 0$ cycles. If for every $\varepsilon > 0$ there exists a collection $\{A_1^\varepsilon, \ldots, A_{m+2}^\varepsilon\}$ of pairwise disjoint subcontinua of X such that for each i, $d_H(A_i^\varepsilon, X) < \varepsilon$, then X is semi-indecomposable.

Suppose on the contrary that X has two disjoint subcontinua, H and K, that have nonempty interior. Let $x_H \in \text{int}(H)$ and $x_K \in \text{int}(K)$. Then there exists $0 < \epsilon < (1/3)\text{d}(H, K)$ such that $B_\epsilon(x_H) \subset \text{int}(H)$ and $B_\epsilon(x_K) \subset \text{int}(K)$. Let $\mathcal{U}$ be a open cover of X with mesh less than ϵ and whose nerve is G.



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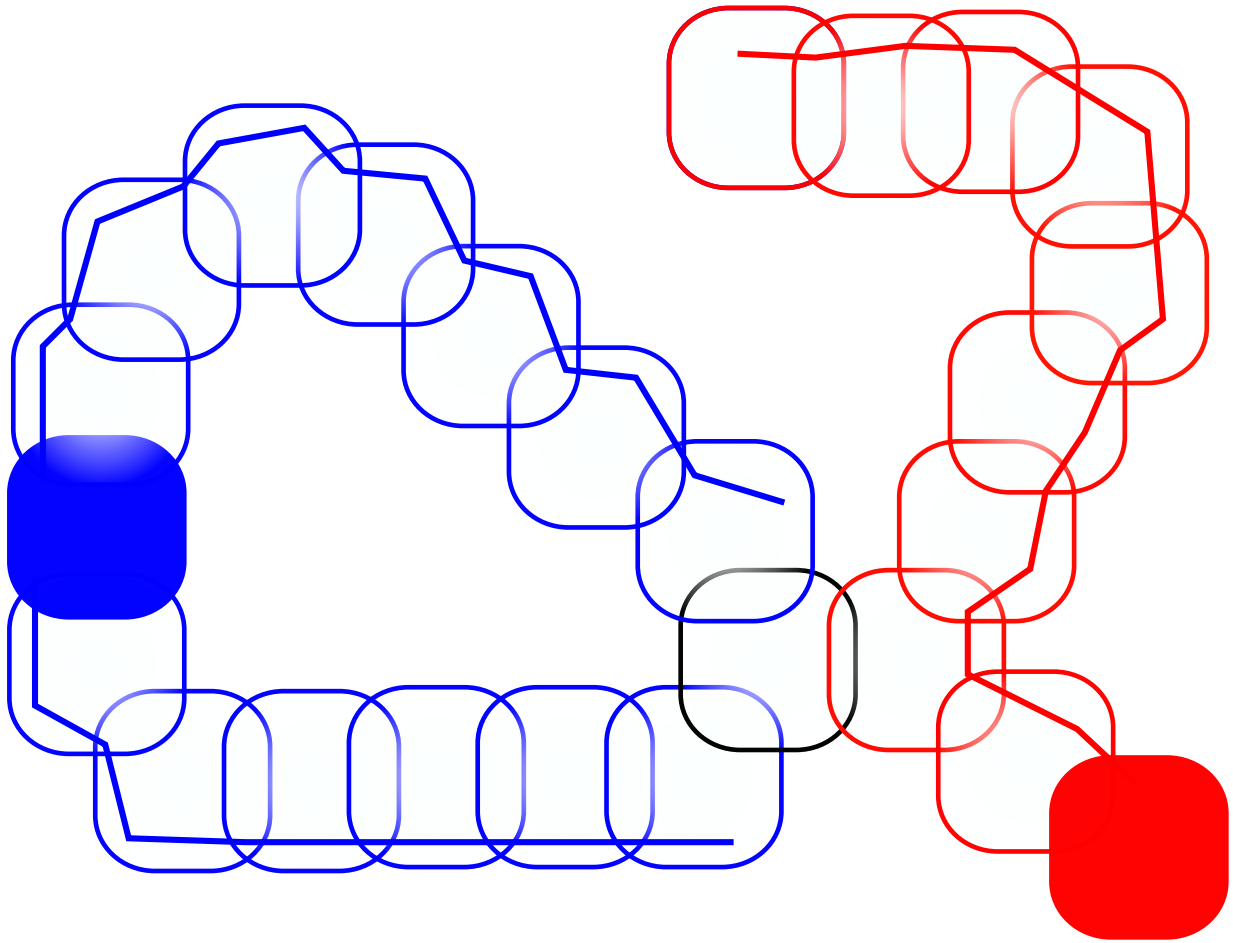
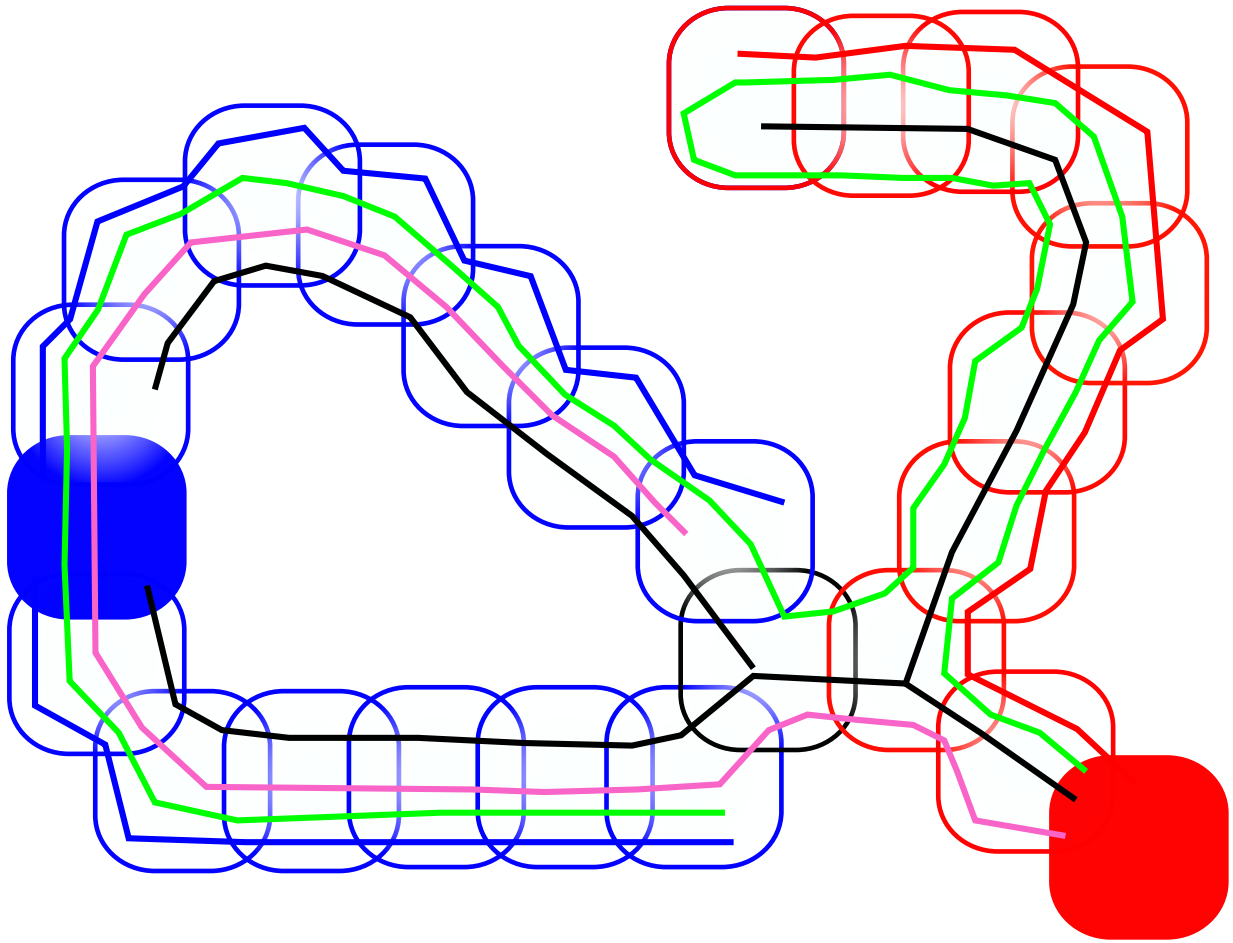


Then there exists $0 < \epsilon < (1/3)\mathrm{d}(H, K)$ such that $B_\epsilon(x_H) \subset \mathrm{int}(H)$ and $B_\epsilon(x_K) \subset \mathrm{int}(K)$. Let $\mathcal{U}$ be a open cover of X with mesh less than ϵ and whose nerve is G.



Let λ be a Lebesgue number for $\mathcal{U}$. By hypothesis, there exist disjoint subcontinua, A_1, A_2, A_3, such that $d_H(X, A_i) < \lambda$ for each i.

Note that $A_i \cap H \neq \emptyset$ and $A_i \cap K \neq \emptyset$ for each i.

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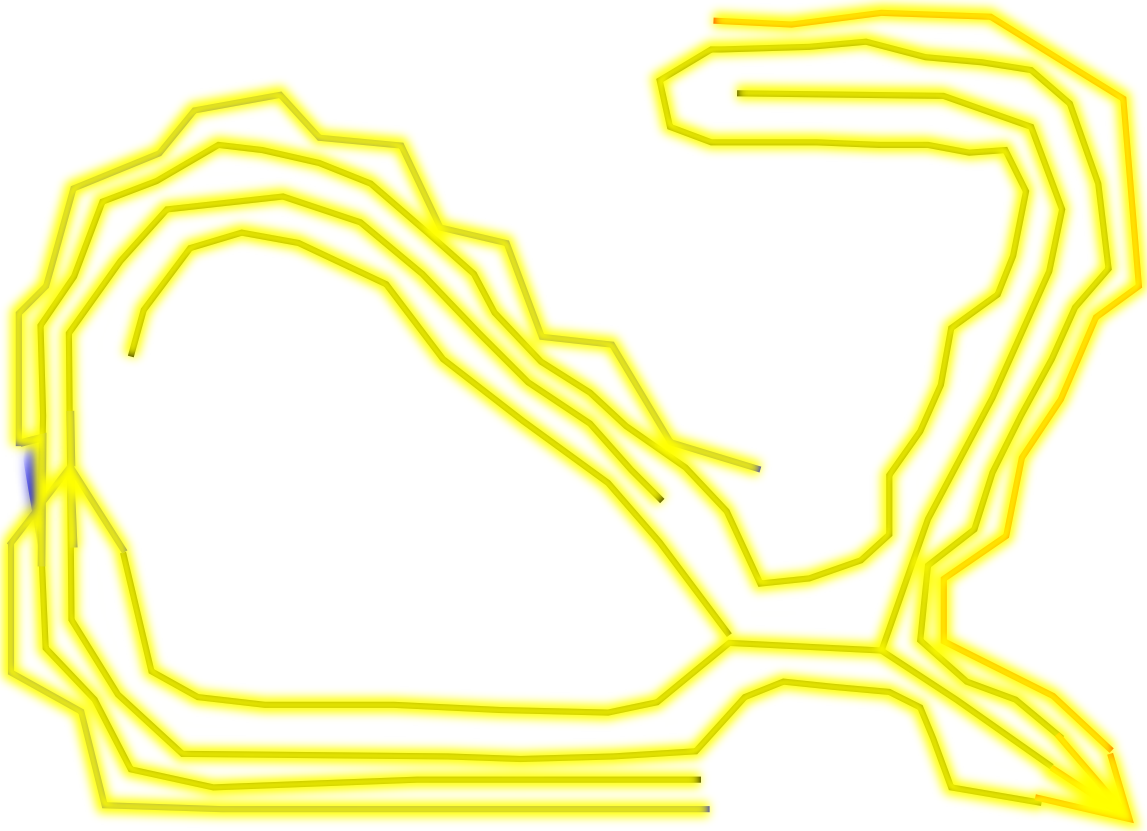
Let $0 < \gamma < (1/3)\mathrm{d}(A_i, A_j)$, $i \neq j$.



Let $\mathcal{V}$ be any open cover with mesh less than γ. Then the nerve of this cover must have at least 3 circle-chains. This contradicts the fact that X is G-like (in this case 1-cyclic.)

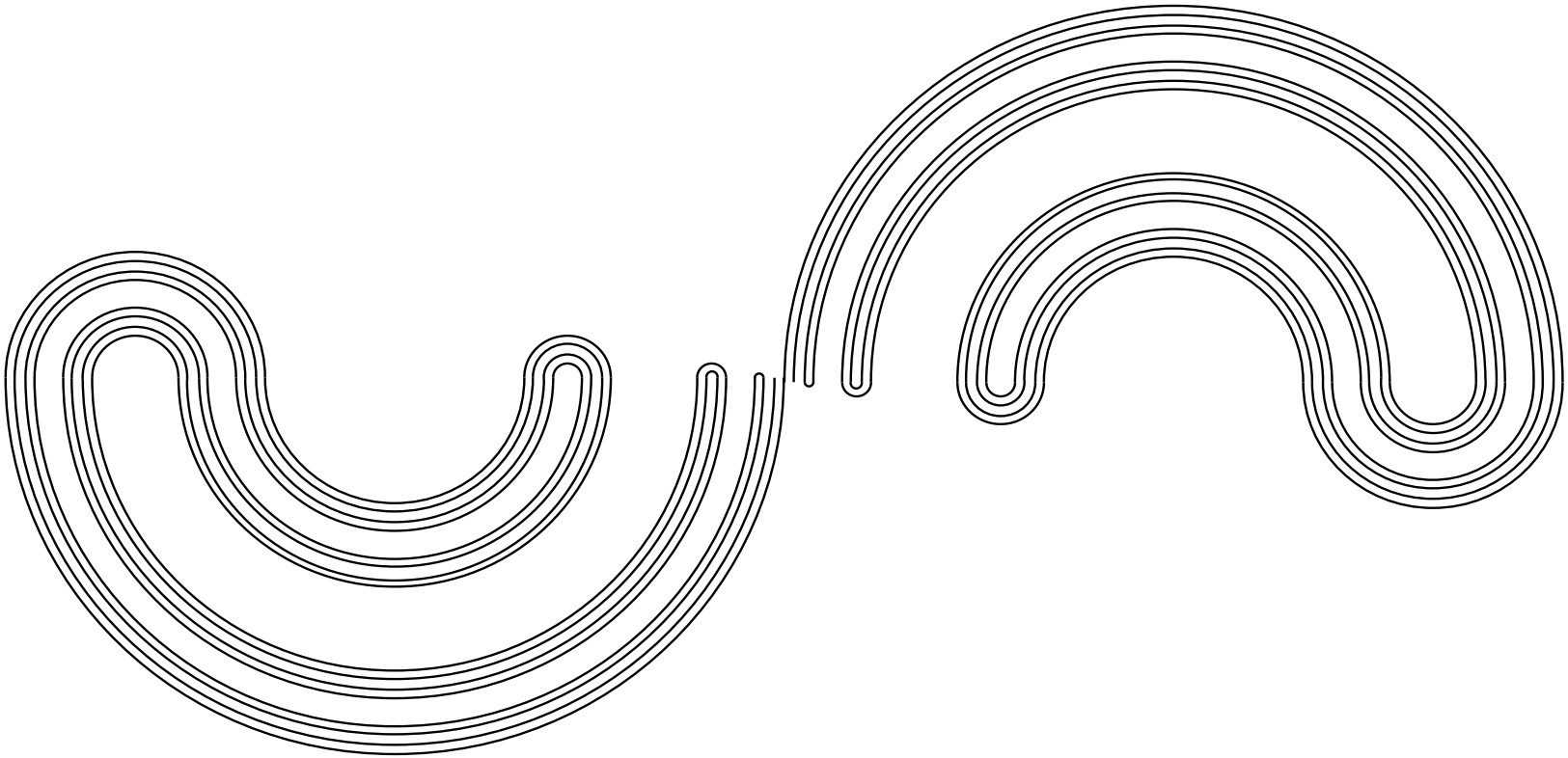
Lemma

Let X be a G-like continuum where G is a graph with $m \geq 0$ cycles. If there exists a mixing homeomorphism $h : X \to X$, then X must be $\frac{1}{m+2}$-indecomposable.

Suppose to the contrary that X is not $\frac{1}{m+2}$-indecomposable, then there exists a collection $\{A_1, \ldots, A_{m+2}\}$ of pairwise disjoint subcontinua of X that have non empty interior. Let $\varepsilon > 0$. Then, since h is mixing, there exists an N such that for each $i \in \{1, \ldots, m+2\}$, $d_H(h^N(A_i), X) \leq d_H(h^N(\text{int}(A_i)), X) < \varepsilon$. Then the collection $\{h^N(A_1), \ldots, h^N(A_{m+2})\}$ satisfy the conditions of Theorem 6. Hence, X is semi-indecomposable and therefore $\frac{1}{m+2}$-indecomposable. This contradiction finishes the proof of the theorem.

Let X be a continuum. A collection of n subcontinua, $\{A_i\}_{i=1}^n$, is an *n-decomposition* for X if $X = \bigcup_{i=1}^n A_i$, but $\bigcup_{i\neq j}^n A_i$ is a proper subset of X for each j.

A continuum X is n-indecomposable if there exists an n-decomposition but no $(n+1)$-decomposition for X.



A 2-indecomposable continuum.

