Shape of minimal sets in aperiodic flows

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May 21-25, 2018
Nipissing University

15th Annual Workshop on Topology and Dynamical Systems

Sponsored by the Fields Institute

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Abstract

In 1950, H. Seifert asked whether every non-singular $\mathbb{R}$-action (flow) on the 3-sphere has a periodic trajectory. The conjecture that the answer is yes became known as the Seifert Conjecture. Seifert proved the conjecture for perturbations of the flow parallel to the Hopf fibration. The Modified Seifert Conjecture asserted the existence of a minimal set of topological dimension ≤ 1.

Since a C^1 counterexample to the Seifert Conjecture given P.A.Schweitzer in 1974, many other examples of aperiodic flows on S^3 appeared in the literature, all of which are based on a construction of a plug with peculiar minimal sets. We are interested in the algebraic properties of the minimal sets from the point of view of Borsuk's Shape Theory and Vietoris-Čech Homology. In particular, we will concentrate on one-dimensional minimal sets obtained by the method of self-insertion.

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minute introduction to shape theory

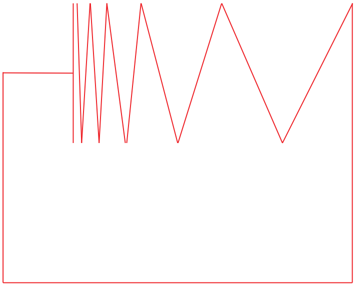

Figure : $\sin \frac{1}{x}$-circle

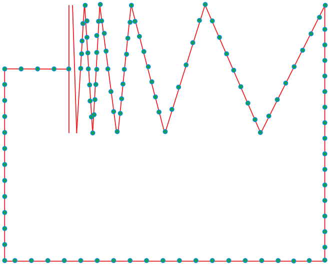

Figure : Vietoris ϵ-cycle

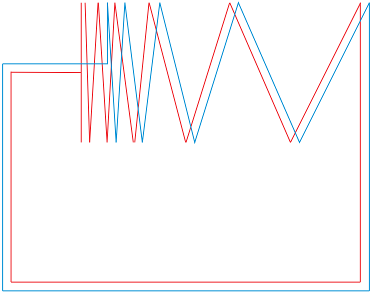

Figure : Approximating circle representing a cycle

Geometric interpretation in Borsuk's shape theory.

Shape

Karol Borsuk 1968 - 1969:
Denote by AR and ANR the classes of metric absolute retracts and absolute neighborhood retracts, resp.
Let $M, N, P \in AR$; $X \subset M$, $Y \subset N$, and $Z \subset P$ be compact.

Definition

A fundamental sequence from X to Y,

$$\underline{f} = \{f_k, X, Y\}_{M,N}$$

is a sequence of maps $f_k : M \to N$, $k = 1, 2, \ldots$, such that for every neighborhood U of Y in N, there is a neighborhood V of X in M such that $f_k|_V \simeq f_{k+1}|_V$ in U for almost all k.

- Composition of fundamental sequences is well-defined:

Definition

Let $\underline{f} = \{f_k, X, Y\}_{M,N}$ and $\underline{g} = \{g_k, Y, Z\}_{N,P}$ be fundamental sequences. The composition $\underline{g} \circ \underline{f}$ is the fundamental sequence $\{g_k \circ f_k, X, Z\}_{M,P}$.

- A map $f : X \to Y$, generates a fundamental sequence

$$\underline{f} = \{f_k, X, Y\}_{M,N}.$$

Usually, the considered absolute retracts are $\mathbb{R}^n$, the Hilbert cube Q, or the Hilbert space H.

Notation: $\underline{f} = \{f_k, X, Y\}$ if $M = N = P$.

