FUNCTORS IN THE ASYMPTOTIC CATEGORIES

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Nipissing Topology Workshop 2018

Asymptotic topology

The asymptotic topology (coarse geometry) deals with the large scale properties of metric spaces and some related structures (e.g., coarse spaces).
Applications: e.g., to Big Data.

Categories

A metric space (X, d) is proper if every closed ball in X is compact.
A map is proper if the preimage of every compact subset is compact.
A map is coarsely proper, if the preimage of every bounded set is bounded.
A map $f\colon (X, d) \to (Y, \rho)$ is coarsely uniform, if there is a non-decreasing function $\phi\colon [0, \infty) > [0, \infty)$ such that $\lim_{t\to\infty} \phi(t) = \infty$ and $\rho(f(x), f(y)) \leq \phi(d(x, y))$ for all $x, y \in X$.
A map f is coarse, if f is coarse uniform and coarse proper.
A map $f\colon (X, d) \to (Y, \rho)$ is asymptotically Lipschitz, if there are λ and s ($\lambda > 0$, $s > 0$) such that $\rho(f(x), f(y)) \leq \lambda d(x, y) + s$, $x, y \in X$.

The objects of the asymptotic category $\mathcal{A}_D$ are proper metric spaces, the morphisms of this category are proper asymptotically Lipschitz maps.

The objects of the asymptotic category $\mathcal{A}_R$ are proper metric spaces (actually, one can consider all metric spaces), the morphisms of this category are coarsely proper, coarse uniform maps.

Geodesic spaces

A metric space X is called geodesic if for any $x, y \in X$ there exists an isometric embedding $\alpha\colon [0, d(x, y)] \to X$ such that $\alpha(0) = x$ and $\alpha(d(x, y)) = y$.
For geodesic spaces:

$$\text{coarse uniform} = \text{asymptotically Lipschitz}$$

Isomorphisms

Two spaces, X and Y, are coarsely equivalent if there exist coarsely proper, coarse uniform maps $f\colon X \to Y$, $g\colon Y \to X$ such that the compositions gf and fg are of finite distance to the identity maps 1_X and 1_Y respectively.

Coarse invariants: asymptotic dimension

Let $\mathcal{A}$ be a family of subsets of a metric space X.
$\mathcal{A}$ is uniformly bounded if

$$\operatorname{mesh} \mathcal{A} = \sup\{\operatorname{diam}(A) \mid A \in \mathcal{A}\} < \infty.$$

For $D > 0$, $\mathcal{A}$ is D-discrete if
$d(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\} \geq D$ for all $A, B \in \mathcal{A}$, $A \neq B$.

Definition (Gromov)

The asymptotic dimension of X does not exceed $n = 0, 1, 2, \ldots$ if for every $D > 0$ there exists a uniformly bounded cover of X which is a union of at most $n + 1$ D-discrete families.

Notation: $\operatorname{asdim} X \leq n$.

Another asymptotic dimensions

If the mesh of the uniformly bounded cover can be chosen as a linear (resp. power) function of D (on $[r_0, \infty)$, for some $r_0 \geq 0$) then we obtain the notion of the asymptotic Assouad-Nagata dimension asdim_{AN} (resp. asymptotic power dimension asdim_P).

The asymptotic dimension with linear control [Dranishnikov] $\ell\text{-}\mathrm{asdim}\, X$ of a metric space X is defined as follows:
$\ell\text{-}\mathrm{asdim}\, X \leq n$ if there is $c > 0$ such that for every $R < \infty$ there is $\lambda > R$ such that $(\lambda, c\lambda)\text{-}\dim X \leq n$.
One can similarly define the asymptotic dimension with power control [Kucab].

Coronas

Let X be a proper metric space. A function $\phi\colon X \to \mathbb{R}$ is slowly oscillating if, for every $r > 0$,

$$\lim_{x\to\infty} \mathrm{diam}(\phi(B_r(x))) = 0.$$

The compactification generated by the algebra of bounded slowly oscillating functions is called the Higson compactification, its remainder is called the Higson corona (denoted υX).
The Higson corona functors acts from the coarse category to the category of compact Hausdorff spaces.

Characterization

A compactification $\bar{X}$ of a pointed proper metric space (X, d, x_0) is isomorphic to the Higson compactification if and only if the following holds:
for every closed A, B in X,

$$(\bar{A} \cap \bar{B}) \setminus X = \emptyset \Leftrightarrow \lim_{r \to \infty} d(A \setminus B_r(x_0), A \setminus B_r(x_0)) = \infty.$$

Another coronas

1) Sublinear corona $\upsilon_L X$ [Dranishnikov-Smith]: for every closed A, B in X, there is $c > 0$ and $r_0 \geq 0$ such that

$$(\bar{A} \cap \bar{B}) \setminus X = \emptyset \Leftrightarrow d(A \setminus B_r(x_0), A \setminus B_r(x_0)) \geq cr, \;\; r \geq r_0.$$

2) Subpower corona $\upsilon_P X$ [Kucab-Z.]: for every closed A, B in X, there is $\alpha > 0$ and $r_0 \geq 0$ such that

$$(\bar{A} \cap \bar{B}) \setminus X = \emptyset \Leftrightarrow d(A \setminus B_r(x_0), A \setminus B_r(x_0)) \geq r^{\alpha}, \;\; r \geq r_0.$$

These constructions determine functors in suitable categories.

$$\upsilon_L X \preceq \upsilon_P X \preceq \upsilon X$$

There are examples showing that this preorder is strict.

