Decomposition towers and their forcing

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Introduction

Given a map f, a point x is called **periodic (of period n)** if points $x, f(x), \ldots, f^{n-1}(x)$ are all distinct while $f^n(x) = x$. From the standpoint of the theory of dynamical systems, this is the simplest type of limit behavior of a point. The description of possible sets of periodic orbits of maps from a certain class is a natural and appealing problem.

In the theory of dynamical systems two maps $f : X \to X$ and $g : Y \to Y$ are said to be **(topologically) conjugate** if there exists a homeomorphism $\psi : X \to Y$ such that $\psi \circ f = g \circ \psi$, i.e. if there exists a change of coordinates transforming the map f into the map g. Conjugate maps are considered equivalent. Sometimes one adds restrictions on the **conjugacy** ψ, such as preserving orientation, and the like.

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Cyclic patterns

Classes of equivalence are then called **cyclic patterns** (since we consider **only** cyclic patterns and permutations, we will call them simply **patterns** and **permutations** from now on). Thus, one comes across a problem of characterizing possible sets of patterns exhibited by interval maps.

A naive question: how does one describe patterns? An obvious answer: by permutations that they are. A drawback: such description is too detailed and complicated. To have more information may not always be better because then the structure of the set of all patterns exhibited by a map is buried under piles and piles of inessential details.

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The Sharkovsky order and theorem

A different approach to patterns is to strip them of all characteristics but one: **THE PERIOD**. This may seem to be too coarse and imprecise (a lot of different patterns will be lumped into big groups), but the result may be more transparent.

And indeed, it is this idea that led to a remarkable result, the Sharkovsky Theorem (A. N. Sharkovsky, 1964). To state it let us first introduce the **Sharkovsky order** for positive integers:

$$\mathbf{3} \succ_s \mathbf{5} \succ_s \mathbf{7} \succ_s \cdots \succ_s \mathbf{2} \cdot \mathbf{3} \succ_s \mathbf{2} \cdot \mathbf{5} \succ_s \mathbf{2} \cdot \mathbf{7} \succ_s \cdots \succ_s \mathbf{4} \succ_s \mathbf{2} \succ_s \mathbf{1}.$$

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The Sharkovsky order and theorem

Denote by $\mathrm{Sh}(\mathbf{k})$ the set of all integers $\mathbf{m}$ with $\mathbf{k} \succeq_{\mathbf{S}} \mathbf{m}$ and by $\mathrm{Sh}(\mathbf{2}^\infty)$ the set $\{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, \dots\}$; denote by $\mathbf{P}(\varphi)$ the set of periods of cycles of a map φ.

Sharkovsky Theorem

If $\mathbf{g} : [\mathbf{0}, \mathbf{1}] \to [\mathbf{0}, \mathbf{1}]$ is continuous, $\mathbf{m} \succ_{\mathbf{S}} \mathbf{n}$ and $\mathbf{m} \in \mathbf{P}(\mathbf{g})$ then $\mathbf{n} \in \mathbf{P}(\mathbf{g})$ and so there exists $\mathbf{k} \in \mathbb{N} \cup \mathbf{2}^\infty$ with $\mathbf{P}(\mathbf{g}) = \mathbf{Sh}(\mathbf{k})$. Conversely, if $\mathbf{k} \in \mathbb{N} \cup \mathbf{2}^\infty$ then there exists a continuous map $\mathbf{f} : [\mathbf{0}, \mathbf{1}] \to [\mathbf{0}, \mathbf{1}]$ such that $\mathbf{P}(\mathbf{f}) = \mathbf{Sh}(\mathbf{k})$.

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The approaches inspired by the Sharkovsky Theorem

The Sharkovsky Theorem introduces a concept of **forcing relation**. It states that if $\mathbf{m} \succ_S \mathbf{n}$ then the fact that an interval map has a point of period **m** **forces** the presence of a point of period **n** among the periodic points of the map. Think of the period of a cycle as its "type" - then the Sharkovsky Theorem result shows how such types of periodic points (i.e., their periods) force each other.

What other types can we associate with interval periodic orbits? Permutations are the most precise, but provide too much detail, and even though there is forcing among them, explicit description of it seems to not lend itself to a transparent picture. On the other hand, periods are the least precise and, hence, unsatisfactory either.

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Decomposition towers: intro

We propose a "middle-of-the-road" kind of type of a periodic orbit called **decomposition towers**. They are inspired by the Spectral Decomposition for interval maps and our recent results on forcing for periods of mixing patterns.

Decomposition towers are much more precise than periods. Yet they do not involve combinatorics (unlike permutations) and are, therefore more transparent. In our view, this is why they admit an explicit description of forcing relation. First though we need definitions.

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Definition

A permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ has **non-trivial block structure** if $n = mq, m > 1, q > 1$ and **blocks** $\{1, \dots, m\}$, $\{m + 1, \dots, 2m\}, \dots, \{qm - m + 1, \dots, qm\}$ are permuted by π. The block $\{1, \dots, m\}$ then is called the **initial** block (of the corresponding block structure).

There are also two extreme trivial block structures of π, with initial blocks (a) $\{1, \dots, n\}$ and (b) $\{1\}$. For each permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ there exists a well-defined **maximal** nested string of pairwise distinct initial blocks denoted

$$\Pi_0 = \{1, \dots, n\} \supset \Pi_1 \supset \dots \supset \Pi_k = \{1\}$$

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It follows that given a bigger initial block and a smaller initial block, the union of some iterations of the smaller initial block is the bigger initial block. Hence p_i is a multiple of p_{i+1}. This justifies the next definition.

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Let π be a permutation and $\{\Pi_0 \supset \Pi_1 \supset \cdots \supset \Pi_k\}$ be the maximal string of pairwise distinct initial blocks. Then the string of integers $\{p_0/p_1, \dots, p_{k-1}/1\}$ is called the **(decomposition) tower** of π.

Evidently, each $p_i/p_{i+1} > 1$ has the meaning of the period of the $i+1$-st initial block in the i-th initial block under the power of the map that fixes the i-th initial block.

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Main results

Define the following order among natural numbers:

$$4 \gg 6 \gg 3 \gg \cdots \gg 4n \gg 4n+2 \gg 2n+1 \gg \cdots \gg 2 \gg 1$$

Definition

Let $\mathcal{N} = \{n_0, \dots, n_{k-1}, n_k\}$, $\mathcal{M} = \{m_0, \dots, m_{r-1}, m_r\}$ be two towers. Add to each of them infinite strings of 1's and denote these extensions by $\mathcal{N}'$ and $\mathcal{M}'$. Let s be the first place at which $\mathcal{N}'$ and $\mathcal{M}'$ are different. Then $\mathcal{N} \gg \mathcal{M}$ ($\mathcal{M} \gg \mathcal{N}$) if $n_s \gg m_s$ ($m_s \gg n_s$).

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Main Theorem

If $\mathcal{N} \gg \mathcal{M}$ and a continuous interval map has a cycle with tower $\mathcal{N}$ then it has a cycle with tower $\mathcal{M}$. Moreover, suppose that $\mathbf{f}$ is a continuous interval map. Then there exists a sequence of integers $\mathcal{N}(\mathbf{f}) = (\mathbf{n_0}(\mathbf{f}), \mathbf{n_1}(\mathbf{f}), \dots)$ such that a tower $\mathcal{M} = (\mathbf{m_0}, \dots, \mathbf{m_k})$ is present among towers of cycles of $\mathbf{f}$ if and only if $\mathcal{N}(\mathbf{f}) \gg \mathcal{M}$.