Definition

Let $\underline{f} = \{f_k, X, Y\}_{M,N}$ and $\underline{g} = \{g_k, X, Y\}_{M,N}$ be fundamental sequences. $\underline{f}$ is shape equivalent to $\underline{g} = \{g_k, X, Y\}_{M,N}$ (or shape homotopic), notation $\underline{f} \simeq \underline{g}$, if for every neighborhood U of Y in N, there is a neighborhood V of X in M such that $f_k|_V \simeq g_k|_V$ in U for almost all k. Equivalently, $\underline{f} \simeq \underline{g}$ provided $f_1, g_1, f_2, g_2, \ldots$ is a fundamental sequence.

Any two fundamental sequences generated by the same map $f : X \to Y$, $\underline{f} = \{f_k, X, Y\}_{M,N}$ and $\underline{g} = \{g_k, X, Y\}_{M,N}$ are shape homotopic.

Definition

A fundamental sequence generated by the identity id_X is denoted by $\underline{\mathrm{id}}_X$.

Definition

Let $\underline{f} = \{f_k, X, Y\}_{M,N}$ and $\underline{g} = \{g_k, Y, X\}_{N,M}$ be fundamental sequences. If $\underline{g} \circ \underline{f} \simeq \underline{\mathrm{id}}_X$ and $\underline{f} \circ \underline{g} \simeq \underline{\mathrm{id}}_Y$, then $\underline{f}$ is a shape equivalence. More precisely, $\underline{f}$ is a shape equivalence if such $\underline{g}$ exists. Clearly, $\underline{g}$ is then a shape equivalence as well.
If such $\underline{f}$ and $\underline{g}$ exists for some M and N, containing X and Y, respectively, then X and Y are shape equivalent (or have the same shape, or X has the shape of Y) and we write $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$.

This setting allows to define shape homotopy groups (fundamental groups) and other notions. Shape homology and cohomology groups are the Vietoris-Čech groups.

Shape Equivalence

- Sh(X)=Sh(Y) implies all Vietoris-Čech homology and cohomology groups are equal.
- Sh(X, x_0)=Sh(Y, y_0) implies all Borsuk (shape, fundamental) homotopy groups are equal.

Example

The shape of a planar continuum depends only on the number of complementary domains.

The Hawaiian Earring, the Cantor Hawaiian Earring $C \times S^1/C \times \{1\}$, the Sierpiński Carpet, all have the same shape.

Movability - Borsuk

Definition

Let $M \in ANR$. A compact set $F \subset M$ is movable in M provided

$$\forall_U \exists_V \forall_W \exists_H \, H : V \times I \to U$$

such that

1. $H(x, 0) = x$ for all $x \in V$,
2. $H(V \times \{1\}) \subset W$.

U, V, W are open neighborhoods of F in M; H is a homotopy.
We may assume that $F \subset W \subset V \subset U$.

Borsuk

Theorem (K. Borsuk 1968-1972)

- *If M and N are ANRs, F is a compact metric space, and $i : F \hookrightarrow M$, $j : F \hookrightarrow N$ are embeddings, then $i(F)$ is movable in M iff $j(F)$ is movable in N.*
- *A compact space F is movable if there exist a metric ANR M and an embedding $i : F \hookrightarrow M$ such that $i(F)$ is movable in M.*
- *Movability is shape invariant.*
- *If $X \subset \mathbb{R}^2$, then X is movable.*

Theorem (D.R. McMillan) If $X \subset$ surface, then X is movable.(1974)

The Denjoy continuum D is movable. $\mathrm{Sh}(D)=\mathrm{Sh}(S^1 \vee S^1)$

One-dimensional continua

Theorem

(A. Trybulec, 1974)

- *If X and Y are one-dimensional metric continua, and $f : X \to Y$ is continuous and onto, then X is movable implies Y is movable.*
- *A movable one-dimensional continuum has the same shape as a one-point union of countably many circles, a planar continuum.*

A one-dimensional Peano continuum is movable. There are examples of non-movable two-dimensional Peano continua (Borsuk).

The Menger Curve has the same shape as the Sierpiński Carpet.

Seifert Conjecture in dimension three

Seifert Conjecture

A non-singular flow on S^3 possesses a circular trajectory.

Modified Seifert Conjecture

A non-singular flow on S^3 possesses a minimal set of topological dimension one.

