Periodic Orbits of Piecewise Monotone Maps

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Joint work with Michał Misiurewicz.

The Basics

A dynamical system is a space X together with a function $f : X \to X$. We study the behavior of $f^n : X \to X$ as $n \to \infty$.

The most basic example of long term behavior are *periodic points*. A point $x \in X$ is periodic if there exists n such that $f^n(x) = x$. The smallest integer n satisfying this property is called the period of the periodic point.

Two dynamical systems $f : X \to X$ and $g : Y \to Y$ are said to be *topologically conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$.

Motivation: Sharkovsky's Theorem

Theorem (Sharkovsky)

Let $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function. Consider the ordering:*

$$\begin{array}{llllll} 3 & >_S & 5 & >_S & 7\dots & \\ 2\cdot 3 & >_S & 2\cdot 5 & >_S & 2\cdot 7\dots & \\ \dots & & & & & \\ 2^n\cdot 3 & >_S & 2^n\cdot 5 & >_S & 2^n\cdot 7\dots & \\ \dots & >_S & 2^2 & >_S & 2^1 & >_S \quad 1 \end{array}$$

If f *has a periodic point of period* p*, then* f *has a periodic point of period* q *for every* $p >_S q$ *in the above ordering.*

What we would like to investigate is an analogue of this theorem for discontinuous maps with "reasonable" structure. We shall restrict our attention to piecewise monotone maps which are "unimodal".

Simplifying

Denote the set of "unimodal" piecewise monotone maps by $\mathcal{F}$.

We are interested only in the periodic orbits of these maps. Therefore, it would be convenient if we had a simple family of "representatives".

We will approach this using kneading theory.

Itineraries

For $x \in J$ we define its *itinerary* $\underline{I}_f(x)$ under the mapping f to be the sequence $I_0(x)I_1(x)I_2(x)\ldots$, where

$$I_0(x) = \begin{cases} R & \text{if } \; x > c, \\ C & \text{if } \; x = c, \\ L & \text{if } \; x < c, \end{cases} \tag{1}$$

and $I_j(x) = I_0(f^j(x))$. We adopt the convention that the itinerary terminates if $I_j(x) = C$ for some j.

We will call a sequence $\underline{A}$ of Rs, Ls, and Cs *admissible* if it is either an infinite sequence of Rs and Ls, or a finite (possibly empty) sequence of Rs and Ls followed by a C.

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- Finite sequences $\underline{A}$ have length $|\underline{A}|$ and are even (odd) if the sequence contains an even (odd) number of R's.
- The parity-lexicographical ordering on sequences $\underline{A}$ of R's, L's, and C's so that for $a, b \in [x_b, y_b]$, $a < b$ if and only if $I(a) < I(b)$.

Kneading Sequences

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- We will define the *left kneading sequence* of f to be $I(f(c_-))$ and the *right kneading sequence* of f to be $I(f(c_+))$.
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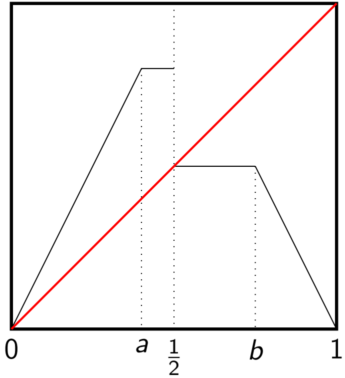
- Kneading sequences determine orbits on a maps core.

Two-sided Truncated Tent maps

Consider a two-sided truncated tent map.



We denote by $\mathcal{TS}$ the parameter space of all parameters $((a, b))$, with $a \in [0, \frac{1}{2}]$ and $b \in [\frac{1}{2}, 1]$. Here we use notation $((a, b))$ for the parameter to avoid confusion with interval (a, b).

A Simplification

Using Kneading Theory, we can prove the following:

Theorem

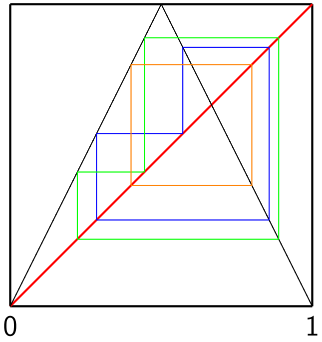
For every $f \in \mathcal{F}$ there exists $(\!(a, b)\!) \in \mathcal{TS}$ such that $K_-(T_{a,b}) = K_-(f)$ and $K_+(T_{a,b}) = K_+(f)$. In particular, $\mathcal{I}(f|_{J_1}) = \mathcal{I}(T_{a,b}|_{J_2})$, where J_1 and J_2 are the cores of f and $T_{a,b}$, respectively.

