Canonical Hyperbolic Laminations

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Student Coworkers

Brandon L. Barry, PhD Dissertation (2015)
On the Simplest Lamination of a Given Identity Return Triangle

Cameron G. Hale, MS Thesis (2020)
Unicritical Laminations and d-gons of Single Critical Moment

Brittany E. Burdette, PhD student (degree expected 2023)
Rotational and Rotation Return Polygons

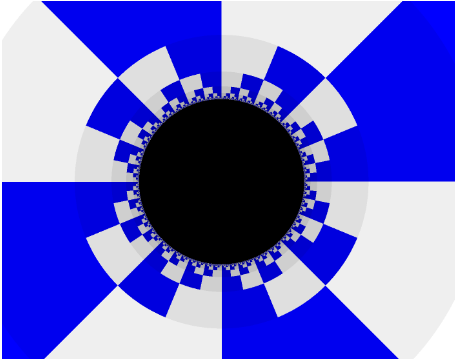
Thomas Sirna, PhD student (degree expected 2024)
Identity Return Polygons

Md. Abdul Aziz, MS Thesis student (degree expected 2023)
Fixed Point Portraits

Outline

The Simplest Julia Set – the Unit Circle $\partial\mathbb{D}$

$P(z) = z^2$ $\qquad re^{2\pi i\, t} \mapsto r^2 e^{2\pi i\, 2t}$



The complement $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ of the closed unit disk is the basin of attraction, B_∞, of infinity.

Complex Polynomials

- Polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree $d \geq 2$:

$$P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

- Compactify $\mathbb{C}$ to $\mathbb{C}_\infty$
- For P, ∞ is attracting fixed point:
  for $z \in \mathbb{C}$ with $|z|$ sufficiently large, $\lim_{n\to\infty} P^n(z) = \infty$.
- Basin of attraction of ∞:

$$B_\infty := \{z \in \mathbb{C} \mid \lim_{n\to\infty} P^n(z) = \infty\}$$

- B_∞ is an open set.

Julia and Fatou Sets

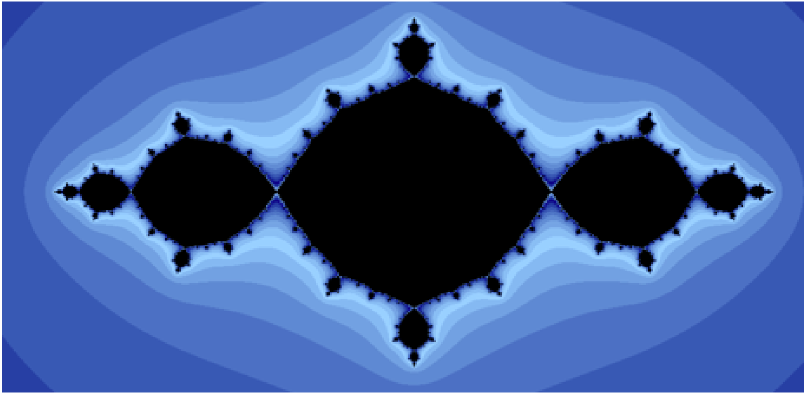
Definitions:

- Julia set $J(P) :=$ boundary of B_∞.
- Fatou set $F(P) := \mathbb{C}_\infty \setminus J(P)$.
- Filled Julia set $K(P) := \mathbb{C}_\infty \setminus B_\infty$.

Facts:

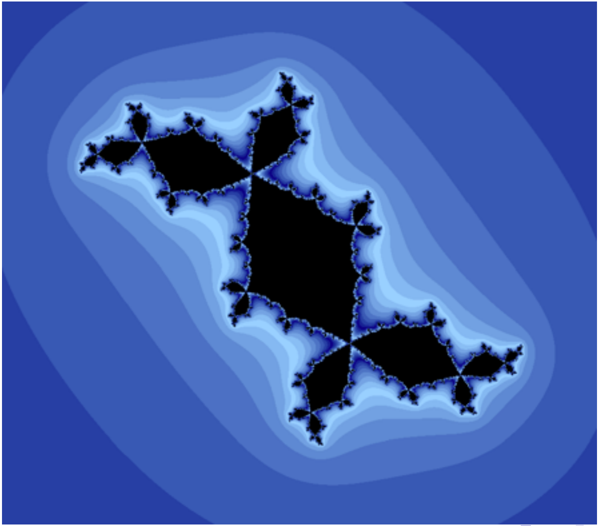
- $J(P)$ is nonempty, compact, and perfect.
- $K(P)$ is full (does not separate $\mathbb{C}$).
- $J(P)$ may separate $\mathbb{C}$.
- Finite attracting orbits are in Fatou set.
- Repelling orbits are in Julia set.
- $J(P)$ is fully invariant under P: $P^{-1}(J(P)) = J(P)$.

