Homogeneity and actions of topological groups

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What topological groups acts continuously and transitively on a homogeneous space?

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What topological groups acts continuously and transitively on a homogeneous space?

If X is a homogeneous space then the group of homeomorphisms $\mathrm{Hom(X)}$ in discrete topology is metrizable (a topological group G is metrizable iff $\chi(G) \leq \aleph_0$). Hence homogeneity can be realized by a metrizable group.

Let X be a coset space. Can X be a coset space of a topological group from some class of groups ((metrizable) compact, Polish, ω-narrow, ω-balanced, etc.)

Topological homogeneity

Let $\mathrm{Hom}(\mathrm{X})$ be the group of homeomorphisms of X.
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A space X is *homogeneous* if for every $x, y \in X$ there is a homeomorphism $g \in \mathrm{Hom}(\mathrm{X})$ such that $g(x) = y$.

G-spaces

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A *G-space* is a triple (G, X, α), where X is a topological space, G is a topological group and an action of G on X is a continuous function $\alpha : G \times X \to X$ such that
$\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for all $g, h \in G$, $x \in X$,
$\alpha(e, x) = x$, where e is the identity of G, for all $x \in X$.

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The action $\alpha : G \times X \to X$ is *transitive* if for every $x, y \in X$ there is $g \in G$ such that $g(x) = y$.
In this case X is homogeneous and we shall say that G *realizes homogeneity* of X.

Coset spaces

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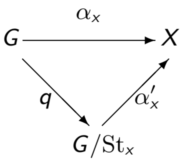
There is a continuous one-to-one map $\alpha'_x : G/\mathrm{St}_x \to X$, $\alpha'_x(g\mathrm{St}_x) = \alpha_x(g)$, and the following diagram is commutative



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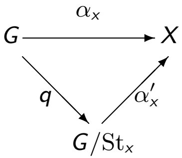

where $\alpha_x(g) = \alpha(g, x)$ and q is the quotient map.

If α'_x is a homeomorphism then X is a coset space of G and the action α is *open* or *micro-transitive* (F. Ancel [1987]). It satisfies the following condition: for any point $x \in X$ and any nbd O of e in G

$$x \in \mathrm{Int}(Ox) \text{ where } Ox = \{y \in X : y = g(x), g \in O\}.$$

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For a compactum X R. Arens [1946] showed that $\mathrm{Hom(X)}$ in compact-open topology is:
(1) Raikov complete topological group;
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V. Fedorchuk [1971] gave an example of a compact homogeneous space which is not a coset space.
J. van Mill [2008] gave an example of a homogeneous Polish space which is not a coset space.

Cardinal invariants of topological groups

A topological group G is called *τ-narrow* if for any nbd O of the identity e in G the cover $\{Og : g \in G\}$ has a subcover of cardinality less than or equal to τ. The minimal cardinal $\tau \geq \aleph_0$ such that G is τ-narrow is called the *index of narrowness* of G. Designation $\mathrm{ib}(G)$.

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Topological groups with $\mathrm{ib}(G) = \aleph_0$ are called *ω-narrow*. They are subgroups of topological products of separable metrizable topological groups.

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The *invariance number* of G is less than or equal to τ (designation $\mathrm{inv}(G) \leq \tau$) if for any nbd O of the identity e in G there exists a family γ of open nbds of e of cardinality less than or equal to τ such that for each $g \in G$ there is $U \in \gamma$ satisfying $g^{-1}Ug \subset O$.

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Every topological group G with $\mathrm{inv}(G) \leq \tau$ can be embedded as a subgroup into a topological product of topological groups of character $\leq \tau$.

Topological groups with $\mathrm{inv}(G) \leq \aleph_0$ are called ω-*balanced*. They are subgroups of topological products of metrizable topological groups.

The Sorgenfrey line and the two arrows compactum

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From L. R. Ford's [1954] result it follows that zero-dimensional homogeneous spaces are coset spaces.