Lemma

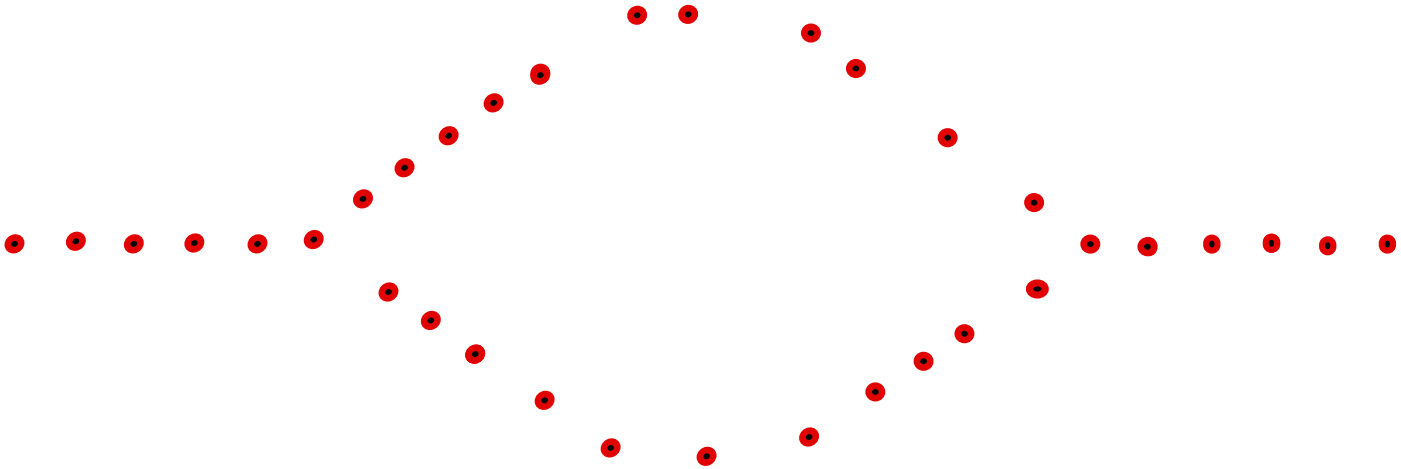
If X is n-indecomposable, then there exists an n-decomposition into indecomposable subcontinua.

Suppose that $\{A_i\}_{i=1}^n$ is a n-decomposition of X. By Zorn's lemma, we may assume that $\{A_i\}_{i=1}^n$ is a minimal decomposition. That is, if $\widehat{A}_i$ is a proper subcontinuum of A_i, then $\widehat{A}_i \cup \bigcup_{j\neq i} A_j \neq X$. Suppose that A_i is decomposable. Then there exists proper subcontinua, A_i^1 and A_i^2, such that $A_i = A_i^1 \cup A_i^2$. Then $A_i^1 \cup \bigcup_{j\neq i} A_j \neq X$, $A_i^2 \cup \bigcup_{j\neq i} A_j \neq X$ but $A_i^1 \cup A_i^2 \cup \bigcup_{j\neq i} A_j = X$. Hence, $\{A_i^1, A_i^2\} \cup \{A_j\}_{j\neq i}$ is an $(n+1)$-decomposition. However, this contradicts the fact that X is n-indecomposable.

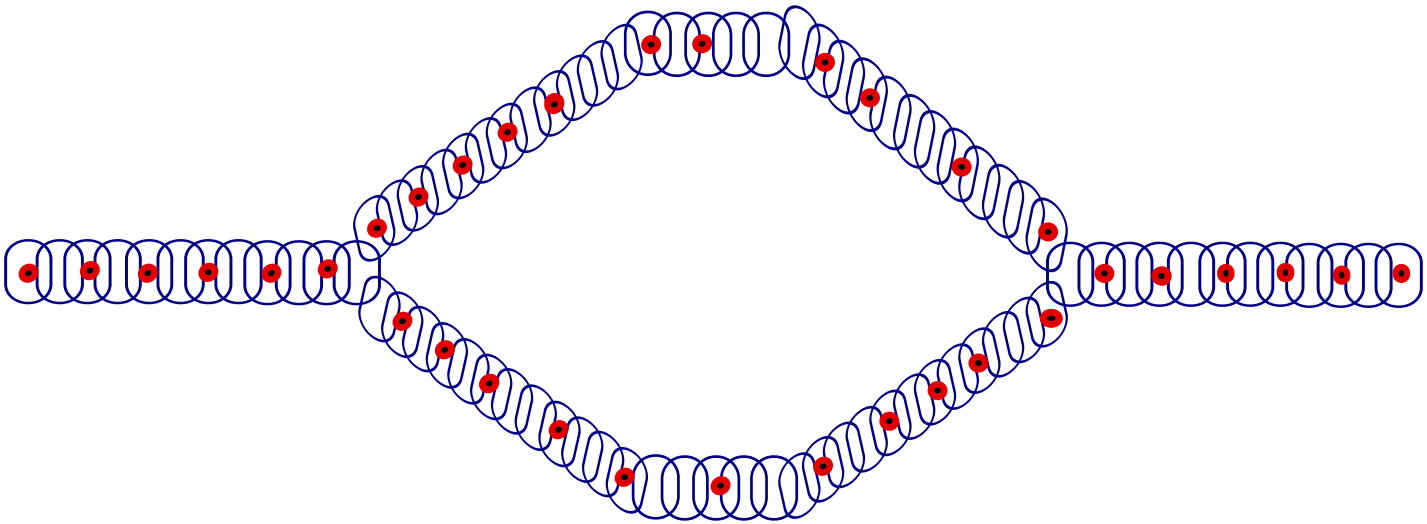
Lemma

Suppose that X is G-like and $\frac{1}{n}$-indecomposable. Then X is k-indecomposable for some $1 \leq k \leq M$ with $M = 2N(n+1) + r$, where G is a graph with N maximal free arcs and r ramification points.

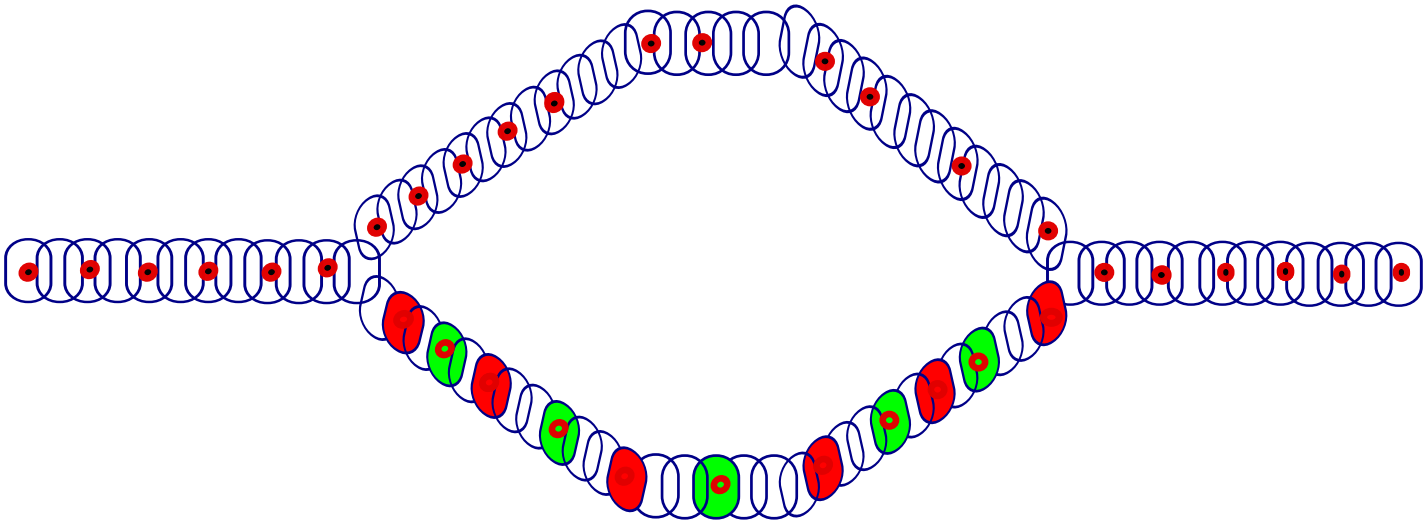
For example let $N = 4$, $r = 2$ and $n = 3$ so $M = 34$. Suppose on the contrary that there exists a 35-decomposition $\{A_i\}_{i=1}^{35}$.



We may assume that there exists $\epsilon > 0$ and a_i such that $B_\epsilon(a_i) \subset A_i - \cup_{j \neq i} A_j$ for each i.



Let $\gamma = \min \mathrm{d}(a_i, A_j)$ for all $i \neq j$, $0 < \delta < (1/3)\min\{\epsilon, \gamma\}$ and $\mathcal{U}$ be a G-cover with mesh less than δ.



Then by the pigeon-hole principle, there exists a free chain that contains at least $2(n+1) = 8$ elements of our decomposition. However, that means that at least 4 must be pairwise disjoint. Since they all have nonempty interior this contradicts the assumption that the continuum is 1/3 indecomposable.

Lemma

Suppose that $\{A_i\}_{i=1}^n$ is an n-decomposition of a G-like continuum X where each A_i is indecomposable. If $h : X \to X$ is a homeomorphism, then for each i there exists an N_i such that $h^{N_i}(A_i) = A_i$.

Theorem

Let X be a G-like continuum where G is a graph with $m \geq 0$ cycles. If $h : X \longrightarrow X$ is a mixing homeomorphism, then X must be indecomposable.

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It follows from Lemmas 7 that X is $1/j$-indecomposable for some j and then 9 that X is k-indecomposable for some k. Hence it follows from Proposition 8 that there exists a k-decomposition, $\{A_i\}_{i=1}^{k}$, of X where each A_i is indecomposable. If $k \geq 2$, then A_1 is a proper subcontinuum of X with nonempty interior. Hence, since h is mixing, $d_H(h^n(A_1), X) \to 0$ as $n \to \infty$. However, it follows from Proposition 10 that there is an N such that $h^N(A_1) = A_1$ for some $N \neq 0$. So $d_H(h^{pN}(A_1), X) = d_H(A_1, X) > 0$ for all p. This contradicts the fact that h is mixing. Hence, X is indecomposable.

However, there exists a mixing homeomorphism on a hereditarily decomposable tree-like continuum!

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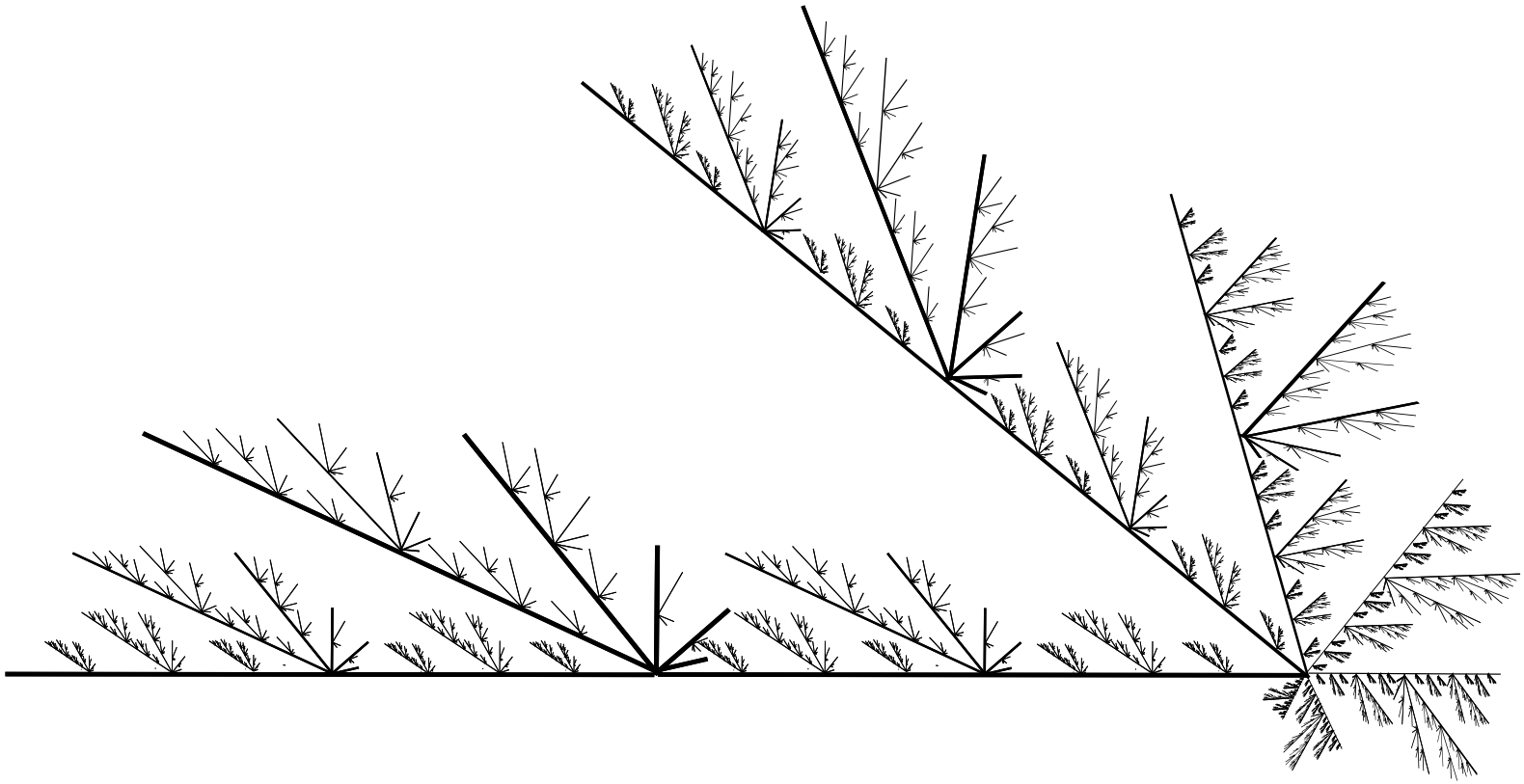
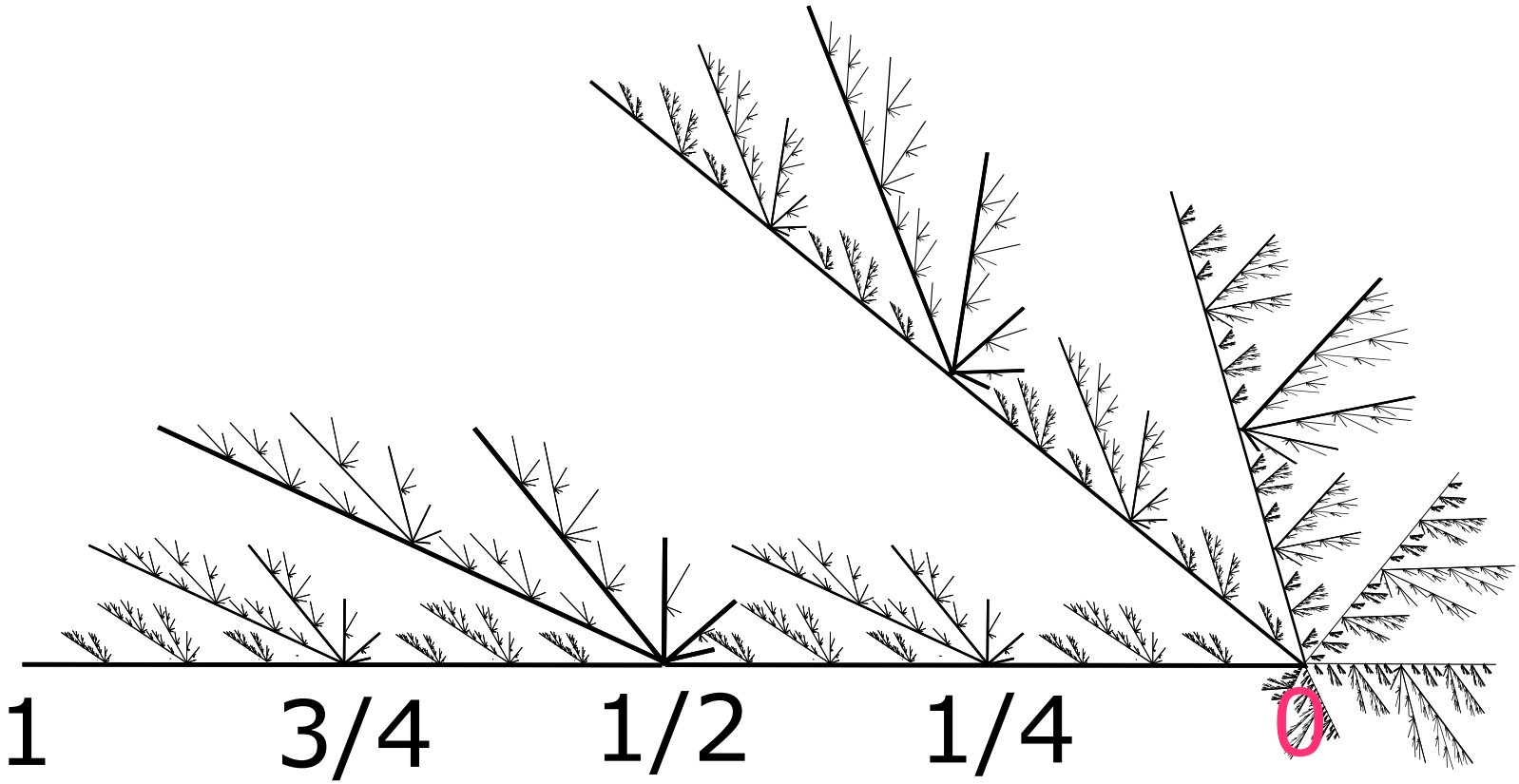


The universal dendrite D_ω.



