Laminations and Critical Portraits

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Motivation

- This work is motivated by the study of connected Julia sets of complex polynomials.
- Here, we take a more combinatorial approach and take several dynamical considerations as facts.
- One consideration is that the operation of the polynomial outside a connected Julia set is determined by the angle d-tupling map, where d is the degree of the polynomial.

We take this as our point of departure.

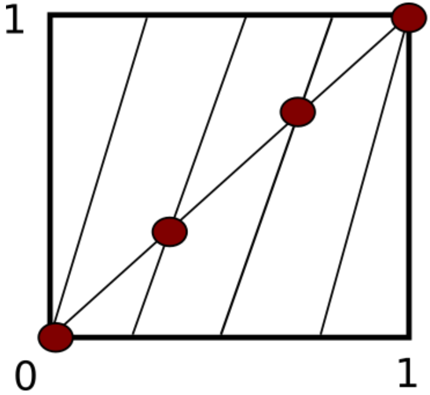
The angle d-tupling map σ_d

Definition

$\sigma_d(t) = dt \pmod 1$ on the unit circle coordinatized by the interval $[0, 1)$.

- $d - 1$ fixed points
- fixed points spaced $\frac{1}{d-1}$ apart

Example σ_4



Critical Sectors and Critical Portraits

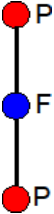
Definition

Let $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_{d-1}\}$ be a collection of pairwise disjoint critical chords. We define $\mathcal{C}$ as a *critical portrait*.
Let $\overline{\mathbb{D}}$ be the closed unit disk. Let E_i be a component of $\overline{\mathbb{D}} \setminus \cup\mathcal{C}$.
We define a *critical sector* as $\overline{E_i}$, where $i = \{1, 2, \ldots, d\}$.

Note: $\overline{E_i} \cap \mathbb{S}^1$ maps onto $\mathbb{S}^1$ one-to-one except at the endpoints of critical chords.

Increasing Specificity

Bicolored Tree



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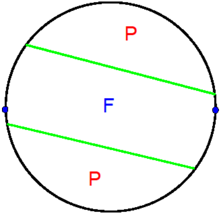
Critical Portrait



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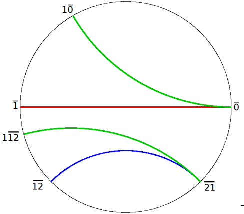
Prelaminational Data



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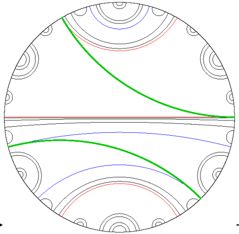

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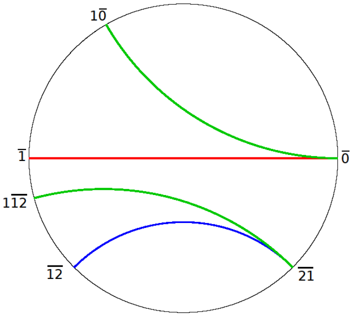
Lamination

Julia Set

Prelaminational data

Definition

Prelaminational data is a critical portrait plus periodic polygons that touch the critical chords.



Invariant Sets Inside Critical Sectors

We call upon some dynamical facts:

- Each critical sector map onto the whole circle, and therefore maps over itself.
- A compact set R remains inside the critical sector.
- R has a rotation number because $\sigma_d|_R$ is one-to-one except at the endpoints of critical chords.

Bicolored Labeling

Definition

If the invariant set in a critical sector has rotation number zero, then it contains a fixed point and the critical sector is labeled F. If the invariant set in a critical sector has a non-zero rotation number, then it is labeled P.

In this work, we only consider rational rotation numbers, and periodic polygons as our rotational sets and for simplicity, we assume critical chords are disjoint, and do not end at fixed points.

Both labels F and P must be used

Theorem

For any given critical portrait, both labels F and P must be used.

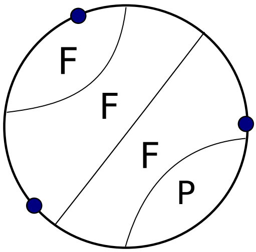
Proof.

For a map of degree d the circle is divided into d critical sectors by $d-1$ critical chords.
Since the map contains $d-1$ fixed points, and there are only d critical sectors, it follows that at least one of these critical sectors will contain no fixed points.
Thus both labels must be used. □

Weak Bicoloring

Theorem

Two F critical sectors may be adjacent, but two P critical sectors may not be adjacent.



Two P Critical Sectors May Not be Adjacent

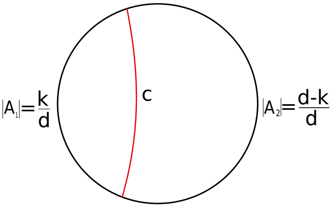
Theorem

Let P be a critical portrait of σ_d, and let P_1, P_2 be adjacent critical sectors. Then P_1 or P_2 must contain a fixed point.

Lemma

Let c be a critical chord with subtended arcs A_1 of length $\frac{k}{d} \leq \frac{1}{2}$ and A_2 of length $\frac{d-k}{d}$. Then A_1 must contain $k-1$ or k fixed points. Consequently, A_2 must contain $d-k$ or $d-k-1$ fixed points respectively.

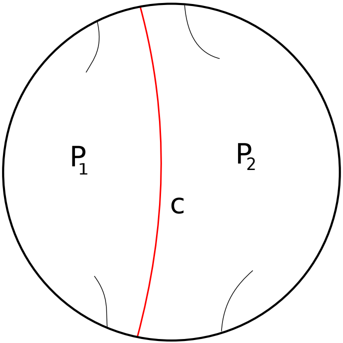
Proof of Lemma



Proof.