Equivariant maps

Let (G, X, α_G) and (H, Y, α_H) be G-spaces. A pair of maps $(\varphi : G \to H,\ f : X \to Y)$ such that $\varphi : G \to H$ is a homomorphism and the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\varphi \times f} & H \times Y \\ \downarrow \alpha_G & & \downarrow \alpha_H \\ X & \xrightarrow{f} & Y \end{array}$$

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Proposition

Let $(\varphi,\ \mathrm{id}) : (G, X, \alpha) \to (H, X, \gamma)$ be an equivariant map of G-spaces. Then we have

(a) *if α is transitive then γ is transitive;*

(b) *if α is open then γ is open.*

Theorem

Let (G, X, α) be a G-space with a transitive action.
Then there exist a G-space (H, X, γ) such that

$$\chi(H) \leq \chi(X) \cdot \mathrm{inv}(G),$$

$$\omega(H) \leq \chi(X) \cdot \mathrm{ib}(G),$$

and an equivarinat map $(\varphi, \mathrm{id}) : (G, X, \alpha) \to (H, X, \gamma)$, where φ is an epimorphism.

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Corollary

Let (G, X, α) be a G-space with a transitive action and $\chi(X) \le \aleph_0$.
If G is ω-balanced (hence, $\mathrm{inv}(G) \le \aleph_0$) then there exist a G-space (H, X, γ) such that

$$\chi(H) \le \aleph_0 \text{ (hence, } H \text{ is metrizable)},$$

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If G is ω-narrow (hence, $\mathrm{ib}(G) = \aleph_0$) then there exist a G-space (H, X, γ) such that

$$\omega(H) \leq \aleph_0 \text{ (hence, } H \text{ is separable metrizable)},$$

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Since a coset space of a metrizable group is metrizable, we have

Corollary

If X is a coset space of an ω-balanced group and $\chi(X) \leq \aleph_0$ then X is metrizable.

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Example

The Sorgenfrey line and the two arrows compactum are not coset spaces of an ω-balanced group.

d-open actions

The action $\alpha : G \times X \to X$ is called

d-open or *weakly micro-transitive* (F. Ancel [1986])
if $x \in \mathrm{Int}(\mathrm{Cl}(Ox))$ for any point $x \in X$ and any nbd O of e in G.

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A map $f : X \to Y$ is *d-open* if for any open $O \subset X$ we have $f(O) \subset \mathrm{Int}(\mathrm{Cl}(f(O)))$.

The terminology is motivated by the fact that an action is d-open iff maps $\alpha_x : G \to X$, $\alpha_x(g) = \alpha(g, x)$, $x \in X$, are d-open.

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$$\text{OPEN ACTION} \subset d\text{-OPEN ACTION}$$

If (G, X, α) is a G-space with a d-open action, then X is a direct sum of clopen subsets (*components of the action*). Each component of the action is the closure of the orbit of an arbitrary point of this component. If the action is open, then X is a direct sum of clopen subsets which are the orbits of the action.

Everywhere below we assume that a d-open action has one component of action.

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$(\mathrm{Hom}(\beta\mathbb{Q}), \beta\mathbb{Q}, \alpha)$ is an example of a d-open action with one component of action, where $\mathrm{Hom}(\beta\mathbb{Q})$ is in compact-open topology.

V. Uspenskii [1987] extended Effros theorem to a transitive action of an ω-narrow group on a space X with a Baire property by donating action's openness in favor of d-openness.

Theorem (V. Uspenskii 1987)

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(c) *if α is d-open then γ is d-open.*

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Corollary

If the homogeneity of a space X with a Baire property and $\chi(X) \leq \aleph_0$ is realized by an ω-narrow group then X is a separable metrizable space.

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Homogeneity of a non-metrizable or non-separable space X with Baire property and $\chi(X) \leq \aleph_0$ cannot be realized by an ω-narrow group.

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Homogeneity of the Sorgenfrey line and the two arrows compactum cannot be realized by an ω-narrow group.

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Example

Homogeneity of the Sorgenfrey line and the two arrows compactum cannot be realized by an ω-narrow group.

The Sorgenfrey line is not a coset space of a Čech complete group.