Example

Theorem (Iwamoto, 2018)

Let $X = [0, \infty)$ be the half open interval with the usual metric. Then the subpower Higson corona $\upsilon_P X$ is a non-metrizable indecomposable continuum.

Proposition (Iwamoto, 2018)

Let $X = [0, \infty)$ be the half open interval with the usual metric. If K is a proper closed subset of $\upsilon_P X$ with non-empty interior in $\upsilon_P X$ then K is disconnected.

Question (Iwamoto): is it true for the Higson corona?

Theorem (Keesling, 1997)

Let A be a σ-compact subspace of υX. Then the closure of A is homeomorphic to the Stone-Čech compactification of A.

Unlikely to the Higson corona, we have the following

Theorem (Kucab-Z.)

There exists a proper unbounded metric space whose subpower corona contains a σ-compact subset which is not C^*-embedded.

Coronas and dimensions

Theorem (Dranishnikov)

$$\operatorname{asdim} X = \dim \upsilon X$$

if $\operatorname{asdim} X$ *is finite.*

(a weaker statement was proved by [Dranishnikov-Keesling-Uspenskij]).

Coronas and dimensions

A metric space X is (asymptotically) connected if there is $C > 0$ such that, for every $x, y \in X$ there is a sequence $x = x_0, x_1, \ldots, x_n = y$ with $d(x_{i-1}, x_i) \leq C$, $i = 1, \ldots, n$.
A metric space is cocompact if there exists a compact subset K of X such that

$$X = \bigcup_{\gamma \in \mathrm{Isom}(X)} \gamma(K),$$

where $\mathrm{Isom}(X)$ is the set of all isometries of X.

Theorem (Dranishnikov-Smith)

For a cocompact connected proper metric space, $\mathrm{asdim}_{AN} X = \dim_L X$ *provided* $\mathrm{asdim}_{AN} X < \infty$.

A similar result holds also for the asymptotic power dimension and asymptotic subpower corona [Kucab-Z.].

An example showing that the cocompactness is essential:

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \geq 1,\ |y| \leq \ln x\}.$$

Products

Given a pointed metric space (X, x_0, d), we define the norm of $x \in X$ as $\|x\| = d(x, x_0)$.
Let (X, x_0), (Y, y_0) be pointed metric spaces. Define

$$X \tilde{\times} Y = \{(x, y) \in X \times Y \mid \|x\| = \|y\|\} = X \times_{\mathbb{R}_+} Y.$$

Cone

Let X be a metric space. The cone CX of X is defined as follows: $CX = X\tilde{\times}\mathbb{R}^2_+/i_+(X)$, where $i_+ : X \to$ is the embedding defined by the formula $i_+(x) = (x, \|x\|, 0)$ (see [Dranishnikov]).

Proposition

The cone $C\mathbb{R}$ is not isomorphic to the half-space $\mathbb{R}^2_+$ in the asymptotic category $\mathcal{A}$.

Join

Let $X \vee Y$ denote the bouquet of pointed metric spaces X and Y . We endow the bouquet with the natural quotient metric. The join $X * \mathbb{R}_+$ is the subspace of $P_2(X \vee \mathbb{R}_+)$ of probability measures with supports of cardinality ≤ 2.

Kantorovich-Rubinstein distance

Let us define the Kantorovich-Rubinstein distance on the join $X * \mathbb{R}_+$ between two probability measures μ and ν, where

$$\mu = \alpha\delta_x + (1-\alpha)\delta_y, \ \nu = \beta\delta_{x'} + (1-\beta)\delta_{y'},$$

$\|x\| = y$, $\|x'\| = y'$, $x, x' \in X$, $y, y' \in \mathbb{R}_+$, is

$$d_{KR}(\mu,\nu) = |\alpha - \beta|(y + y') + \min\{\alpha,\beta\}d(x,x') \\ +(1 - \max\{\alpha,\beta\})|y - y'|.$$

Proposition

The join $\mathbb{R}^n * \mathbb{R}_+$ is isomorphic to the half-space $\mathbb{R}^{n+1}_+$ in the asymptotic category $\mathcal{A}$.

This and the previous proposition provides a negative answer to the question from [A. Dranishnikov, Asymptotic topology, Russian Math. Surveys 55 (2000), no. 6, 71-116] whether CX and $X * \mathbb{R}_+$ are coarsely equivalent.

A non-geodesic asymptotically zero-dimensional example is also possible:

$$X = \{n^2 \mid n \in \mathbb{N}\} \subset \mathbb{R}$$

Probability measures

Let (X, d, x_0) be a pointed metric space. Let $\tilde{P}_2(X)$ denote the set of probability measures in X of the form $\alpha\delta_x + (1-\alpha)\delta_y$, where $\|x\| = \|y\|$.