Basillica $z \mapsto z^2 - 1$

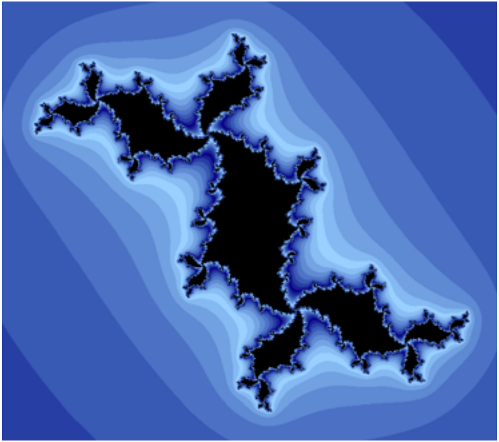


Julia set pictures by Fractalstream

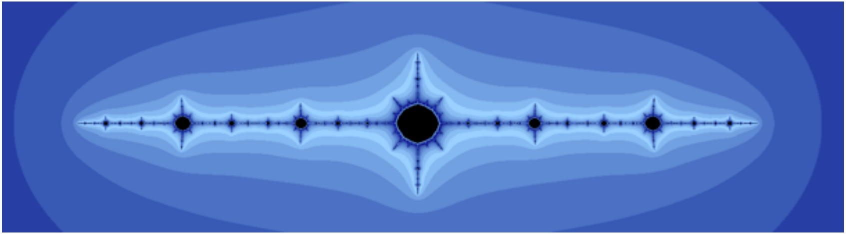
Douady Rabbit $z \mapsto z^2 - 0.12 + 0.78i$



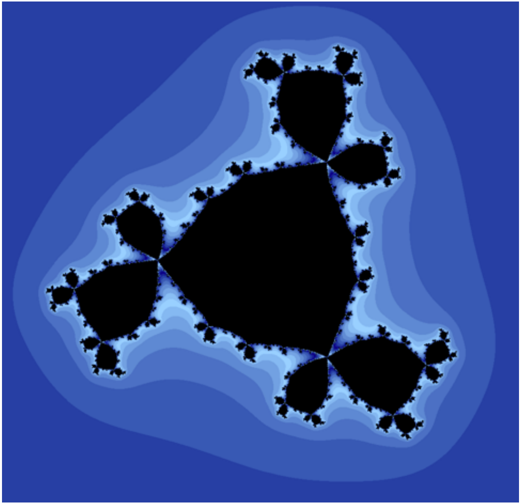
Twisted Rabbit $z \mapsto z^2 + 0.057 + 0.713i$



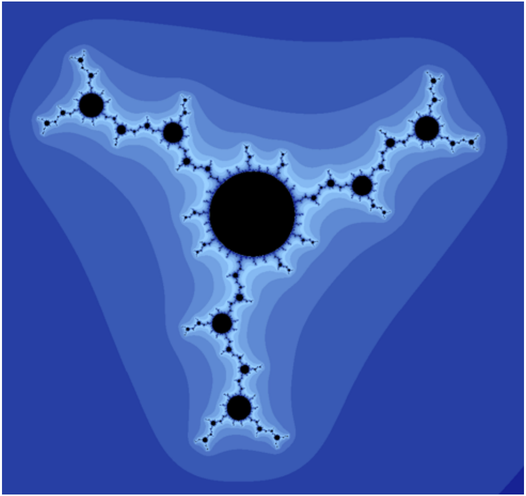
Airplane $z \mapsto z^2 - 1.75$



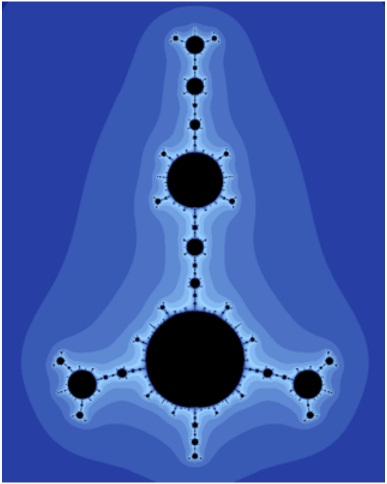
Cubic Rabbit $z \mapsto z^3 + (0.545 + 0.539i)$