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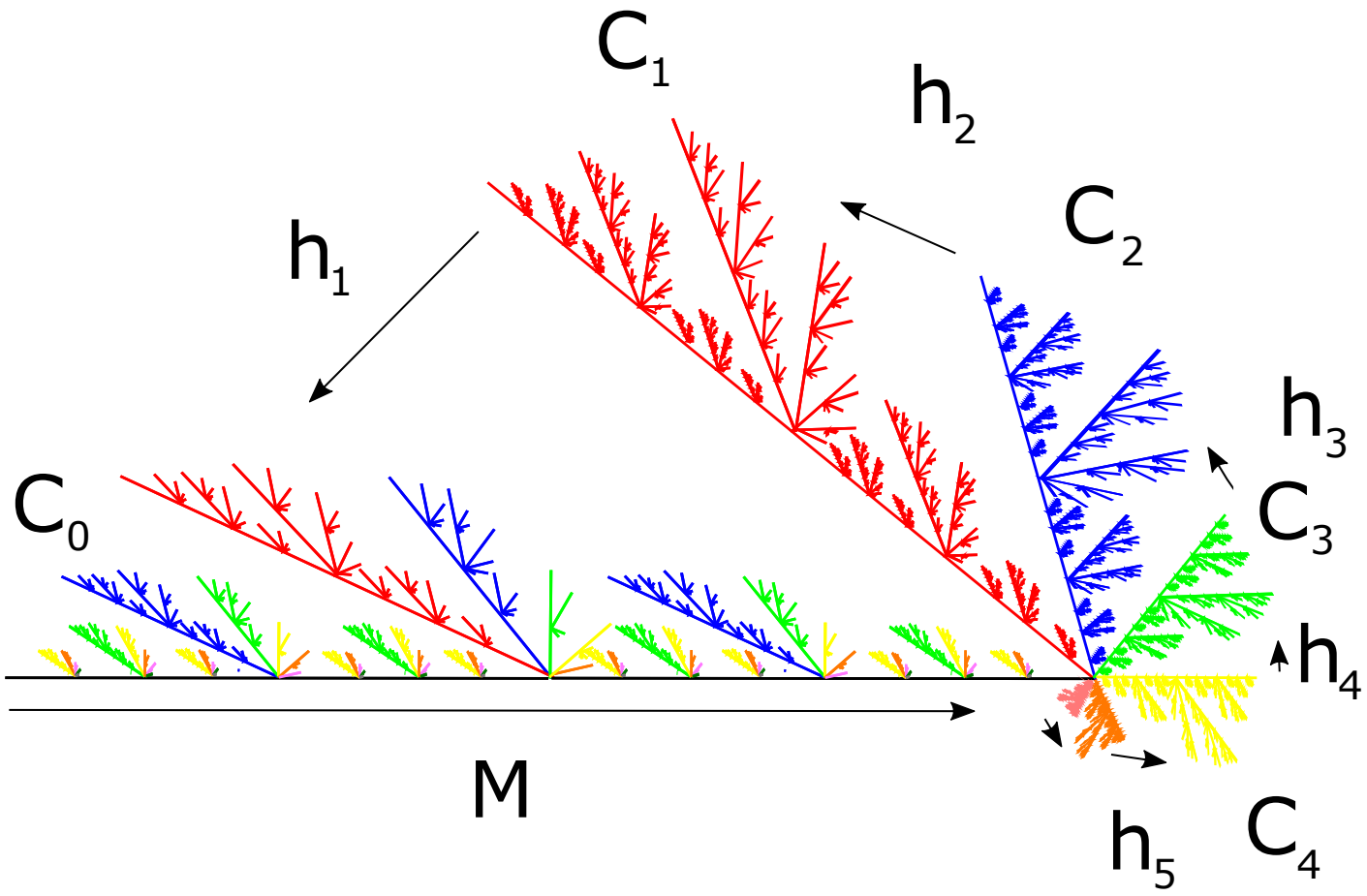
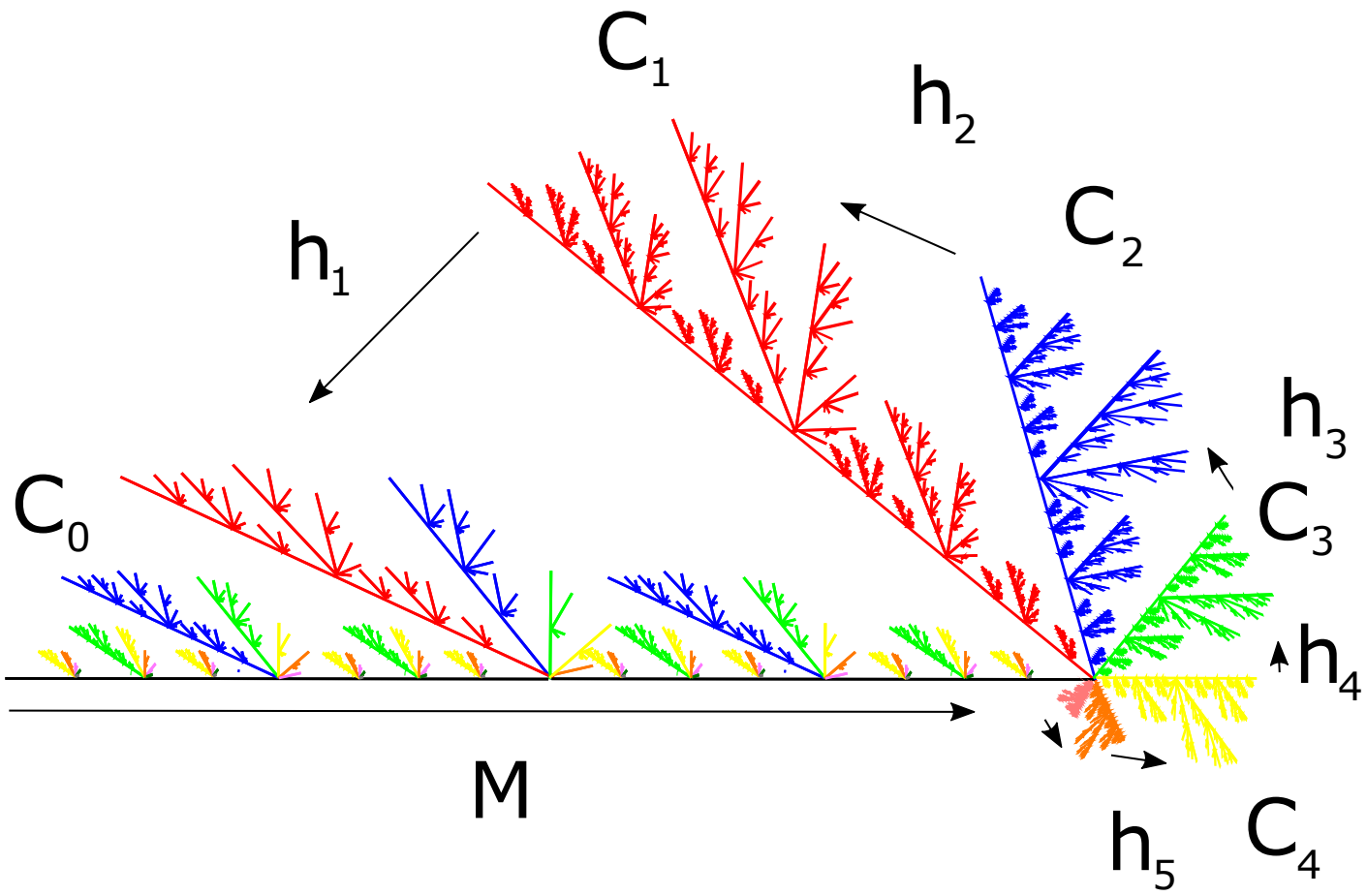
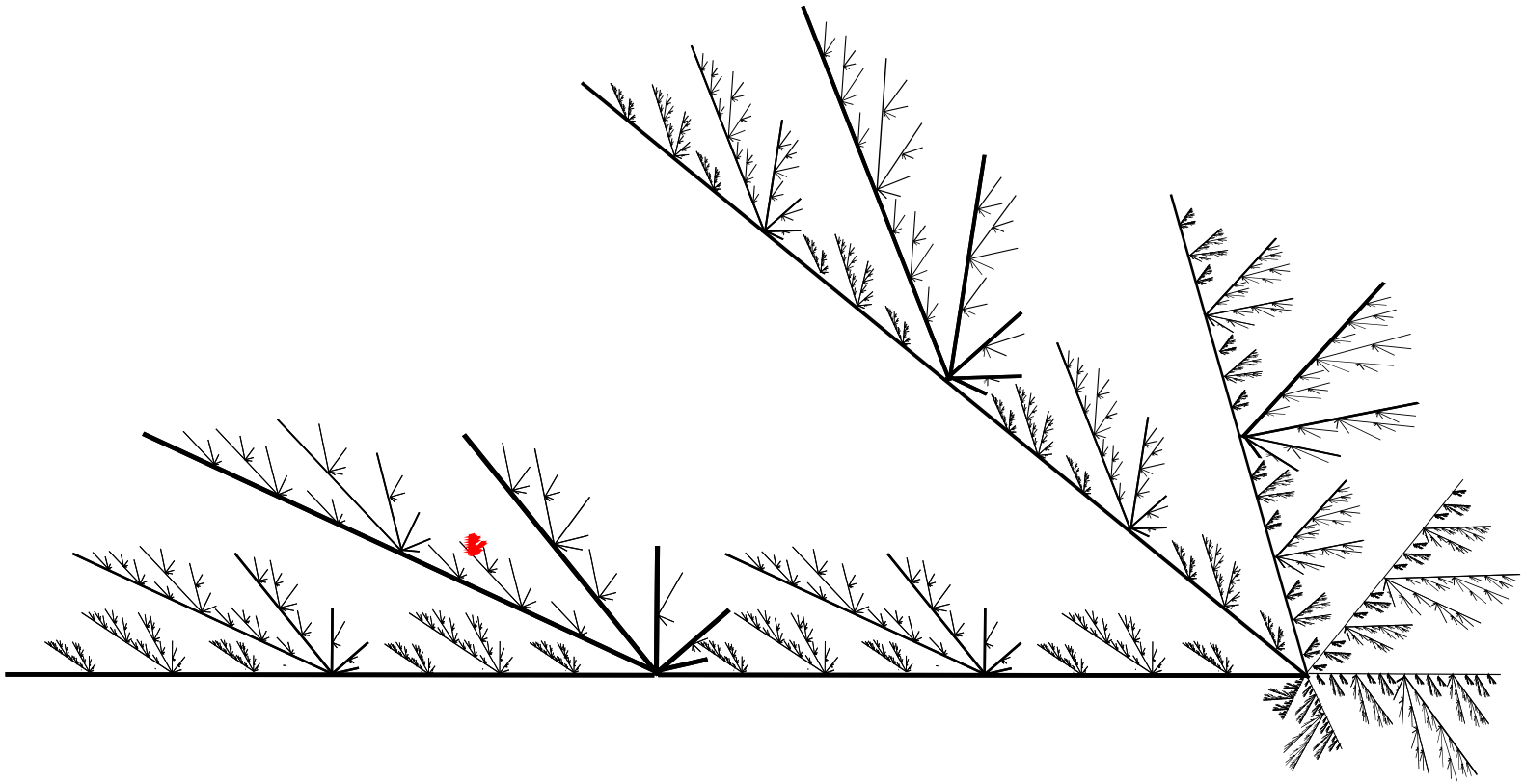


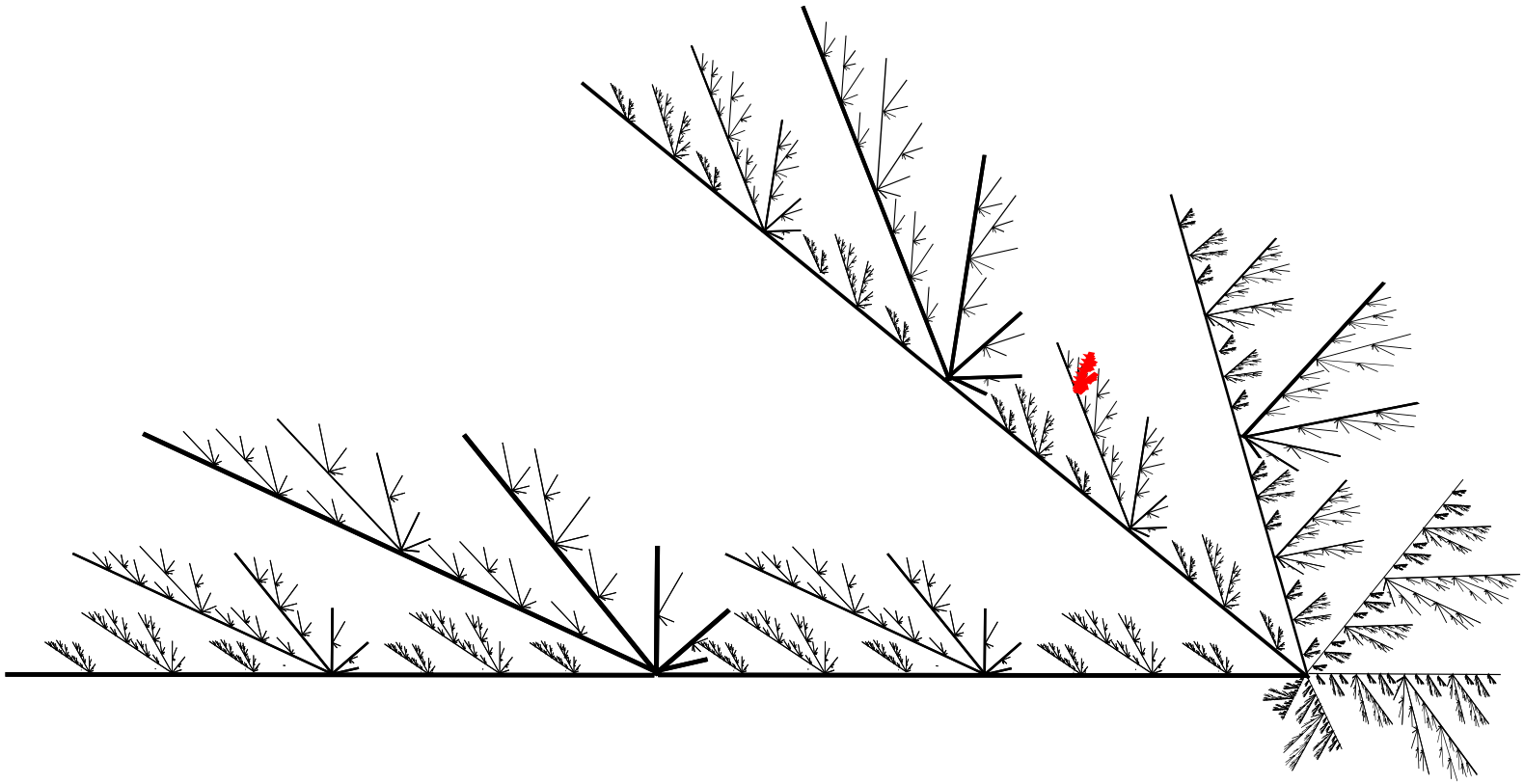
Each $h_i : C_i \longrightarrow C_{i-1}$ is a homeomorphism.
$M : C_0 \longrightarrow \cup_{i=1}^{\infty} C_i$ is a local homeomorphism away from $[0, 1]$.
$g : D_\omega \longrightarrow D_\omega$ by combining h_i and M.



$g : D_\omega \longrightarrow D_\omega$ is mixing.



$g : D_\omega \longrightarrow D_\omega$ is mixing.
In fact it is also exact and weakly positively continuum-wise expansive.



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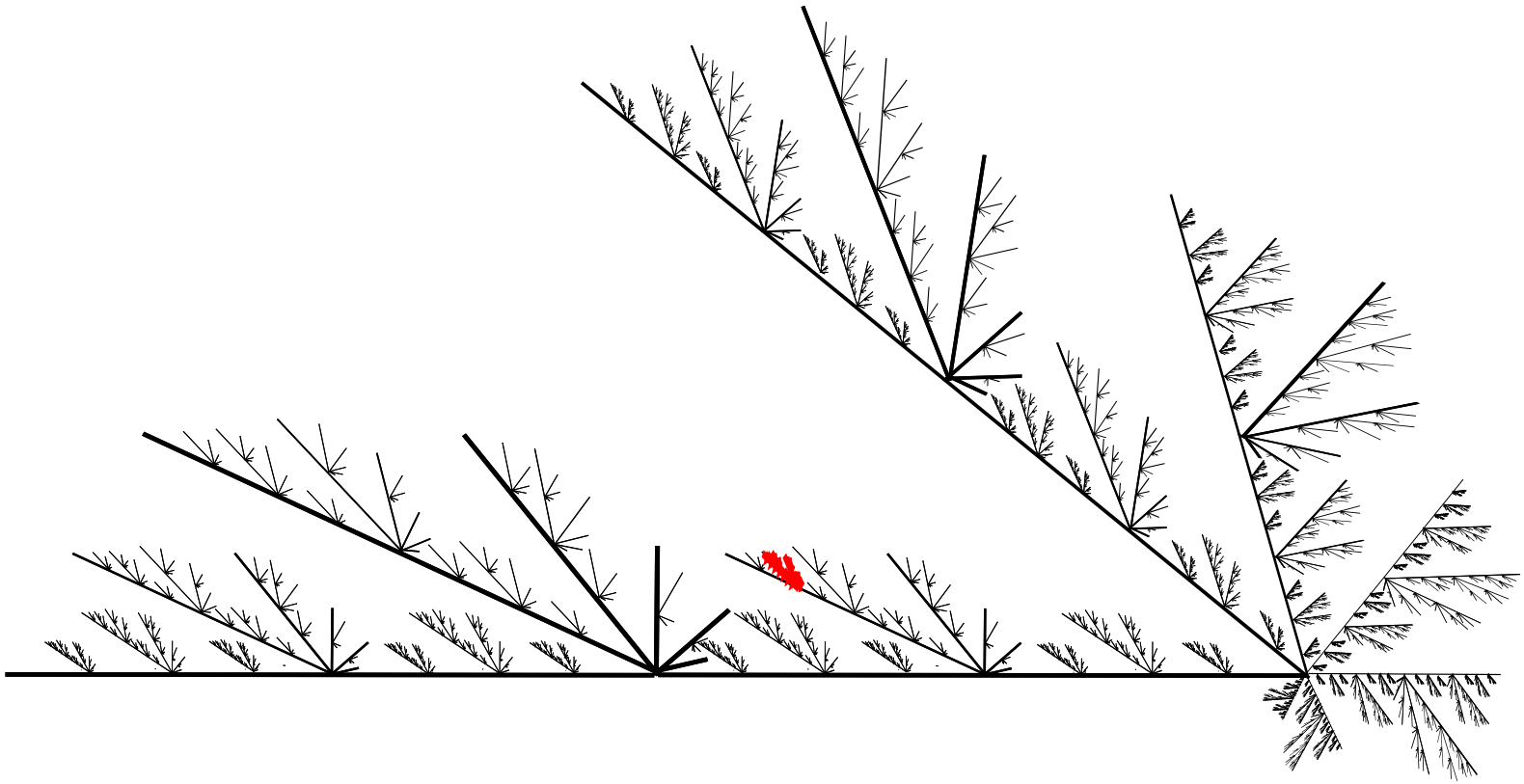