Proposition

$\tilde{P}_2(\mathbb{R}^2)$ and $\mathbb{R}^4$ are coarsely equivalent.

Hyperspaces

For every metric space X by $\exp X$ we denote the hyperspace of X. By d_H we denote the Hausdorff metric on $\exp X$:

$$d_H(A, B) = \inf\{\varepsilon > 0 | A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\}.$$

(Remark: this (extended) metric can be defined also in the set $\mathrm{CL}(X)$ of nonempty closed subsets of X, just let $\inf \emptyset = \infty$.)
For every $n \in \mathbb{N}$, by $\exp_n X$ we denote the subspace of $\exp X$ consisting of the subsets of cardinality $\leq n$.
The hyperspace of compact convex subsets in $\mathbb{R}^n$ is denoted by $cc\,(\mathbb{R}^n)$.

Hyperspaces of euclidean spaces

Theorem

The hyperspaces $\exp \mathbb{R}^n$ *and* $\mathrm{cc}\, \mathbb{R}^n$ *are not coarsely equivalent.*

The hyperspace of subcontinua of X will be denoted by $\exp^c X$,

$$\exp^c X = \{A \subset \exp X \mid A \text{ is connected}\}.$$

Theorem

The hyperspace $\exp^c \mathbb{R}^n$, $n \geq 2$, *is not a geodesic space.*

Theorem

The hyperspaces $\exp \mathbb{R}^n$ *and* $\exp^c \mathbb{R}^n$ *are not coarsely equivalent.*

One can prove a similar statement for the hyperbolic spaces $\mathbb{H}^n$.

Symmetric and hypersymmetric powers

Let $\exp_n X = \{A \in \exp X \mid |A| \leq n\}$. E. Shchepin calls $\exp_n X$ the n-th hypersymmetric power of X.
Let G be a subgroup of the symmetric group S_n. Then G naturally acts on the product X^n and by SP^n_G we denote the orbit space of this action.

Theorem

1) $\operatorname{asdim}(SP^n_G X) \le n \operatorname{asdim} X$;
2) $\operatorname{asdim}_{AN}(SP^n_G X) \le n \operatorname{asdim}_{AN} X$;
3) $\operatorname{asdim}_P(SP^n_G X) \le n \operatorname{asdim}_P X$.

Hypersymmetric powers

Similar results can be also proved for the hypersymmetric powers $\exp_n X$.

Asymptotic dimensions with linear (power) control

Also, these estimates hold for ℓ-asdim and its power counterpart.

Probability measures

The set $P(X)$ of probability measures of compact support is endowed with the Kantorovich-Rubinstein metric:

$$d_{KR}(\mu,\nu) = \sup\left\{\left|\int_X \phi d\mu - \int_X \phi d\mu\right| \mid \phi \text{ is 1-Lipschitz}\right\}$$

Let $\mathcal{A}(0)$ denote the subcategory of $\mathcal{A}$ whose objects are asymptotically zero-dimensional spaces.

Theorem

There exists a proper metric space X such that the space $P(X)$ is not an absolute extensor in the category $\mathcal{A}(0)$.

Idempotent measures

Idempotent mathematics: one of the operations $+$ or $\cdot$ is replaced by an idempotent one (say, $\max = \oplus$).
We obtain the notion of idempotent measure as a counterpart of the notion of probability measure.
A typical idempotent measure of finite support is

$$\mu = \oplus_{i=1}^{n} \alpha_i \delta_{x_i},$$

where $\alpha_1, \dots, \alpha_n \in [0, 1]$, $\oplus_{i=1}^{n} \alpha_i = 1$, $x_1, \dots, x_n \in X$.
Let (X, d, x_0) be a pointed metric space. Let $\Delta(X, x_0) = \{(x, t) \in X \times [0, \infty) \mid 0 \leq t \leq \|x\|\}$.

Metrization

In order to find a suitable metrization of the set $I(X)$ of idempotent measures of compact support, one identifies every idempotent measure with the subgraph of its density function and then use the Hausdorff metric.
E.g., to $\mu = \oplus_{i=1}^{n} \alpha_i \delta_{x_i}$ there corresponds

$$(X \times \{0\}) \cup \bigcup_{i=1}^{n} \{x_i\} \times [0, \alpha_i] \in \mathrm{CL}(\Delta(X)).$$

However

The obtained metric space is bounded and therefore is not interesting for the asymptotic topology (coarse geometry). We are going to modify the metric on the set of idempotent measures of finite supports.

Another metric

Identify $\mu = \oplus_{i=1}^{n} \alpha_i \delta_{x_i}$ with $(X \times \{0\}) \cup \bigcup_{i=1}^{n} \{x_i\} \times [0, \alpha_i \|x_i\|] \in \mathrm{CL}(X)$ and then consider the Hausdorff.

Now, $I(X)$ is the completion of the obtained space.

Theorem

There exists a proper metric space X such that $I(X)$ is not an absolute extensor in the category $\mathcal{A}(0)$.

Question

For which X, the spaces $P(X)$ and $I(X)$ are coarsely equivalent?

Thank you!