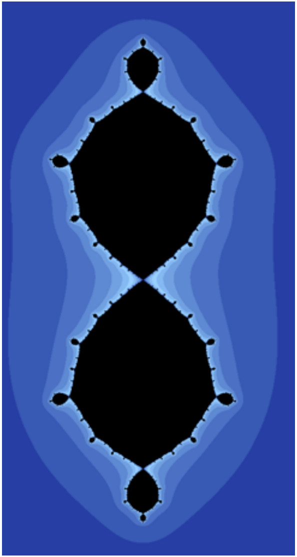
Helicopter $z \mapsto z^3 + (-0.2634 - 1.2594i)$



CN Tower $z \mapsto z^3 - 3cz^2 + 2c$, $c = 0.60126i$



Cubic Bug $z \mapsto z^3 + (\frac{\sqrt{2}}{2} i) z^2$

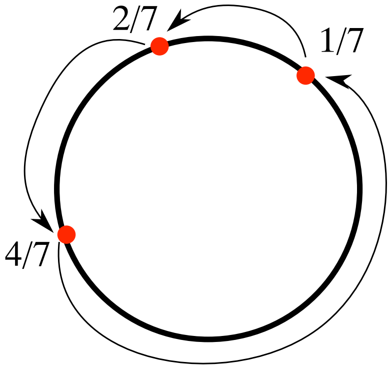


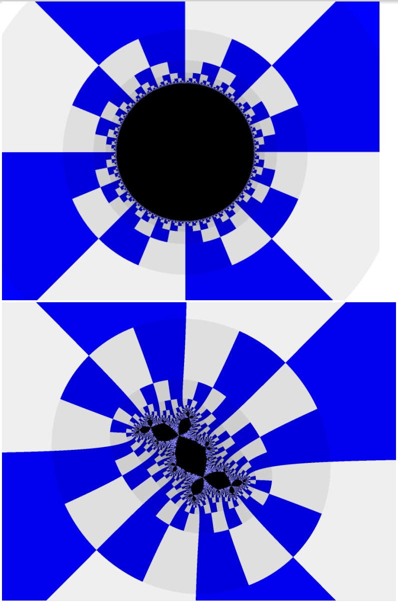
Fat Ant $z \mapsto z^3/3 + cz^2/2$, $c = 1.73506 + 0.899551i$

Pinching – Henry Moore – Art Gallery of Ontario

Dynamics on the Circle

- Consider $P(z) = z^d$ on the unit circle $\partial\mathbb{D}$.
- $z = re^{2\pi t} \mapsto r^d e^{2\pi i(dt)}$.
- Angle $2\pi t \mapsto 2\pi(dt)$.
- "Forget" 2π: then $t \mapsto dt \pmod 1$ on $\partial\mathbb{D}$.
- We measure angles in revolutions.
- Points on $\partial\mathbb{D}$ are coordinatized by $[0, 1)$.
- For a real number s, $s \pmod 1$ is the *fractional part* of s.

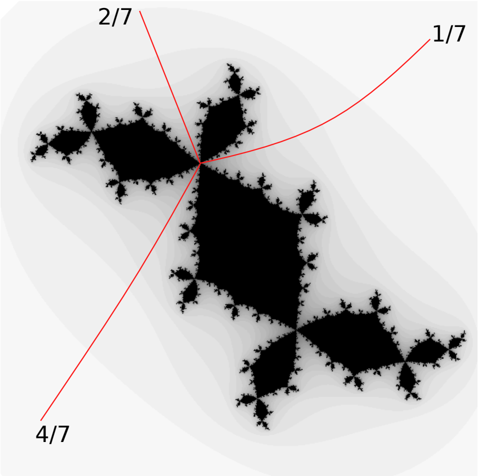




Bottcher Theorem

$$\begin{array}{ccc} \mathbb{D}_\infty & \xrightarrow{z \mapsto z^d} & \mathbb{D}_\infty \\ \downarrow \phi & & \downarrow \phi \\ B_\infty & \xrightarrow[P]{} & B_\infty \end{array}$$