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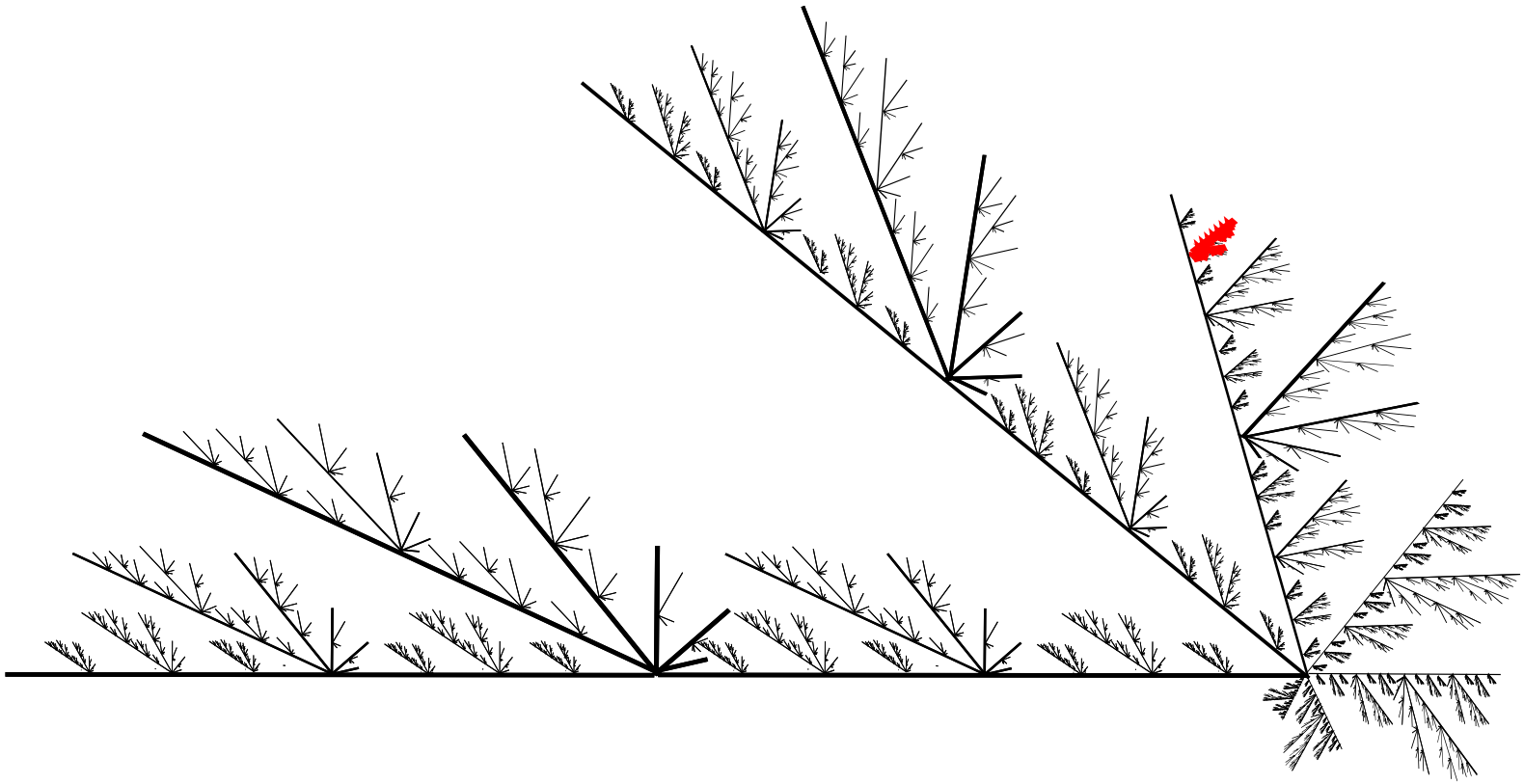

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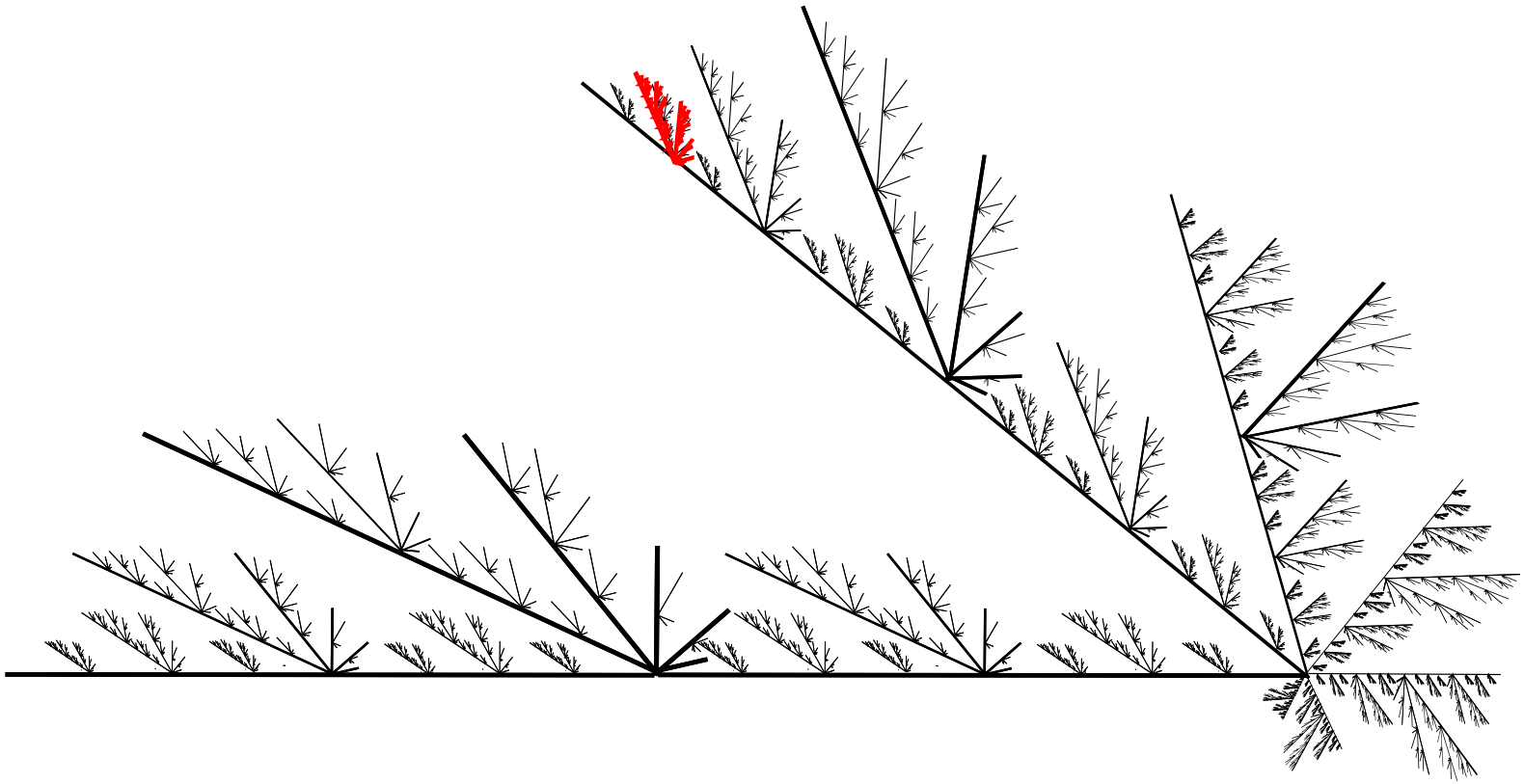

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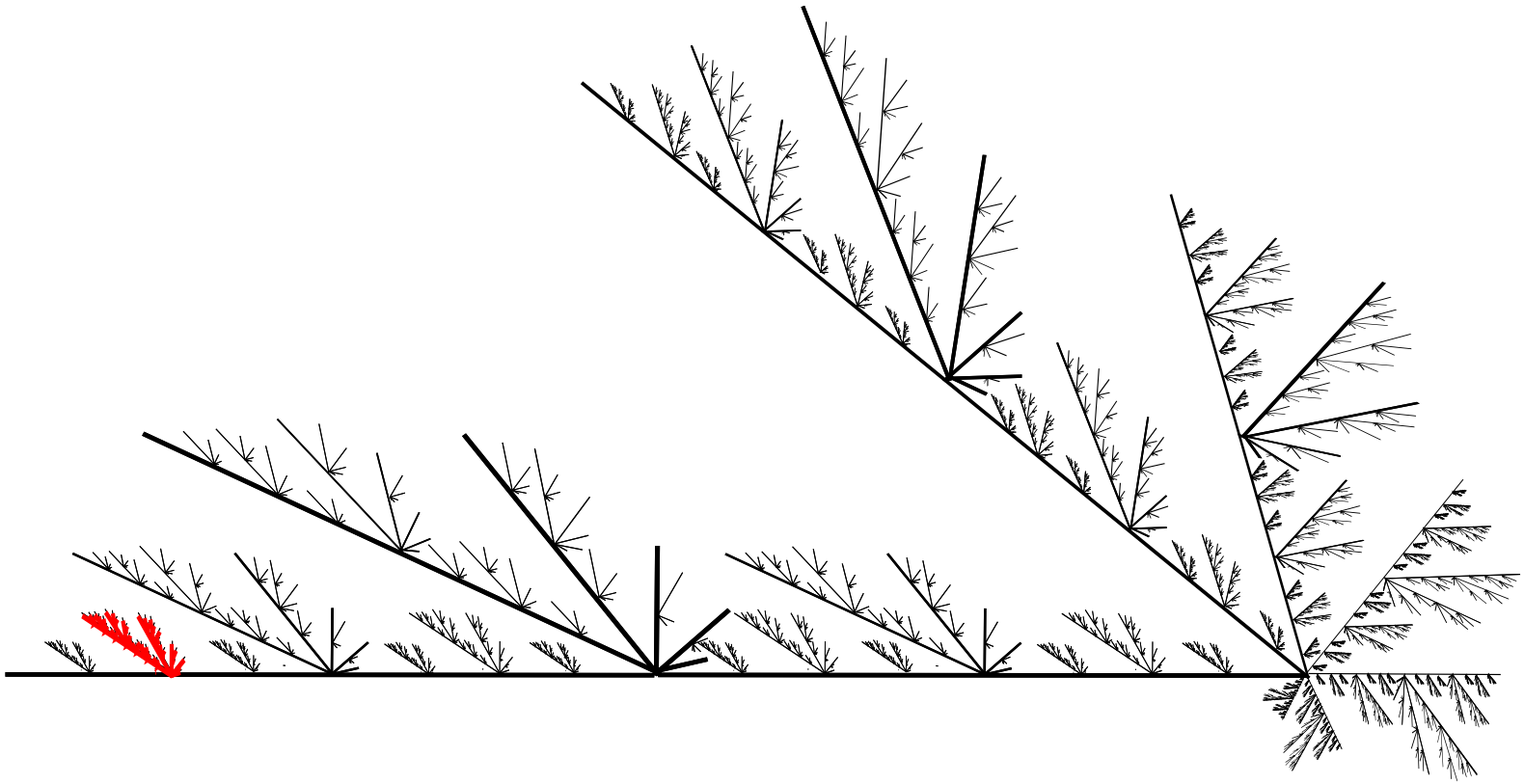

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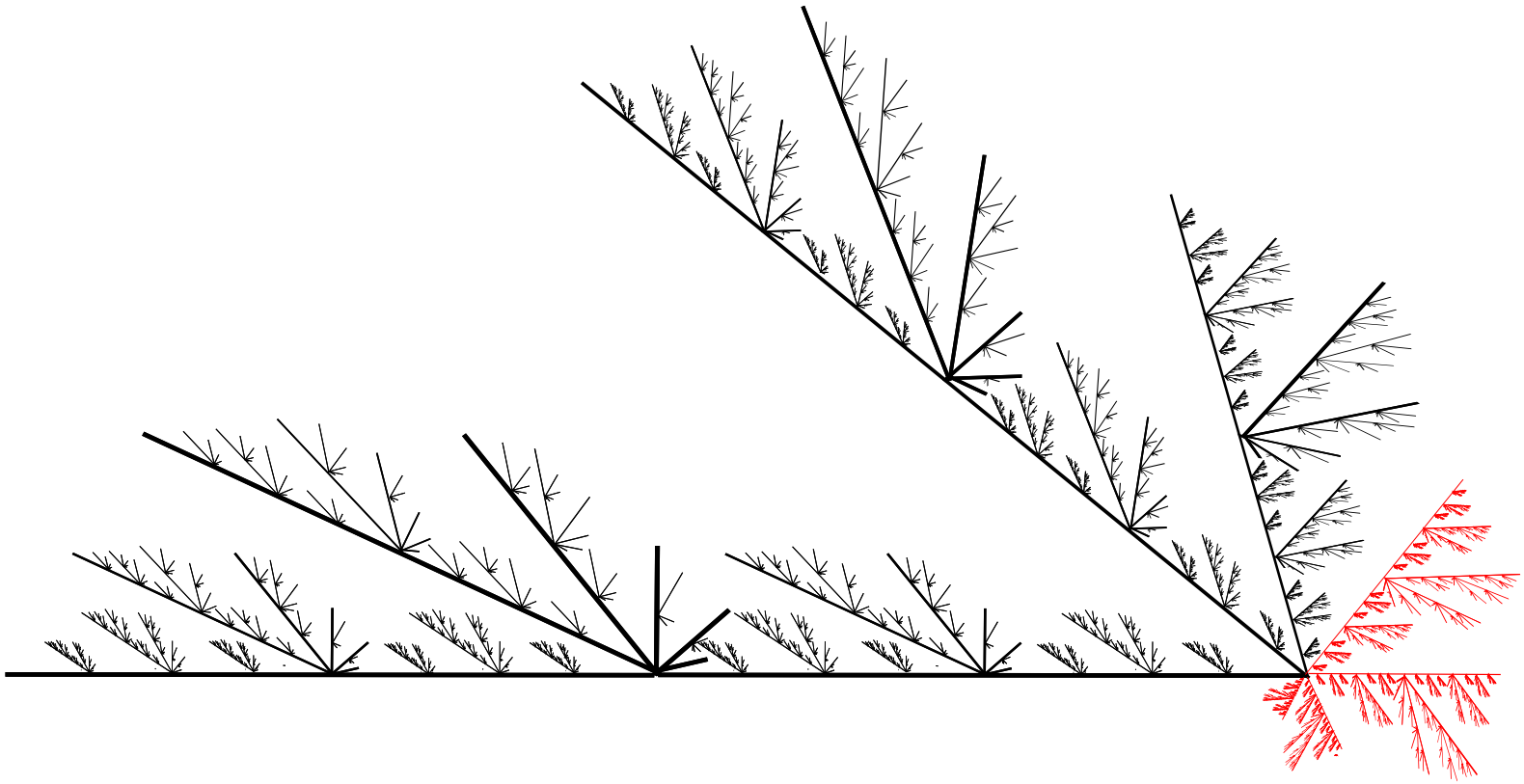

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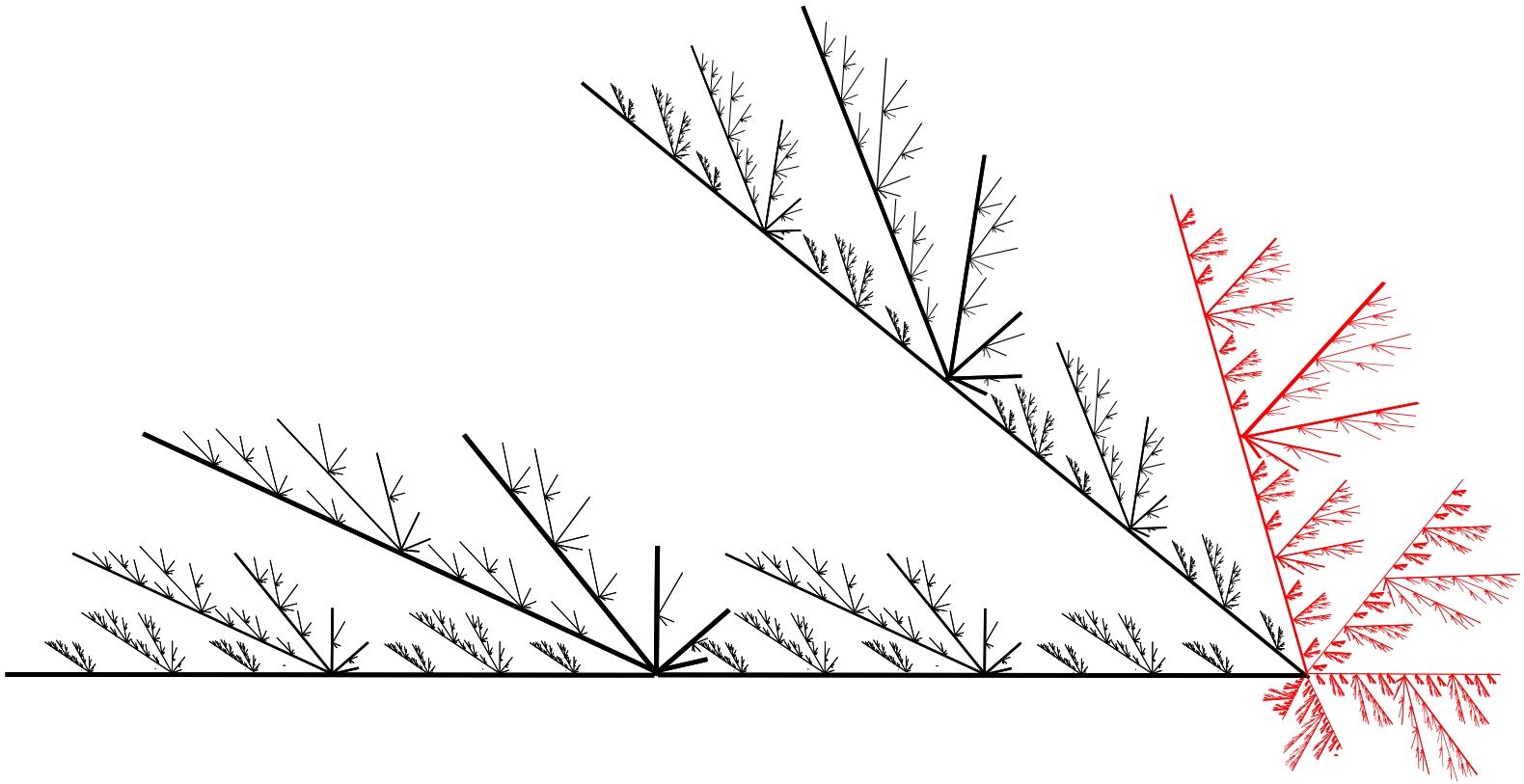