Here $\mathcal{I}$ denotes the set of all itineraries for a given map.

Since we wish to study the periodic orbits of $f \in \mathcal{F}$, it suffices instead to study a map $T_{a,b}$ which has the same kneading sequences as f.

Sharkovsky: Getting a Geometric Understanding

Using a continuous truncated tent map, the Sharkovsky order $>_S$ can be understood to be the order periods are lost as the map is truncated further.



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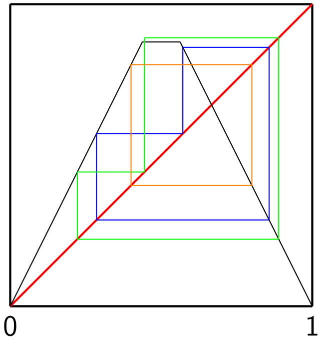

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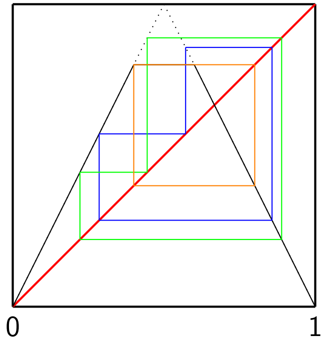
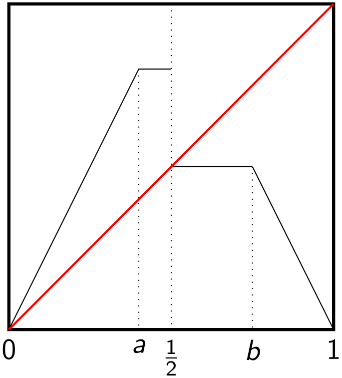


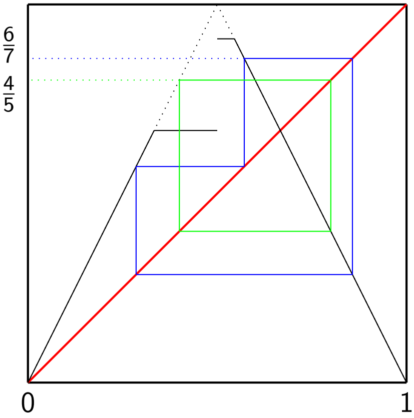
A Two-Parameter Analogue

The discontinuous analogue to this construction is to consider a two-sided truncated tent map.



Getting the Picture

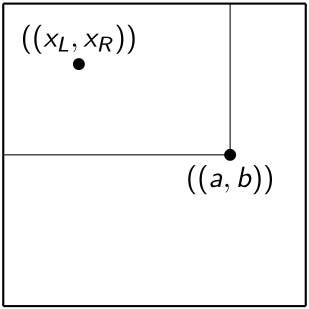
A simple example shows that the Sharkovsky ordering $>_S$ immediately falls apart when considering discontinuous maps.



Definition of a Peak

- Since the Sharkovsky order does not help us, we need to understand how periodic orbits force one another in this context.
- Let Q be a periodic orbit under the full tent map T, $x_L = \max\{x \in Q \mid x < \frac{1}{2}\}$, and $x_R = \min\{x \in Q \mid x > \frac{1}{2}\}$.
- The parameter $(\!(x_L, x_R)\!)$ is called the *peak* associated to the periodic orbit Q.
- The peak acts as a threshold for the periodic orbit in the parameter space $\mathcal{TS}$.

Peaks



The map $T_{a,b}$ has a periodic orbit which corresponds to the peak $((x_L, x_R))$.

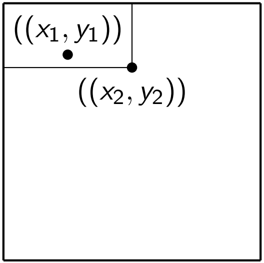
Peaks(continued)

There is a simple geometric relationship to understand when one periodic orbit forces another.

Lemma

Let Q_1 and Q_2 be periodic orbits under T with peaks $((x_1, y_1))$ and $((x_2, y_2))$, respectively. Then T_{x_2,y_2} has periodic orbit Q_1 if and only if $x_1 \leq x_2$ and $y_1 \geq y_2$.

This gives us the following type of picture:



The Set of Peaks

This is an illustration of all peaks up to period 20. We denote the set of peaks by $\mathcal{P}$.