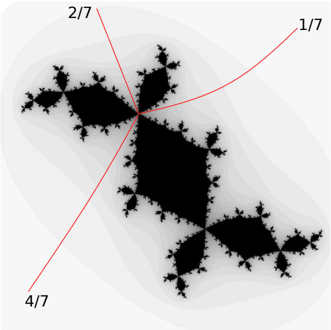
External Rays $P(z) = z^2 + (-0.12 + 0.78i)$



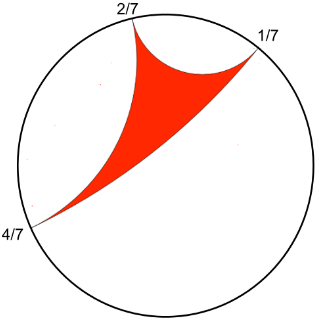
External Rays to Lamination

- Laminations were introduced by William Thurston as a way of encoding connected polynomial Julia sets.

Coincident external rays

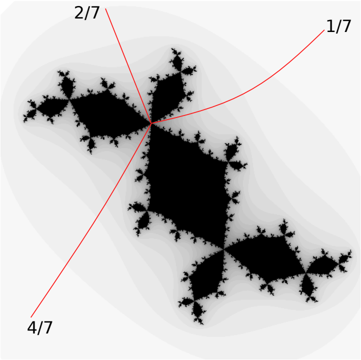


Rabbit triangle



The Rabbit Lamination

The rabbit Julia set

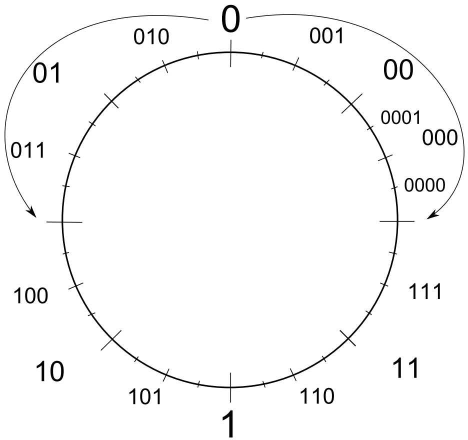


The rabbit lamination

We usually call the chords comprising a lamination the *leaves* of the lamination.

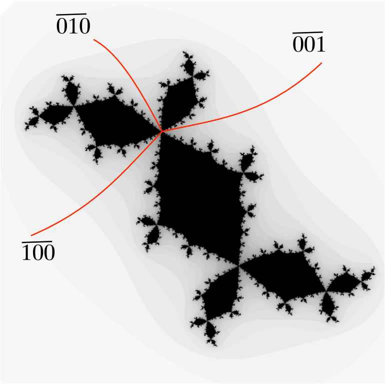
σ_2 Binary Coordinates

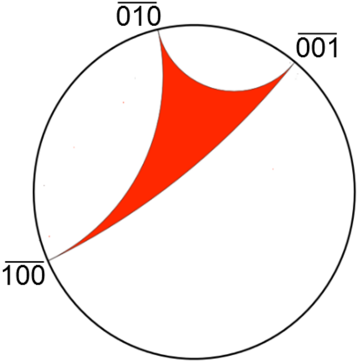
Location dynamically defined.



σ_2

Binary Coordinates and Rabbit



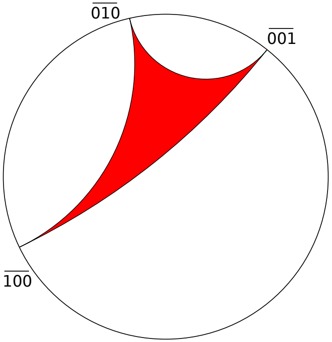


In binary coordinates, σ_2 is the “forgetful” shift.
The overline means the coordinates repeat.

Generating a Lamination from Finite Data

Definition (Pullback Scheme)

A *pullback scheme* for σ_d is a collection of d branches $\tau_1, \tau_2, \ldots, \tau_d$ of the inverse of σ_d whose ranges partition $\partial\mathbb{D}$.

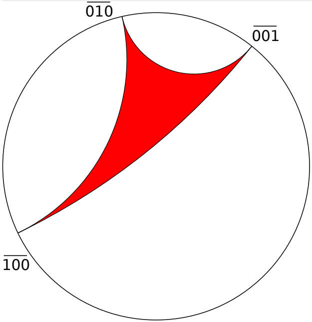


Data: Forward invariant lamination.