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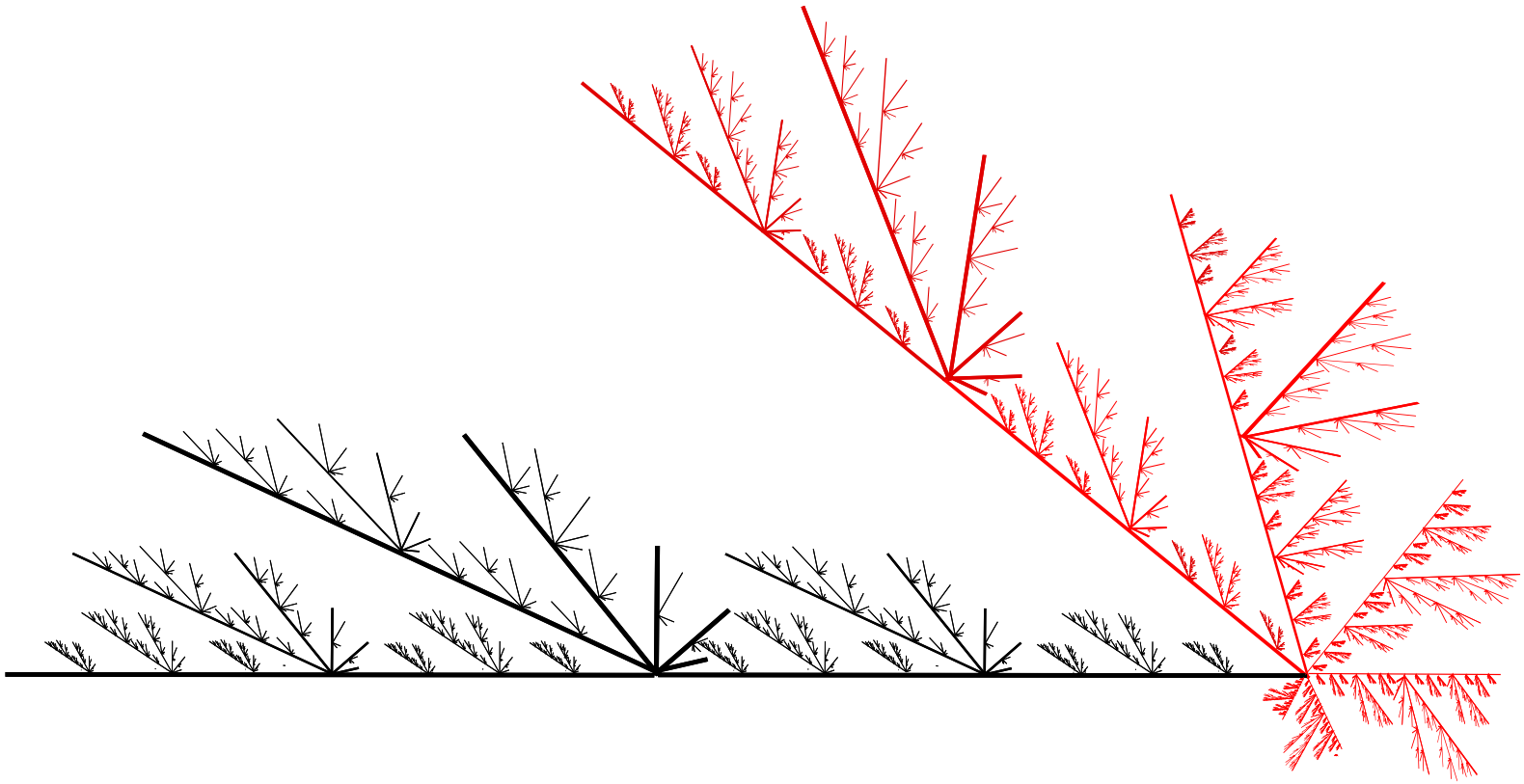

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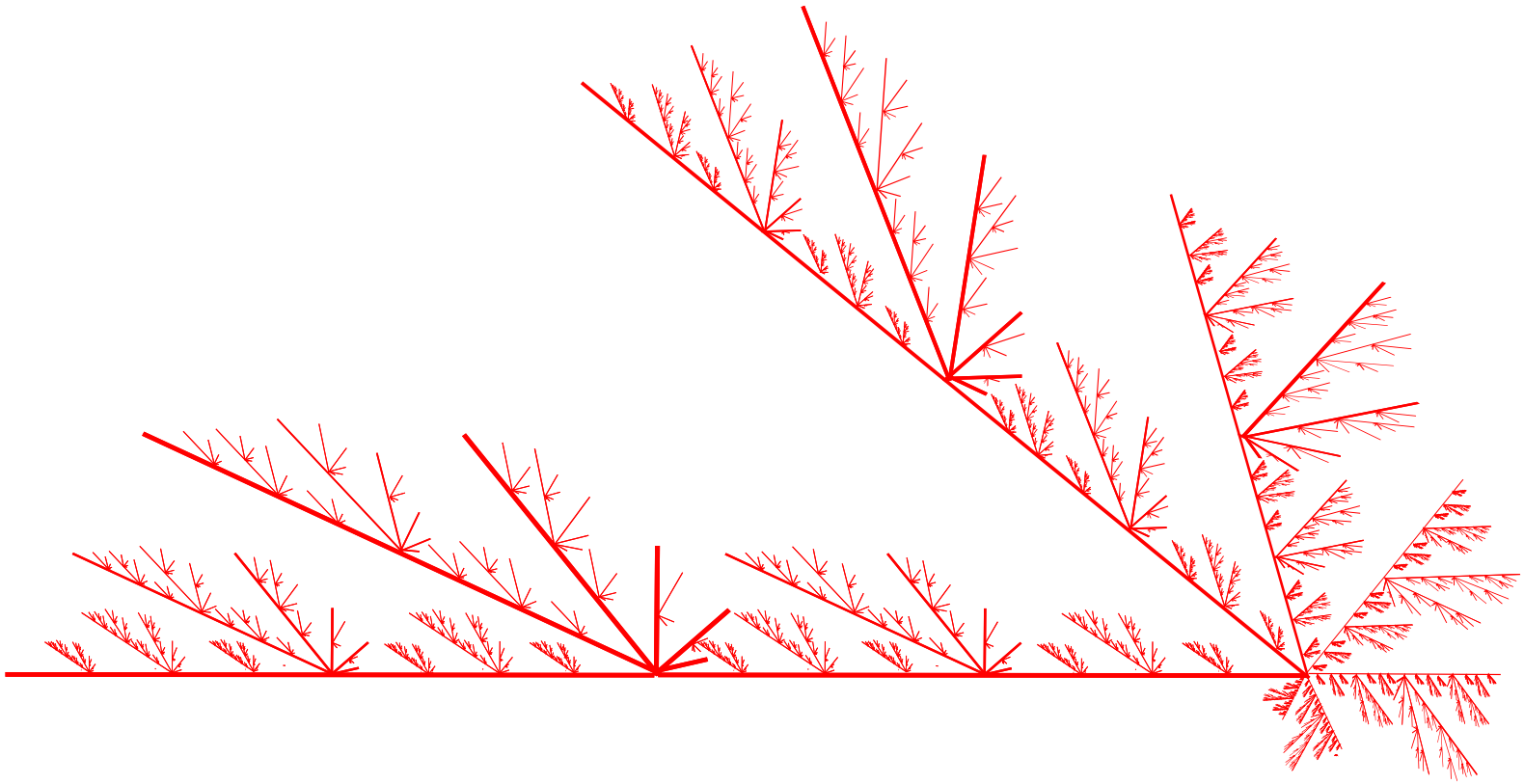

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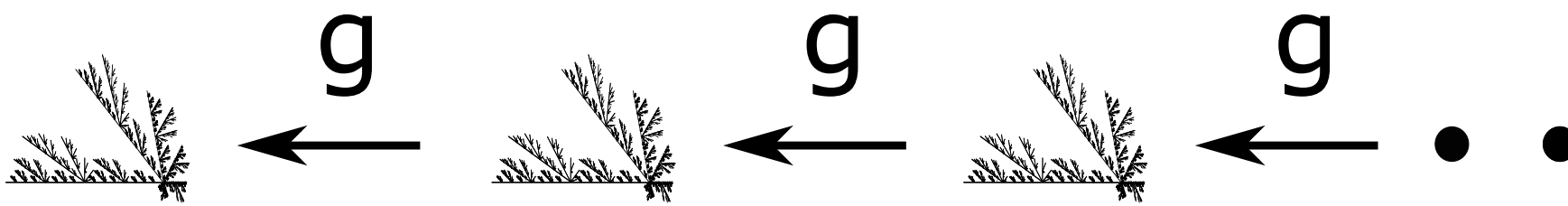
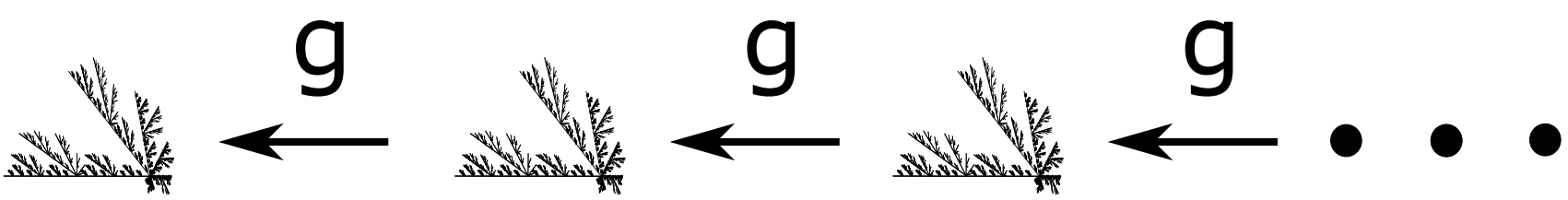


Let $\widehat{D} = \varprojlim\{\widehat{D}_\omega, g\}_{n=1}^{\infty}$ be an inverse limit of universal dendrites.

Since g is mixing (weakly positively continuum-wise expansive), the shift homeomorphism $\widehat{g}(\langle x_i\rangle_{i=1}^{\infty}) = \langle x_i\rangle_{i=2}^{\infty}$ is mixing (weakly positively continuum-wise expansive) .

Since dendrites are tree-like, $\widehat{D}$ is tree-like.

Claim: $\widehat{D}$ is hereditarily decomposable.



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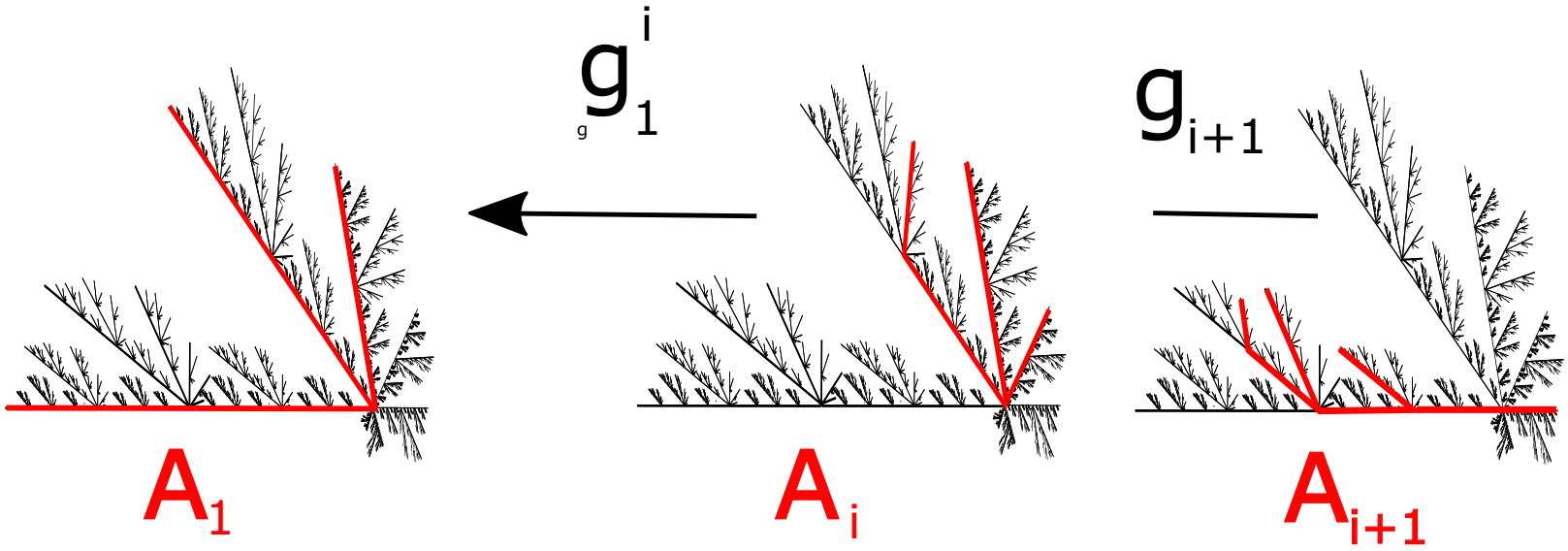

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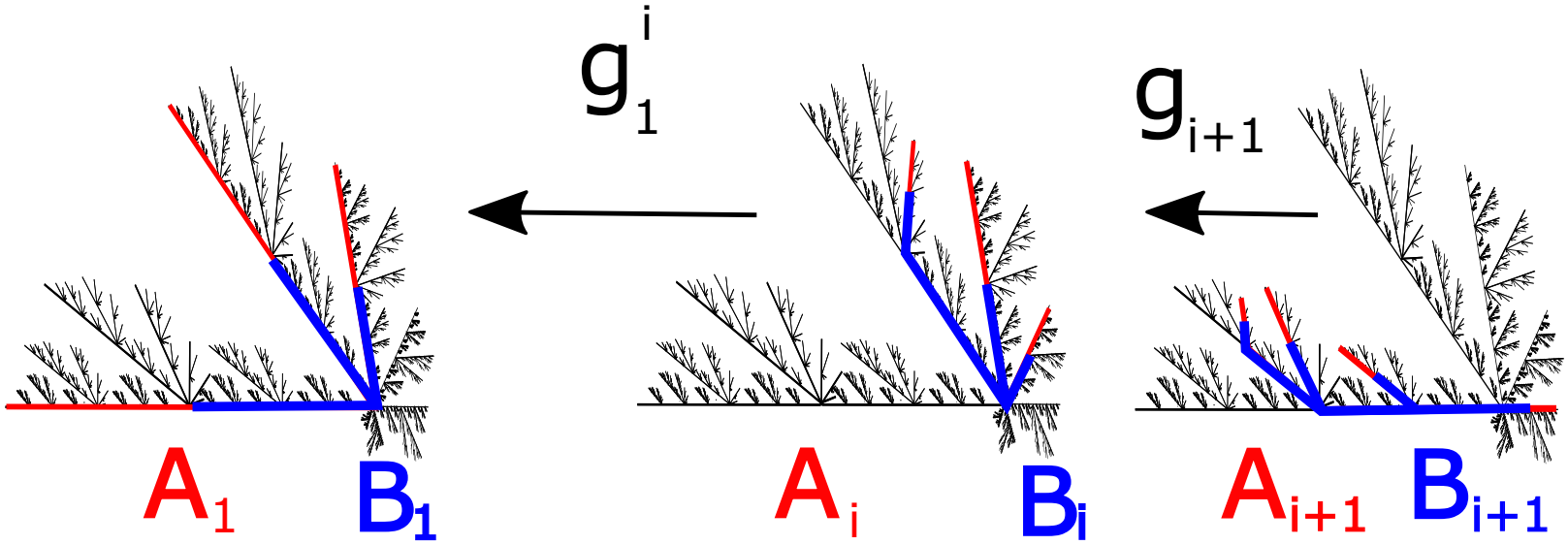

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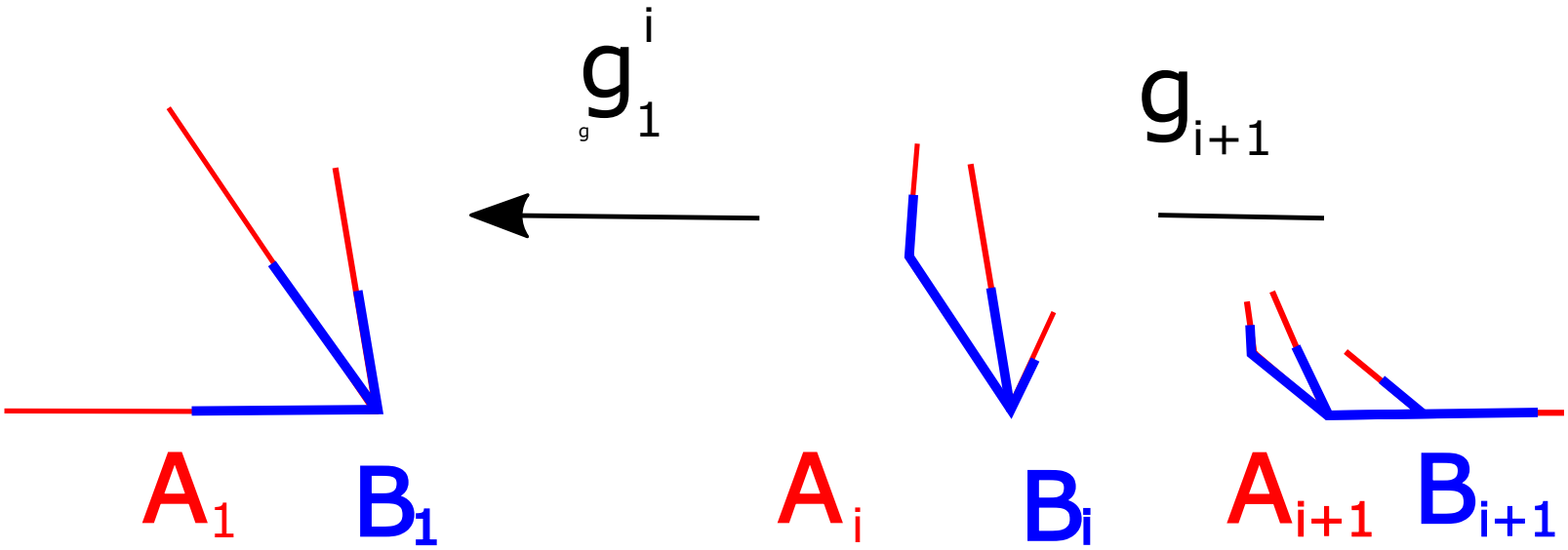