Definition

A minimal set is an non-empty, compact, invariant set that is minimal in this respect.

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Theorem (F. W. Wilson, 1966)

A non-singular flow on a manifold of dimension $n \geq 3$ can be modified in a C^∞ fashion so that every minimal set is an $(n-2)$-dimensional torus $S^1 \times \cdots \times S^1$.

Counterexamples to the Seifert Conjecture

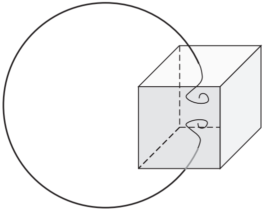
Denjoy minimal sets:

- P.A. Schweitzer (1973) - C^1
- J. Harrison (1984) – $C^{2+\delta}$
- G. Kuperberg (1996) - volume preserving, Hamiltonian on the line bundle
- V. Ginzburg, Başak Gürel (2003) - C^2 Hamiltonian on $\mathbb{R}^4$

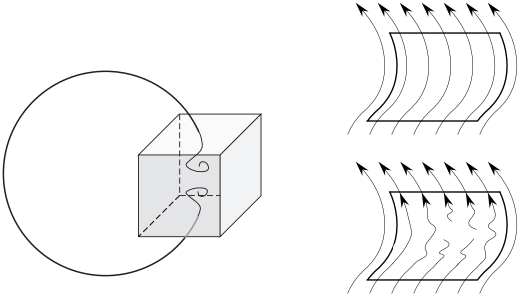
Self-insertion constructions:

- K.K. (1993) - C^∞
- G. Kuperberg and K.K. (1994) - C^ω (Modified Seifert Conjecture); minimal set of topological dimension two
- G. Kuperberg (1994) - PL, continuous; minimal set of topological dimension one

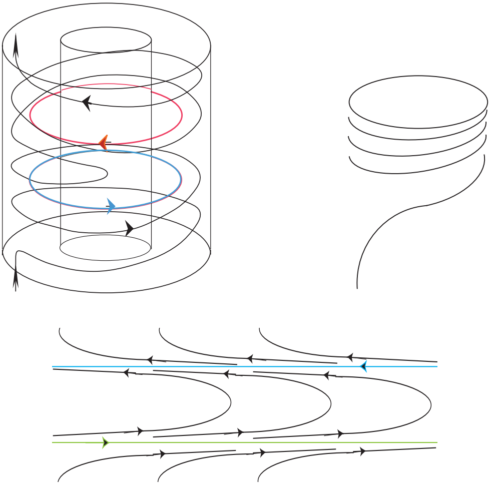
Plug insertion



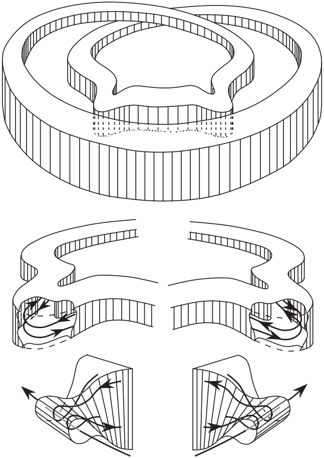
Plug insertion



Wilson-type plug



C^∞ , C^ω



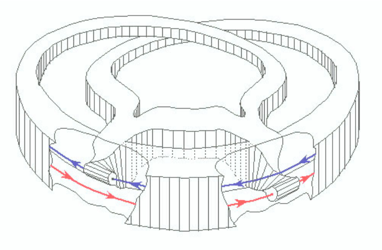


Figure : A smooth plug 1993

G. Kuperberg and K.K. (1994): Insertion yielding

- C^ω construction
- a unique minimal set of topological dimension two

Hurder-Rechtman

The dynamics of generic Kuperberg flows
Steven Hurder, Ana Rechtman
Astérisque 377 (2016), viii+250 pages

The unique minimal set

- is not of the shape of a polyhedron
- satisfies the Mittag-Leffler condition

generic: insertion formulas are polynomial of degree 2

Mittag-Leffler

Definition

Let $M \in ANR$, $F \subset M$ compact, W, V, U open sets containing F.