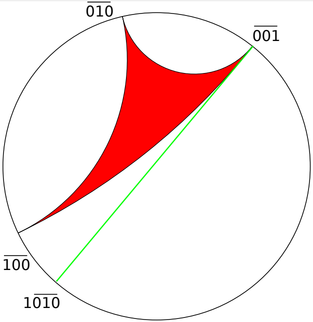
Pullback Scheme

Definition (Guiding Critical Chords)

The generating data of a pullback scheme are a *forward invariant periodic collection of leaves* and a collection of d interior disjoint *guiding critical chords*.

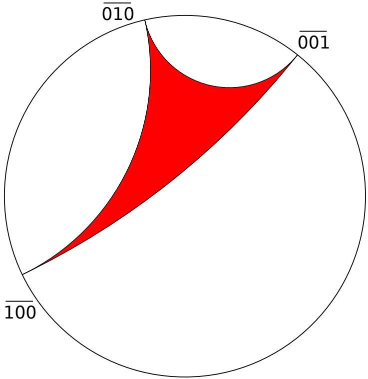


Data: Forward invariant lamination.



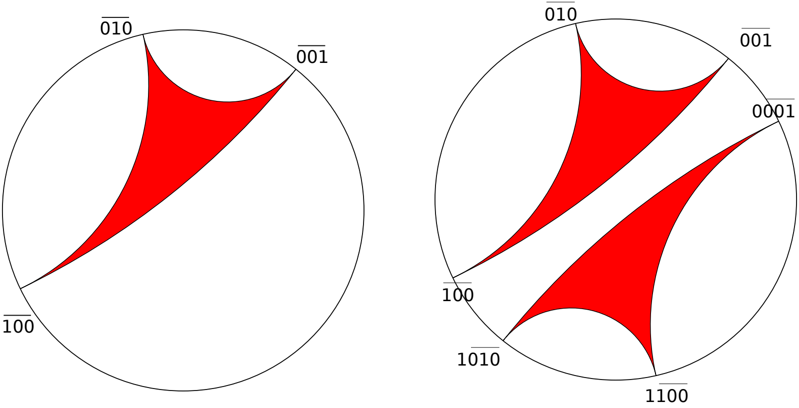
Guiding critical chord(s).

Pullback Scheme

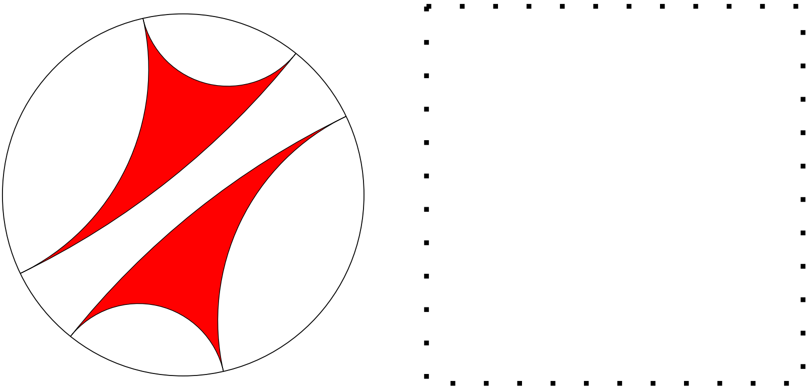


What maps to $\overline{010}$ under σ_2?