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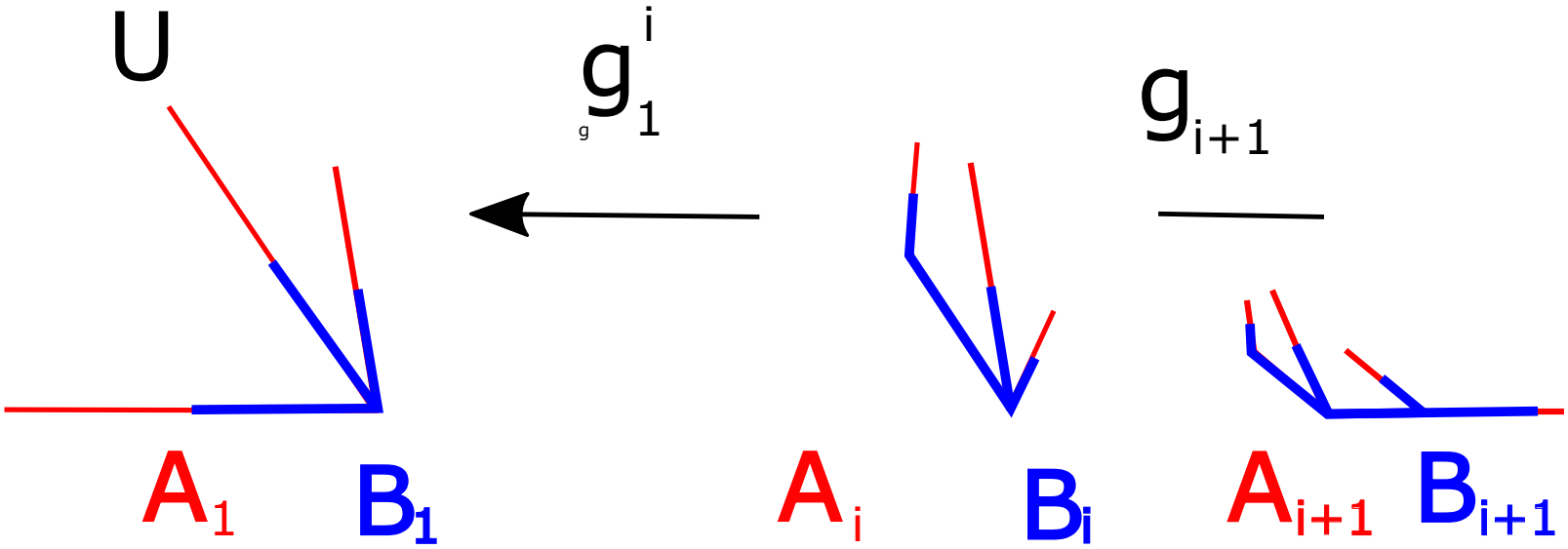

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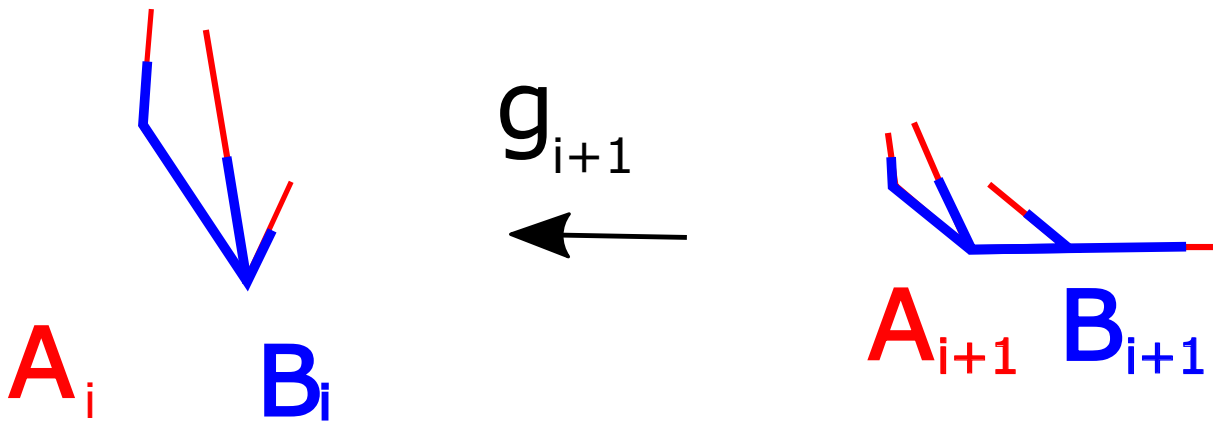

Claim: $g_i^{-1}(B_i) = B_{i+1}$.

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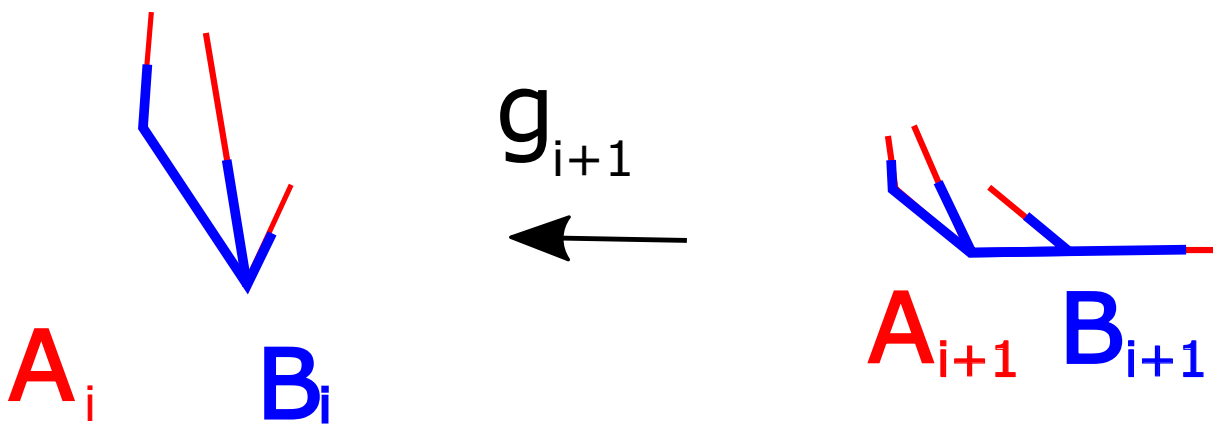

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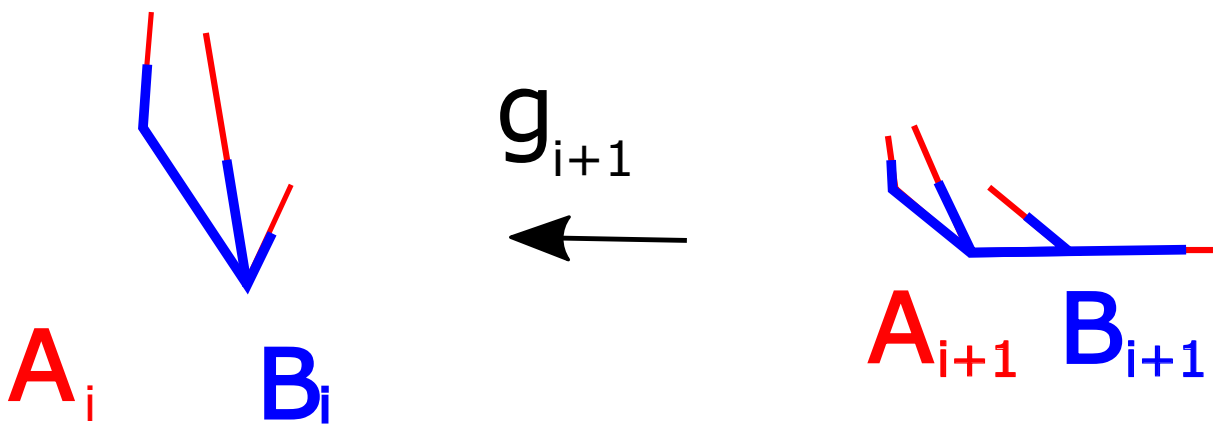

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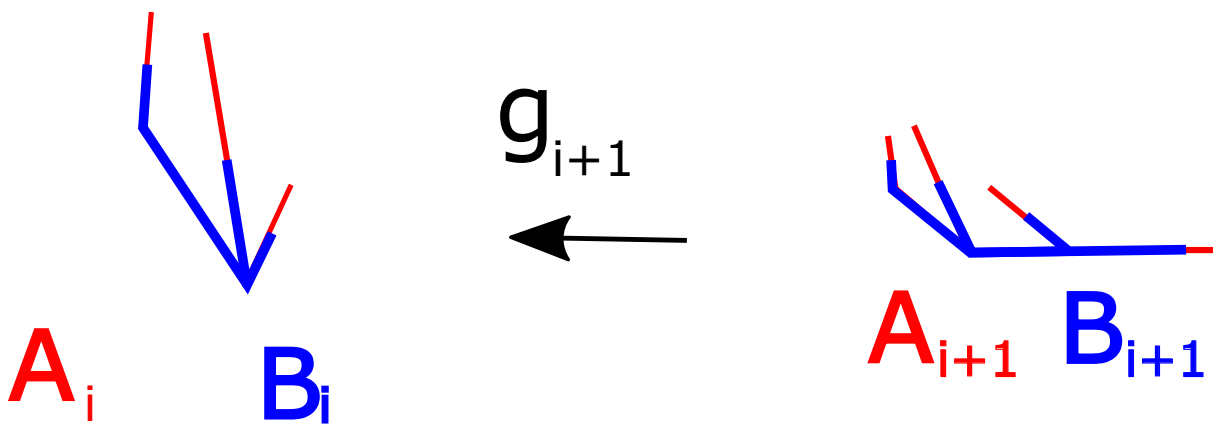

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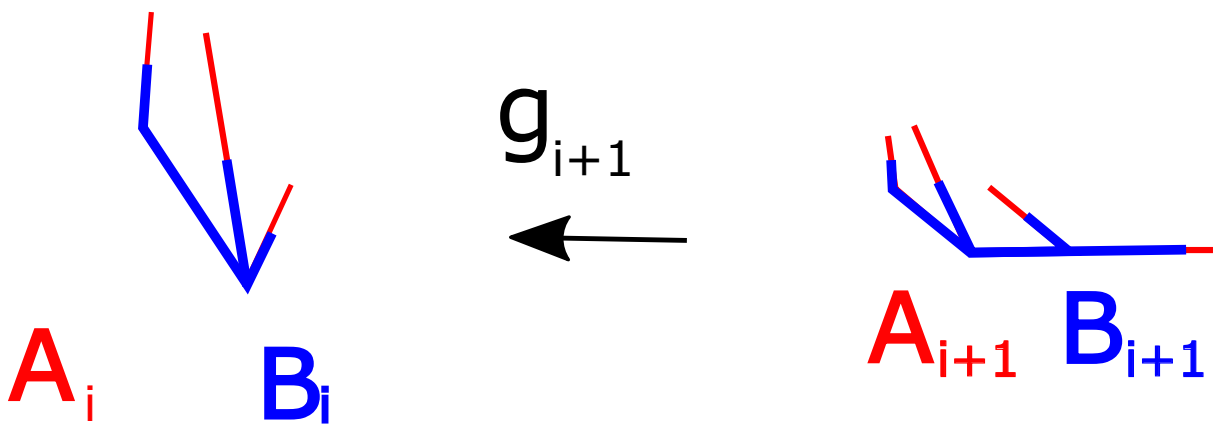

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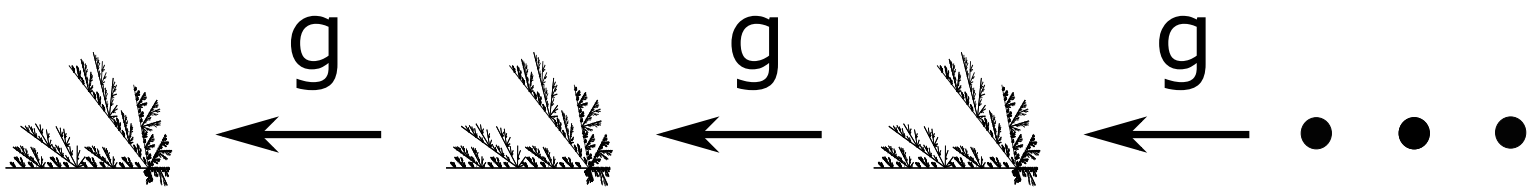

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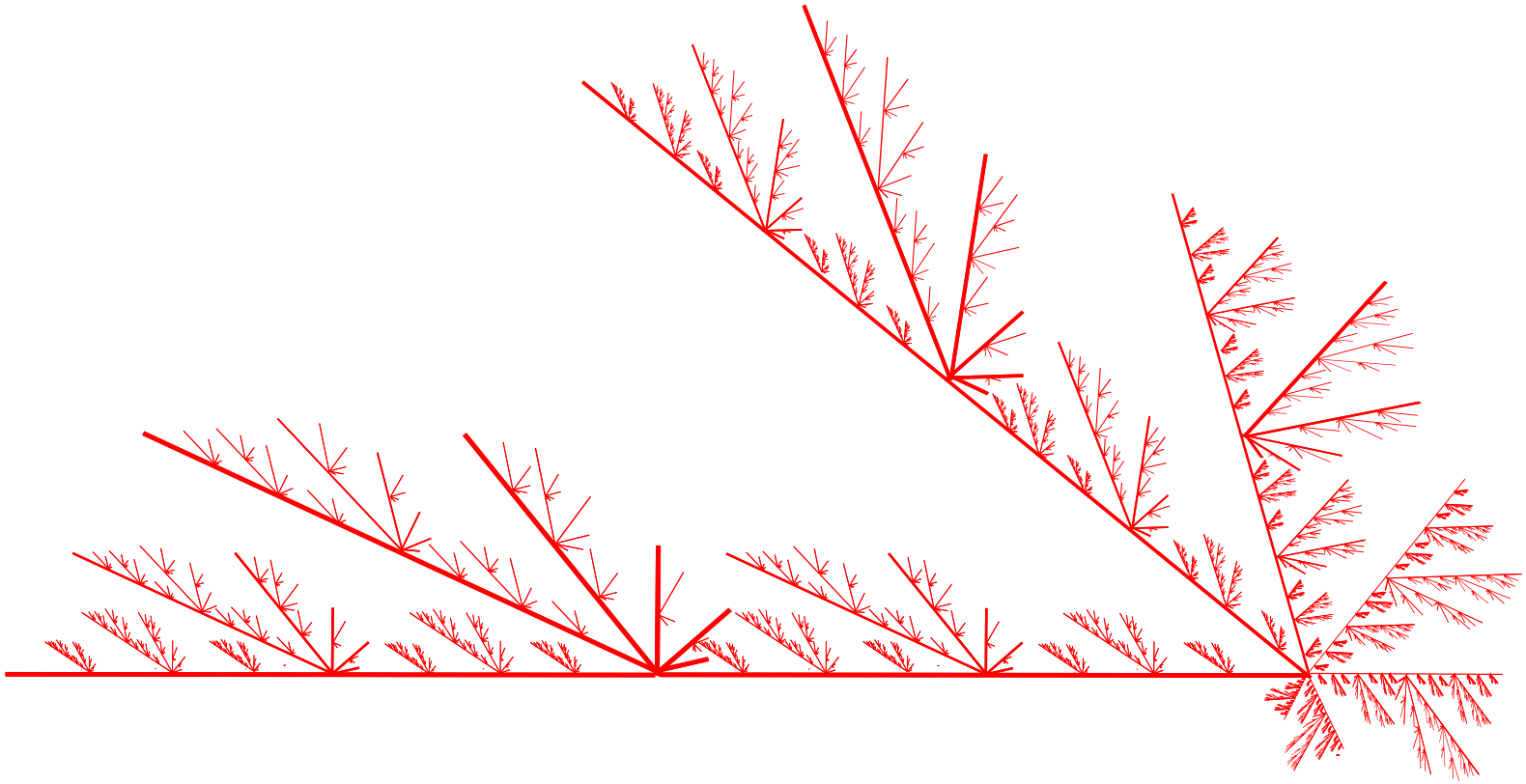