A compact set $F \subset M$ satisfies the Mittag-Leffler condition in M provided $\forall_U \exists_{V \subset U} \forall_{W \subset V}$

$$\mathrm{im}\{\check{H}_1(W,\mathbb{Z}) \to \check{H}_1(U,\mathbb{Z})\} = \mathrm{im}\{\check{H}_1(V,\mathbb{Z}) \to \check{H}_1(U,\mathbb{Z})\}$$

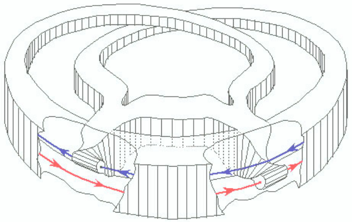
for the inclusion maps $W \hookrightarrow V \hookrightarrow U$.

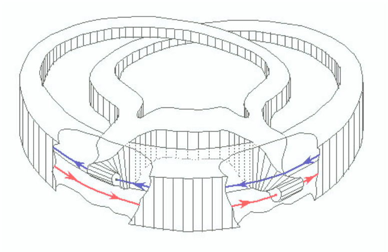
Solenoids non-movable, not Mittag-Leffler

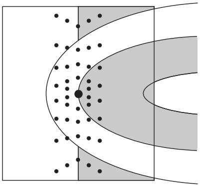
Case-Chamberlin continuum L non-movable, Mittag-Leffler

$L = \varprojlim(S^1 \vee S^1, f_n)$, $f_n(a) = aba^{-1}b^{-1}$, $f_n(b) = a^2b^2a^{-2}b^{-2}$,

where a and b are the natural generators of $\pi_1(S^1 \vee S^1)$.







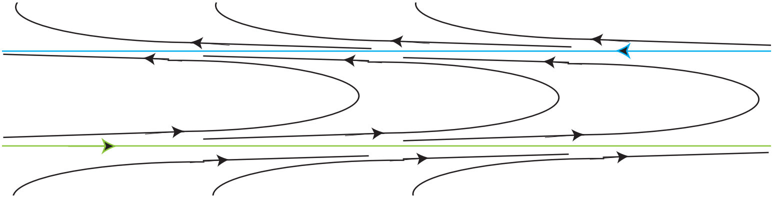


Figure : Reeb component

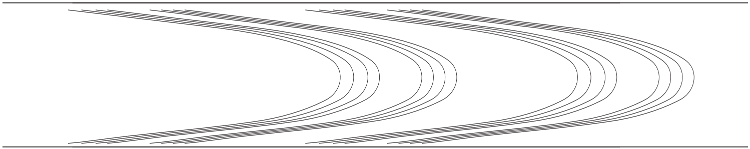


Figure : Cantor Reeb

Controlling the reentry of trajectories to obtain a one-dimensional plug.

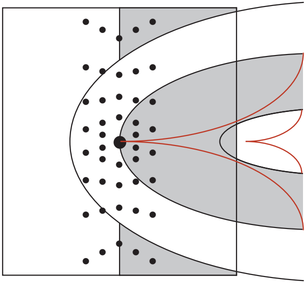


Figure : Sharp C^0 insertion vs. C^∞ insertion

A piece-wise linear plug construction

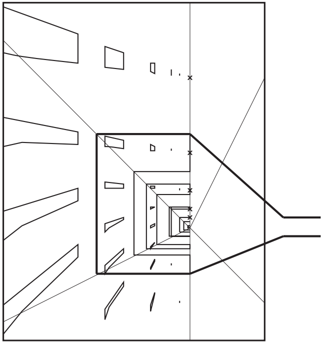


Figure : Controling piece-wise linear insertion.

Open Problems

1. Are the minimal sets in the self-insertion construction movable?
2. Are the minimal sets in the self-insertion construction always Mittag-Leffler?
3. Are there C^1 self-insertion constructions yielding one-dimensional minimal sets?

Thank you for listening!