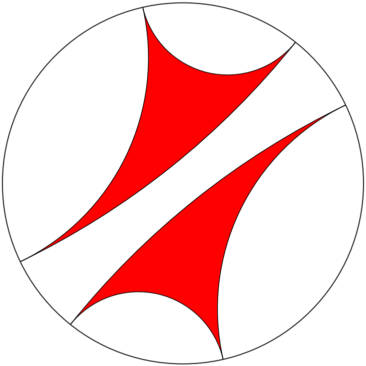
Pullback Scheme

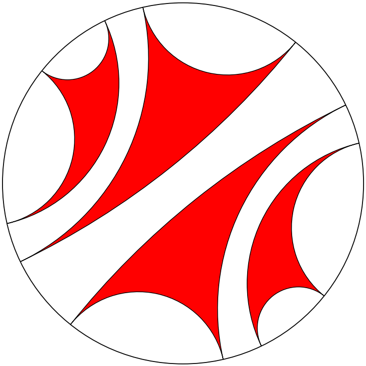


Pullback Scheme

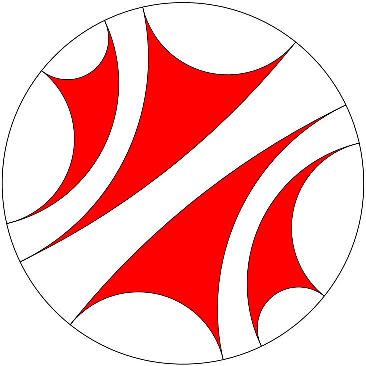


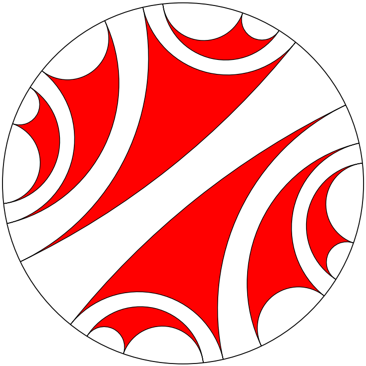
Pullback Scheme



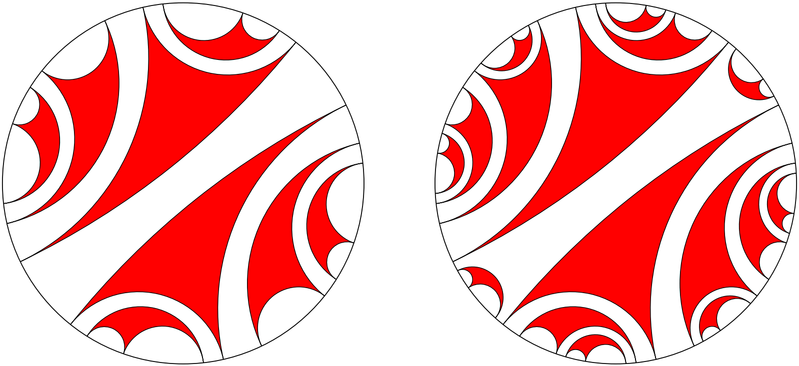


Pullback Scheme

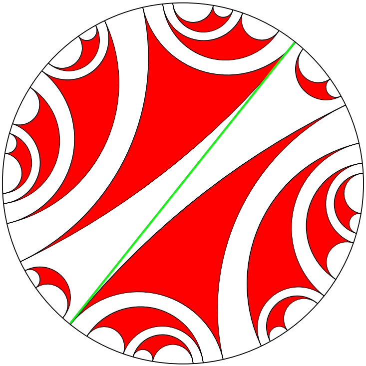




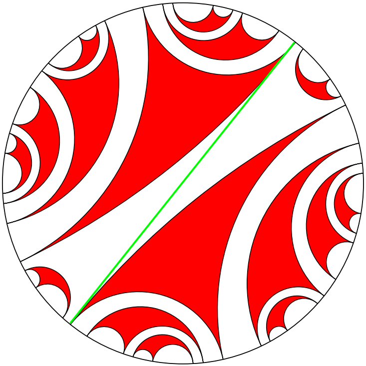
Pullback Scheme

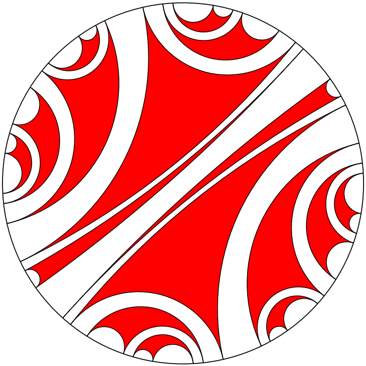


Importance of Guiding Critical Chord



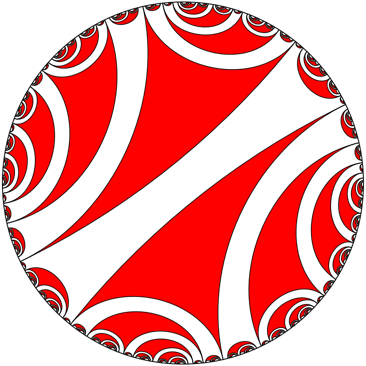
Ambiguity



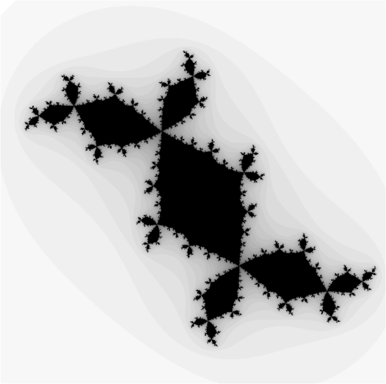


Quadratic Lamination and Julia Set

Rabbit Lamination



Rabbit Julia Set



Quotient space in plane $\Longrightarrow$ homeomorphic to rabbit Julia set.