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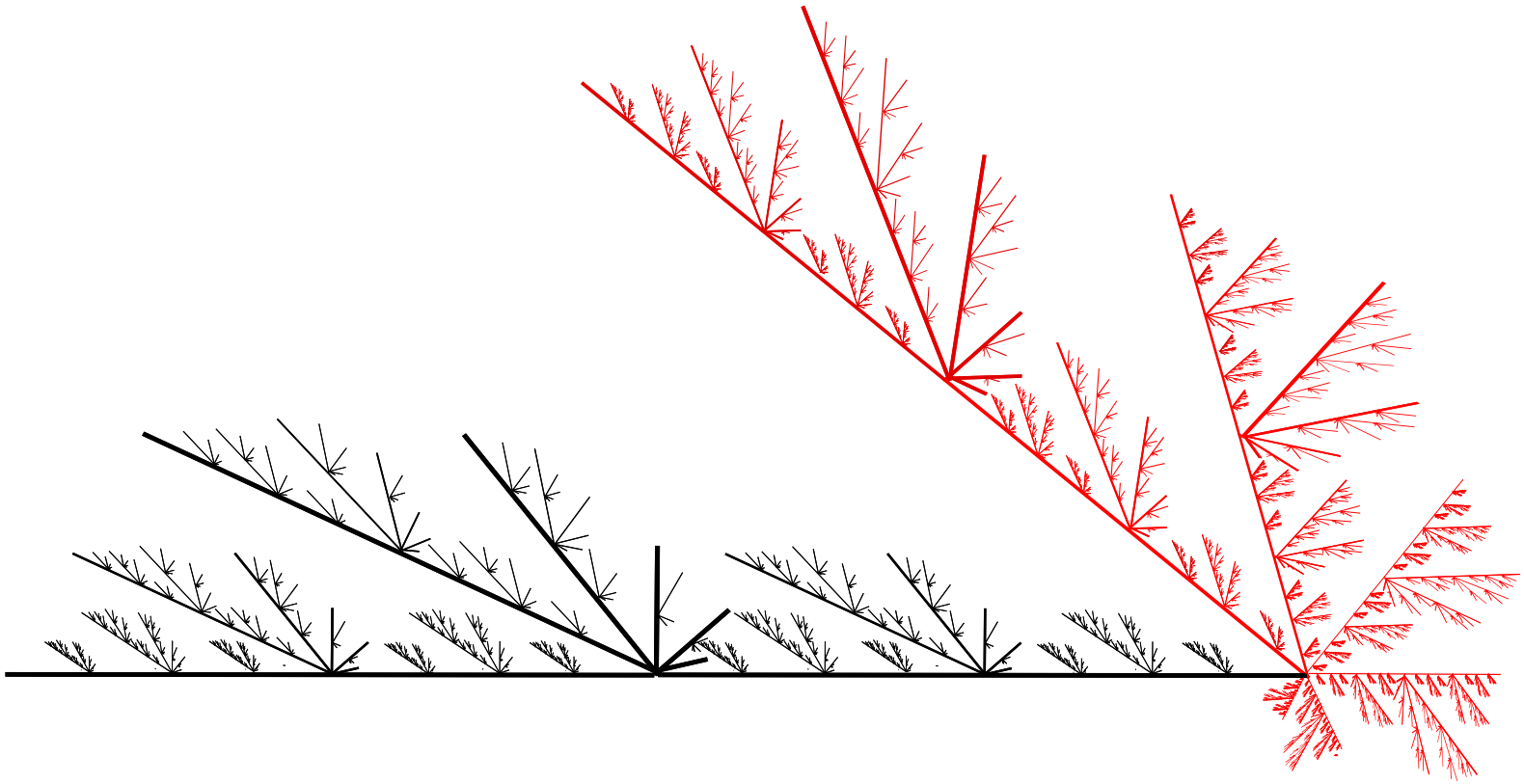

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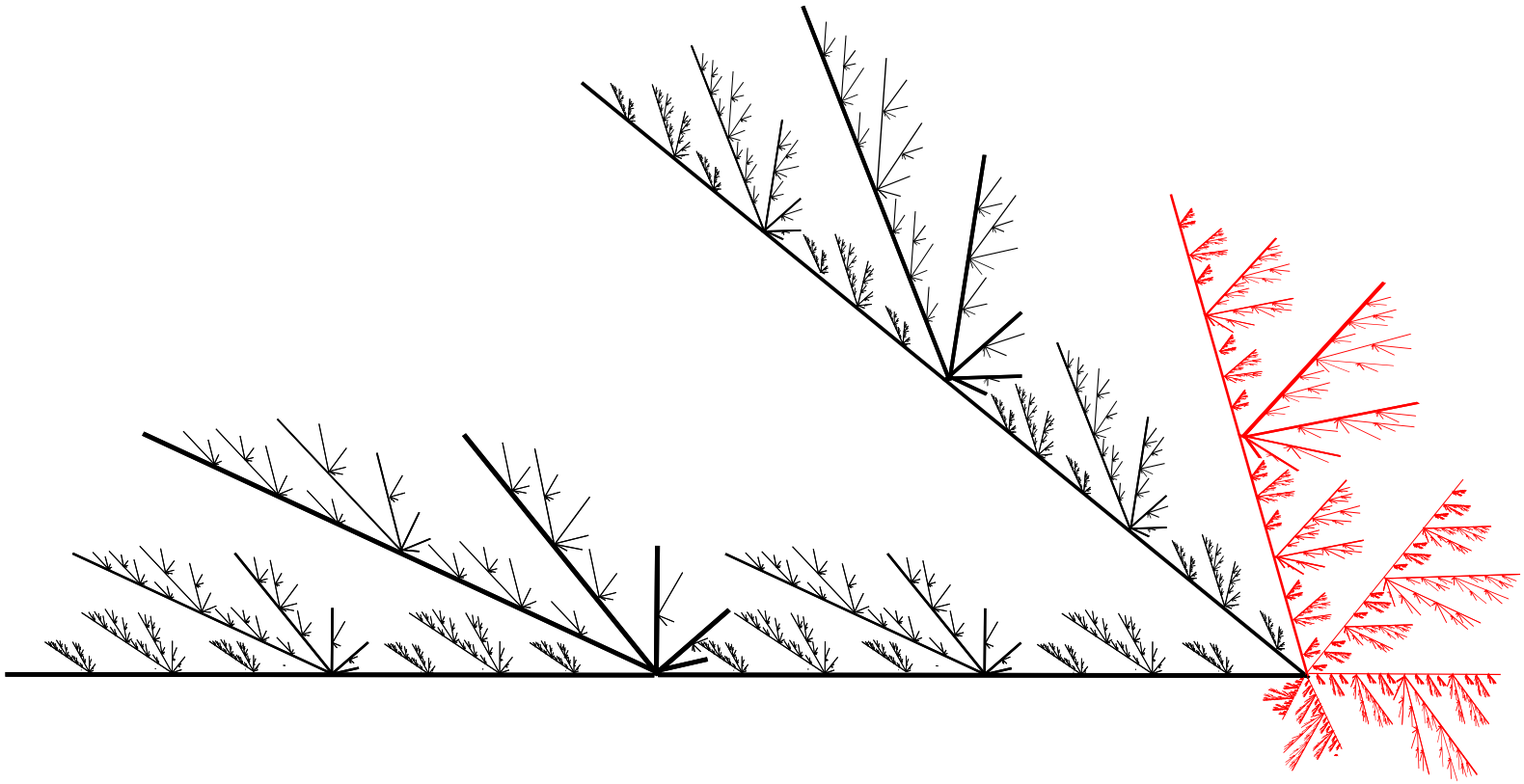

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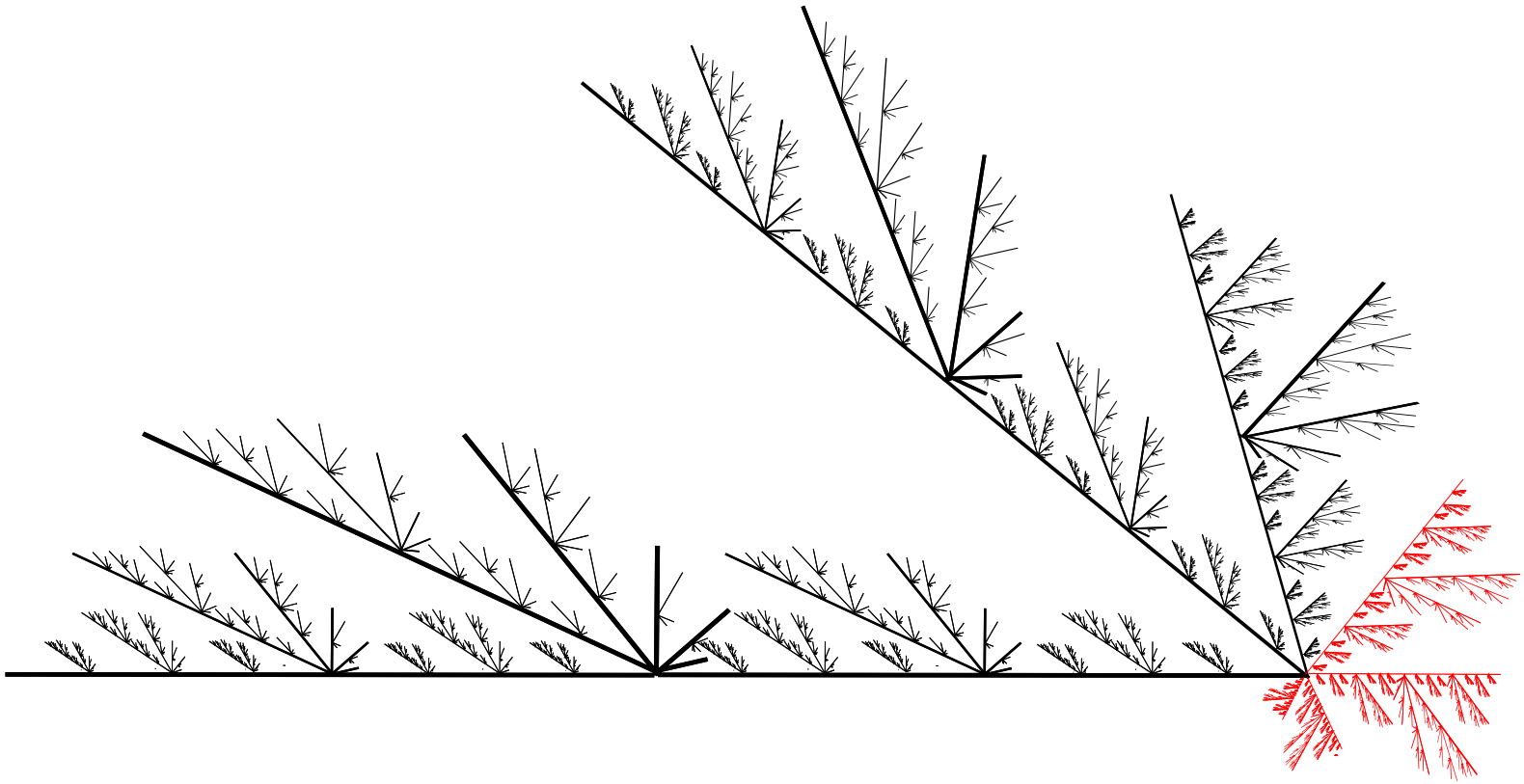

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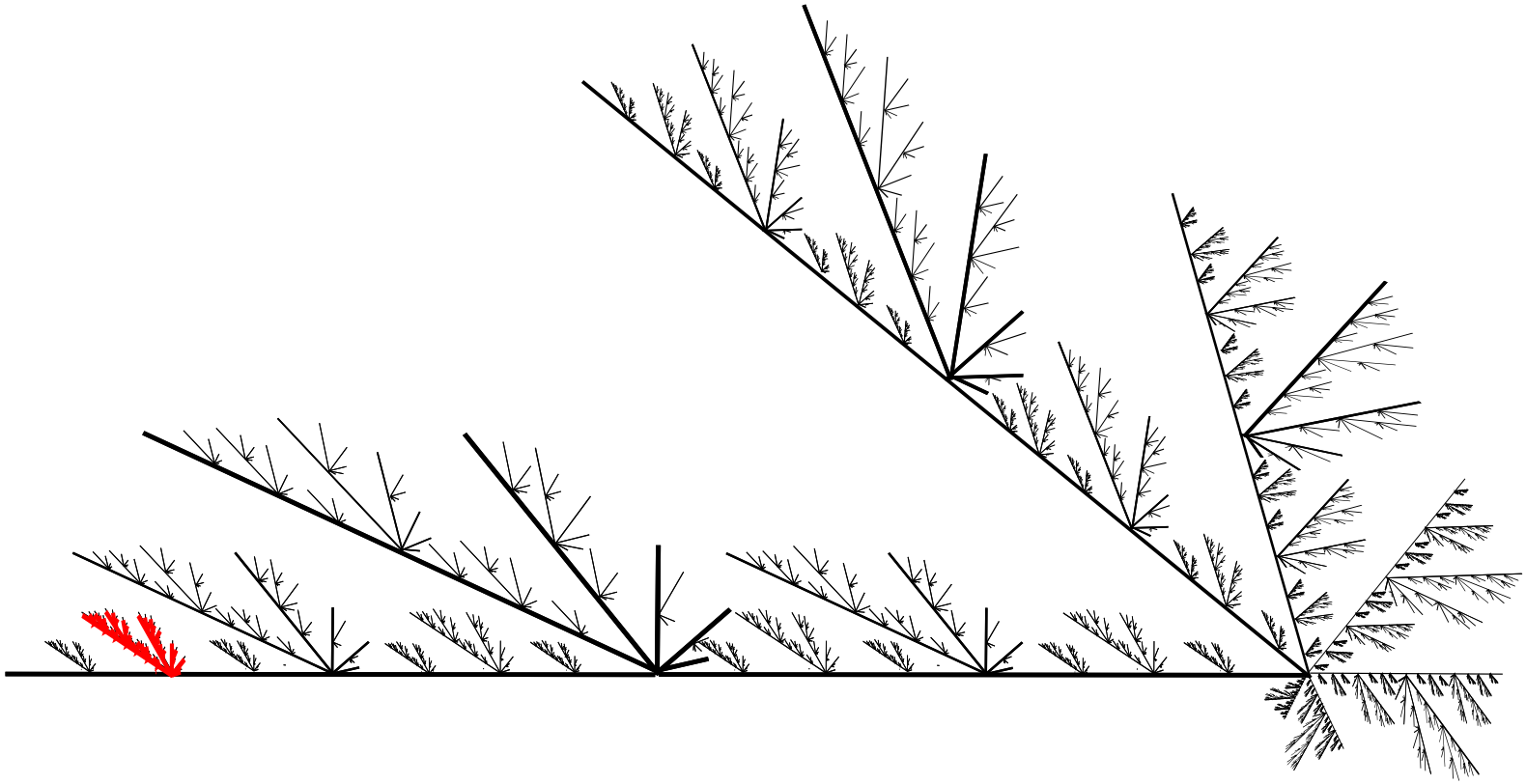

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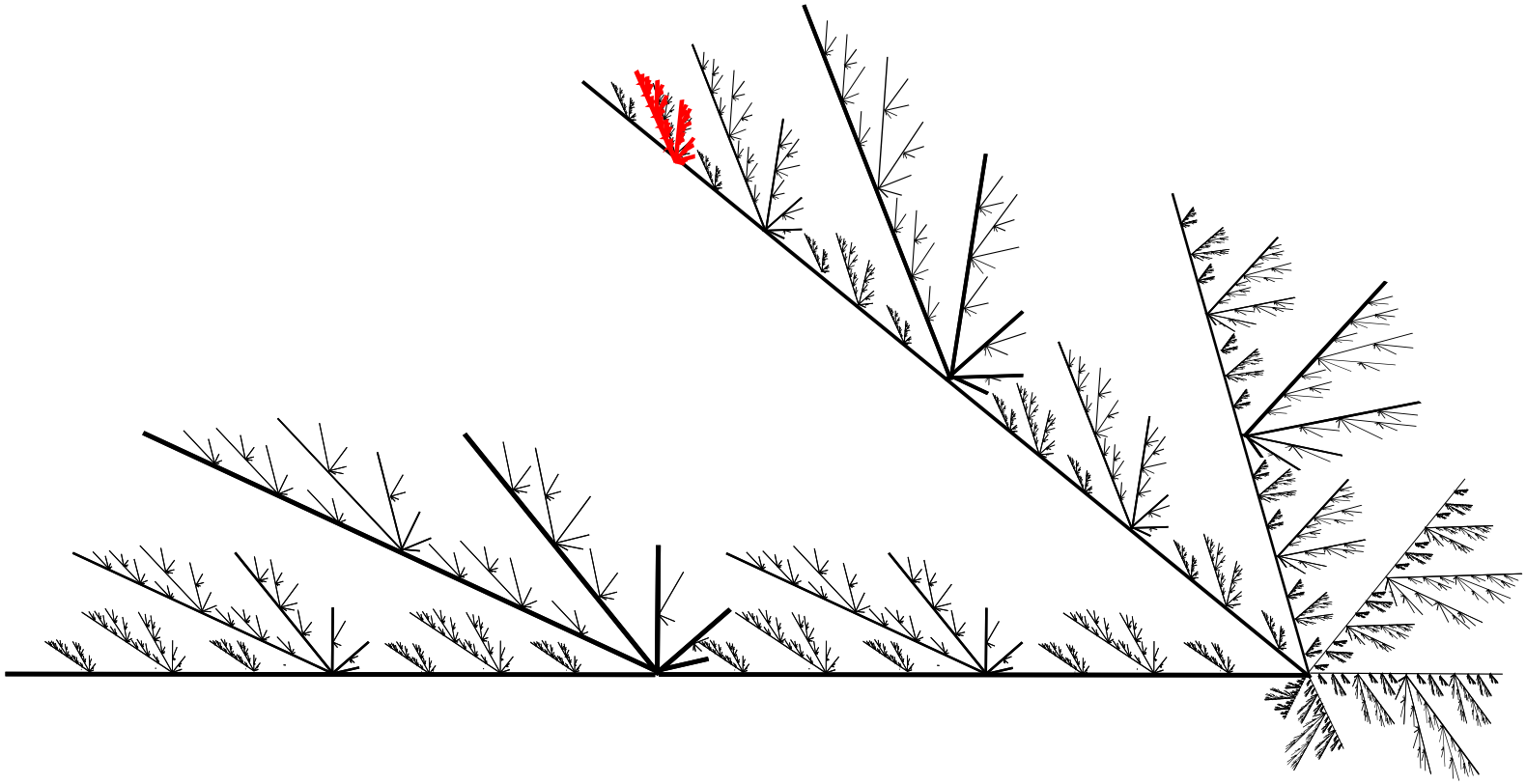