Since fixed points are spaced $\frac{1}{d-1}$ apart and $\frac{k-1}{d-1} < \frac{k}{d} < \frac{k}{d-1}$, A_1 contains at most k fixed points, and at least $k-1$ fixed points. A_2 follows because there are $d-1$ fixed points total. □

Proof of Theorem



Proof of Theorem

Proof.

Suppose $|c| = \frac{k}{d} \leq \frac{1}{2}$ where c separates P_1 and P_2.
BWOC, suppose neither P_1 nor P_2 contains a fixed point.

- Total length of critical chord arcs of $Bd(P_1)$, excluding c is $\frac{k-1}{d}$
- Total length of critical chord arcs of $Bd(P_2)$, excluding c is $\frac{d-k-1}{d}$

Consequently, there can a maximum of $k-1$ fixed points on the P_1 side, and a maximum of $d-k-1$ fixed points on the P_2 side. This means that there are only $d-2$ fixed points placed, thus a contradiction of there being $d-1$ fixed points. □

Orbits commute with adding a fixed point

Theorem

Orbits under σ_d commute with rotation by a fixed point, or equivalently, σ_d commutes with adding $\frac{1}{d-1}$.

Corollary

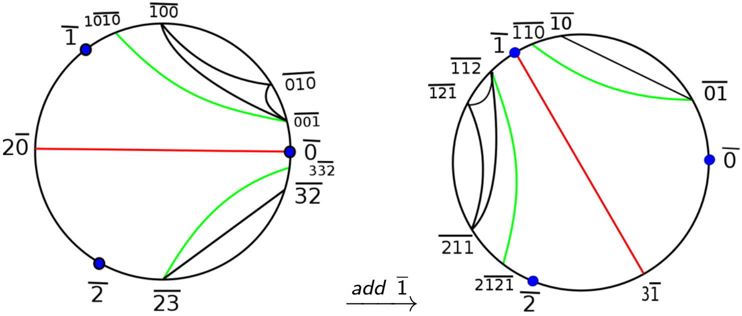
Critical portraits commute with adding a fixed point.

Proof of Theorem

Proof.

Recall $t \to dt (mod 1)$.
We want to show: $t + \frac{1}{d-1} \to dt + \frac{1}{d-1} (mod 1)$.
Given an orbit $t \to dt \to d^2 t \to \ldots \to t$,
now adding $\frac{1}{d-1}$ and applying σ_d, $t + \frac{1}{d-1} \to d(t + \frac{1}{d-1})$
$= dt + \frac{d}{d-1}$
$=dt + \frac{d-1+1}{d-1} \pmod 1$
$= dt + \frac{1}{d-1}$.
Repeat through orbit. □

Example of Commuting Critical Portraits for σ_4



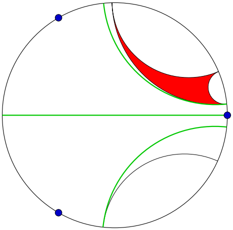
Original Critical Portrait

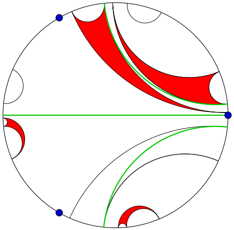
Rotated Critical Portrait

Pulling Back Laminational Data

Definition

The pullback operation on a single point is
$\sigma_d^{-1}(t) = \{\frac{t}{d}, \frac{t+1}{d}, \ldots, \frac{t+d-1}{d}\}$.
A pullback step is the act of pulling back all chords in the periodic prelaminational data such that no two chords cross each other nor the critical chords.





Pullback Commutes with Adding a Fixed Point

Lemma

σ_d^{-1} commutes with adding a fixed point $\frac{1}{d-1}$.

Proof.

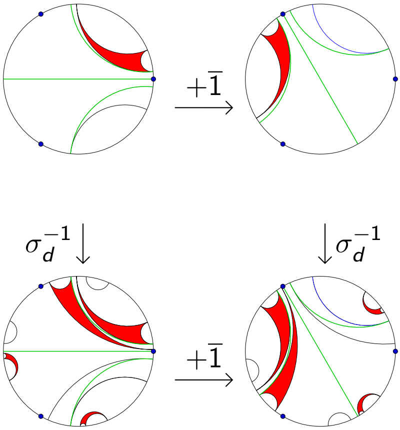
following some algebra:

$\sigma_d(\sigma_d^{-1}(t + \frac{1}{d-1})) = d(\frac{t+k}{d} + \frac{1}{d(d-1)}) = t + \frac{1}{d-1} \pmod 1$ □

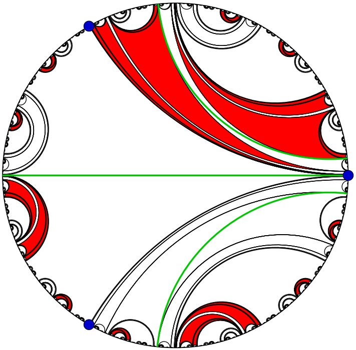
Pullback Commutes with Adding a Fixed Point

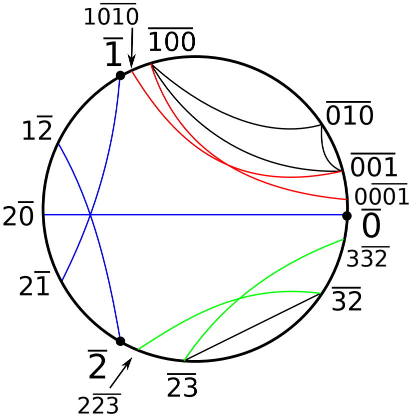
Theorem

The σ_d pullback step commutes with adding a fixed point.



Ambiguity

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Question

When we change our choice of guiding critical leaf still touching the periodic data why do we, or do we not get the same lamination?

Thank you!