The two arrows compactum is not a coset space of a subgroup of a topological product of Čech complete groups.

Coset spaces of (separable) metrizable groups

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Theorem

The following conditions are equivalent for a (separable) metrizable space X:

(a) *X is a coset space of a (separable) metrizable group G;*

(b) *there is a (totally bounded) metric ρ on X such that the action of the group of uniform equivalences with respect to ρ in the topology of uniform convergence is open and transitive.*

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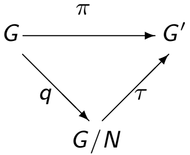
Necessity follows from the result of L. Kristensen [1958] and sufficiency from the result of R. Arens [1946].

Quotient group as a universal element

Let N be a closed normal subgroup of a topological group G, $q : G \to G/N$ be the quotient map. Then for any continuous homomorphism $\pi : G \to G'$ with $N \subset \mathrm{Ker}\pi$ there exists a unique continuous homomorphism $\tau : G/N \to G'$ such that $\pi = \tau \circ q$.



Uniform quotient space as a universal element

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On the quotient set $X/E(\mathcal{U})$ with respect to this partition the *quotient uniformity* $\bar{\mathcal{U}}$ is defined. It is the greatest uniformity on $X/E(\mathcal{U})$ such that the quotient map $q : X \to X/E(\mathcal{U})$ is uniformly continuous.

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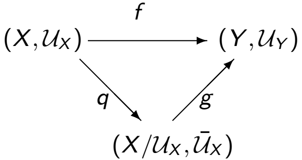
In this case the map q is called a *uniform quotient map*. *Uniform quotient space* $X/\mathcal{U}$ is the quotient set $X/E(\mathcal{U})$ with topology induced by the quotient uniformity $\bar{\mathcal{U}}$.

Uniform quotient space as a universal element

Let $\mathcal{U}_X$ be a pseudouniformity on X. Then for any uniformly continuous map $f : (X, \mathcal{U}_X) \to (Y, \mathcal{U}_Y)$ there exists a unique uniformly continuous map $g : (X/\mathcal{U}_X, \bar{\mathcal{U}}_X) \to (Y, \mathcal{U}_Y)$ such that $f = g \circ q$.



Equiuniform quotient spaces as a universal element (Results of E. Martyanov)

A (pseudo) uniformity $\mathcal{U}$ on a G-space X is called *(pesudo) equiuniformity* if it is *invariant* (each $g \in G$ is uniformly continuous) and *bounded* (that is for any $u \in \mathcal{U}$ there exist $O \in N_G(e)$ and $v \in \mathcal{U}$ such that $Ov = \{OV : V \in v\} \succ u$).

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Let $(G, (X, \mathcal{U}_X), \alpha)$ be a G-space and $\mathcal{U}_X$ be an equiuniformity.
If $\pi \colon G \to G'$ is an open epimorphism with kernel N, then $\mathcal{U}_\pi$ a pseudouniformity on X with base consisting of covers $\varkappa_O = \{OHx : x \in X\}$, $O \in N_G(e)$. Put $\mathcal{U}_N = \mathcal{U}_X \bigwedge \mathcal{U}_\pi$. $\mathcal{U}_N$ is a pseudouniformity.

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A G-space $(G', (X/\mathcal{U}_N, \overline{\mathcal{U}_N}), \gamma)$ is an *equiuniform quotient space*, and the map $(\pi, f) : (G, (X, \mathcal{U}_N), \alpha) \to (G', (X/\mathcal{U}_N, \overline{\mathcal{U}_N}), \gamma)$ is an *equivarinat quotient map*.

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Equiuniform quotient space $(G', (X/\mathcal{U}_N, \overline{\mathcal{U}_N}), \gamma)$ is a universal element for all equivariant map $(\kappa, h) : (G, (X, \mathcal{U}_X), \nu) \to (K, (Y, \mathcal{U}_Y), \alpha)$ such that $\ker\pi \subset \ker\kappa$ and $\phi : (X, \mathcal{U}_N) \to (Y, \mathcal{U}_Y)$ is uniformly continuous.