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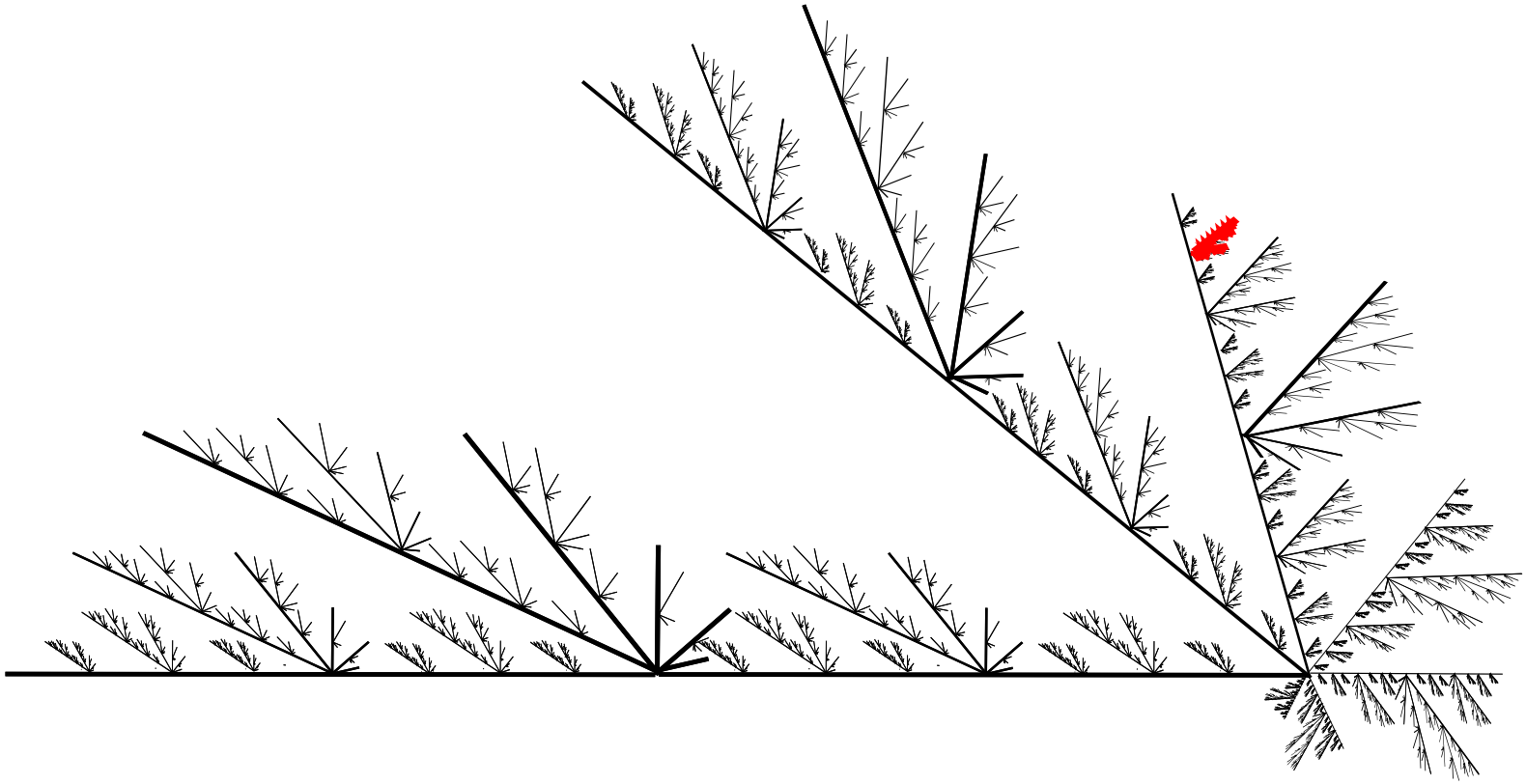

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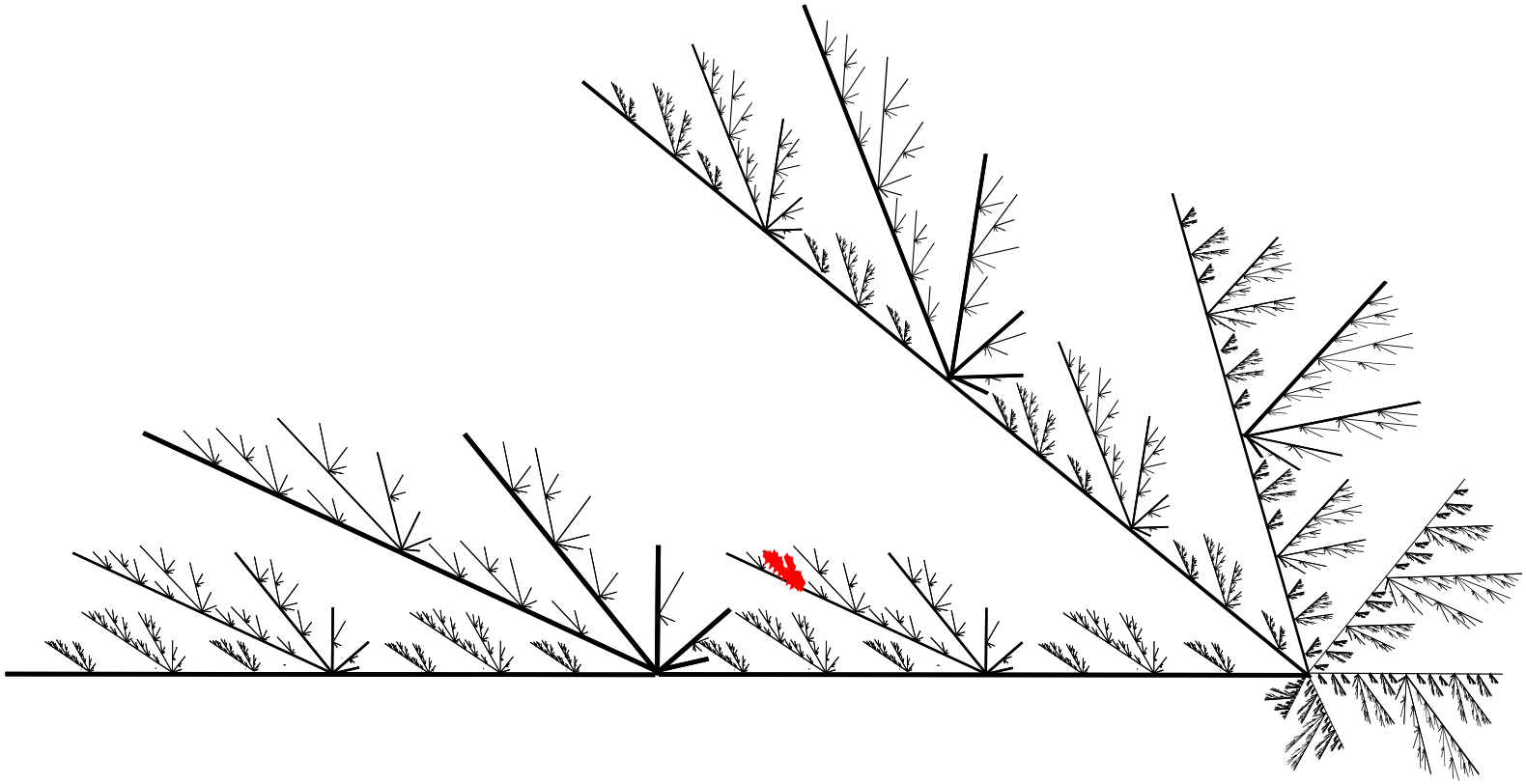

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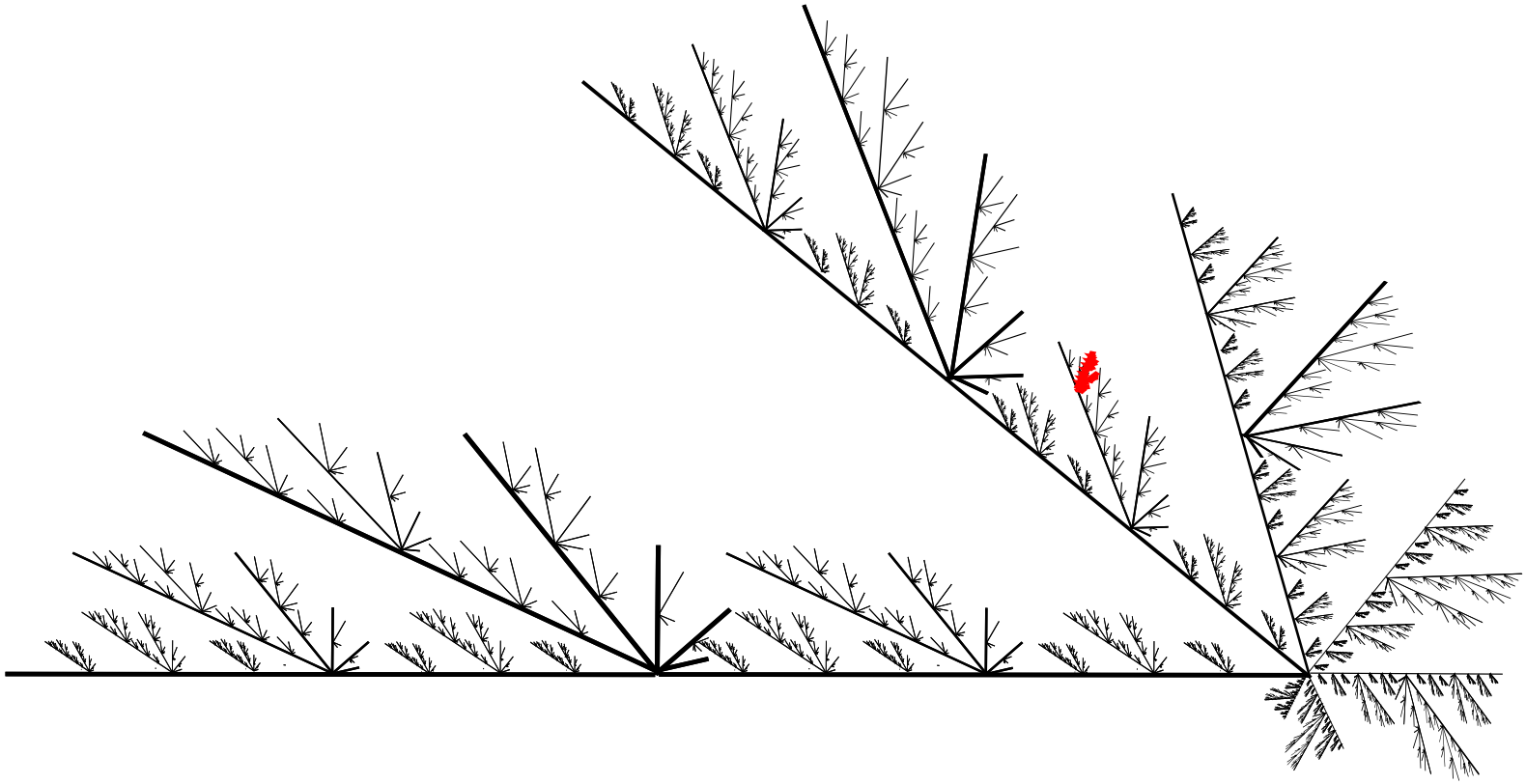

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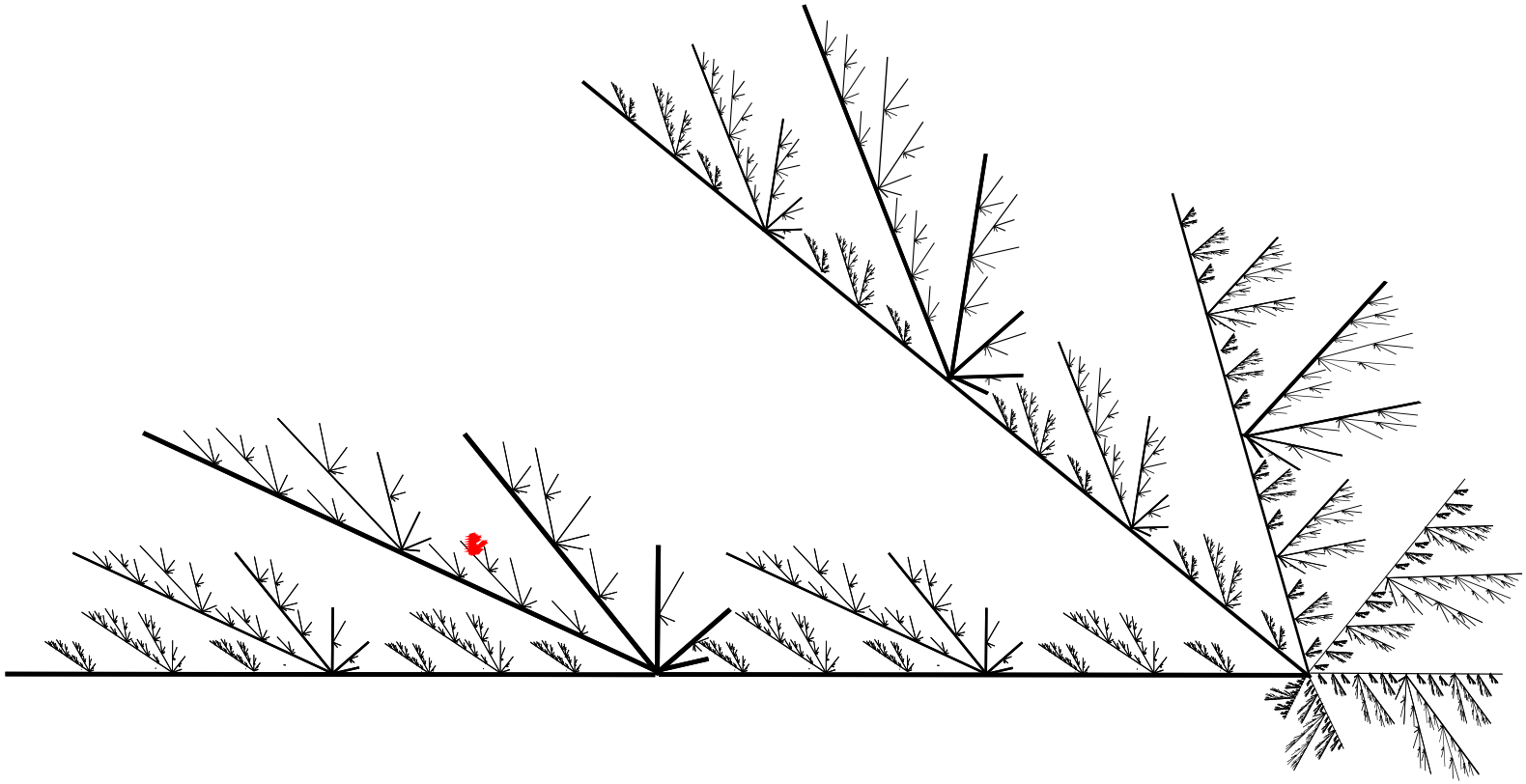

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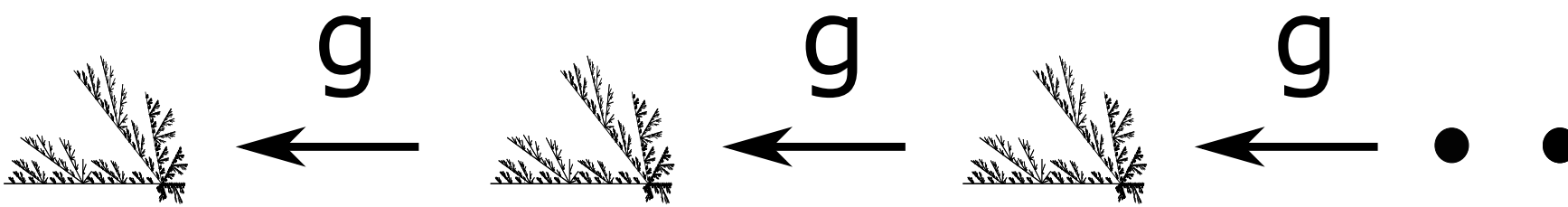

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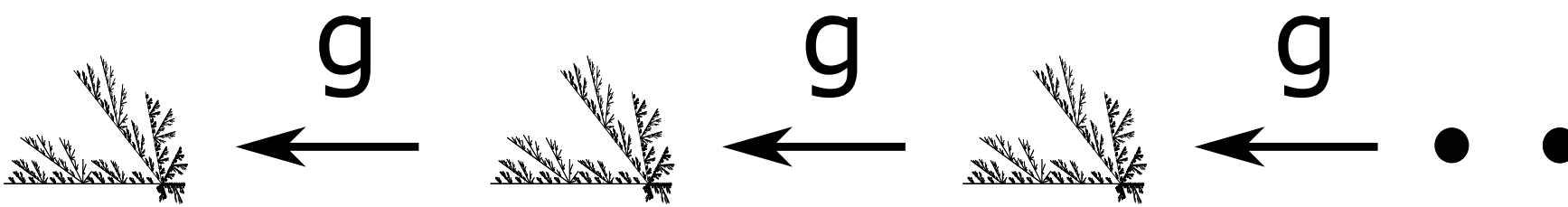
It is 1/2 indecomposable.

Conjecture If X is a one-dimensional continuum that admits a mixing homeomorphism, then X is $1/n$ indecomposable for some n.



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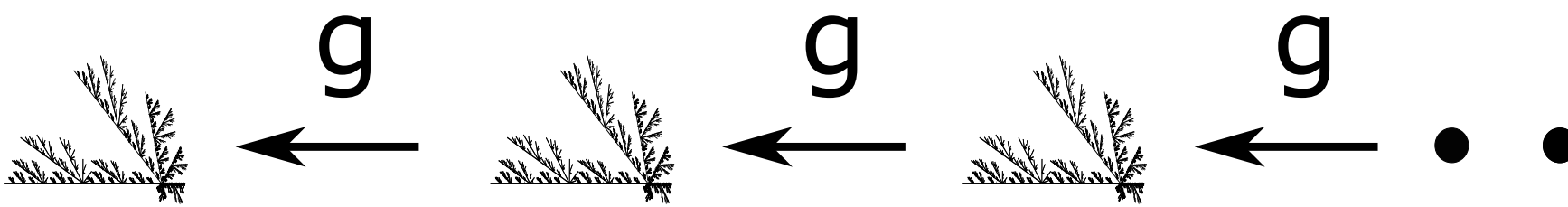

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