Semiconjugate dynamics

Laminations of the Unit Disk

Definition

- A *lamination* $\mathcal{L}$ is a collections of chords of $\overline{\mathbb{D}}$, which we call *leaves*, with the property that any two leaves meet, if at all, in a point of $\partial\mathbb{D}$, and
- such that $\mathcal{L}$ has the property that

$$\mathcal{L}^* := \partial\mathbb{D} \cup \{\cup\mathcal{L}\}$$

  is a closed subset of $\overline{\mathbb{D}}$.
- We allow *degenerate* leaves – points of $\partial\mathbb{D}$.

Note that $\mathcal{L}^*$ is a continuum: compact, connected metric space.

Gaps in a Lamination

Definition

A *gap* in a lamination $\mathcal{L}$ is the closure of a component of $\mathbb{D} \setminus \cup\mathcal{L}$. A gap is *critical* iff two points in its boundary map to the same point. A gap with finitely many leaves in its boundary is called a *polygon*. A gap whose intersection with the circle contains a Cantor set is called a *Fatou* gap.

- The rabbit Julia sets has two kinds of gaps:
  - triangles, of which one is invariant and rotational, and
  - Fatou gaps;
  - the central Fatou gap is a period 3 critical gap.

Critical Portrait

Definition

A chord under σ_d is called *critical* if both its endpoints (and so the whole chord) maps to a single point on the circle.

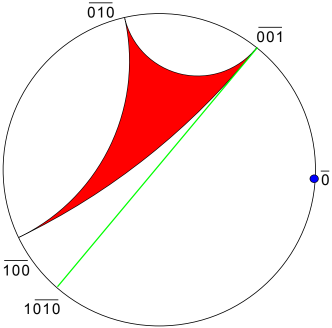
Definition

A maximal collection of critical chords for σ_d, meeting at most at endpoints, is called a *critical portrait*.

- A polygon of chords, consisting entirely of critical chords, is called an *all-critical* polygon.

Critical Portrait for Rabbit Lamination

The critical chord and one endpoint determine the lamination.



- The rabbit triangle's vertices are the only periodic orbit that stays in the left half.
- The fixed point $\overline{0}$ is the only periodic orbit that stays in the right half.

Canonical Hyperbolic Lamination

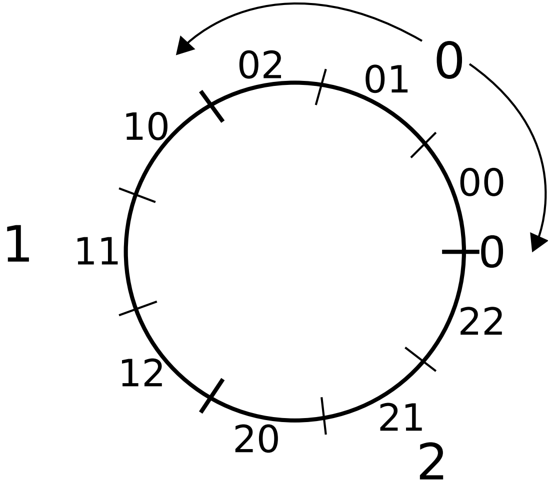
Definition

Given lamination data for σ_d consisting of a collection of periodic polygons and guiding critical portrait, we call a pull-back lamination whose Fatou gaps
(1) are bordered by sides of the given polygons, and
(2) contain the guiding critical portrait,
a *canonical hyperbolic lamination* for the given data.

- There is no claim that a canonical hyperbolic lamination is unique, though that would be a desirable consequence of a good definition.
- **Theorem.** For σ_3, there is always a canonical hyperbolic lamination. (See Brandon Barry's dissertation.)

σ_3 and Ternary Coordinates

Ternary coordinates correspond to shift σ_3.



Sibling Portrait

Definition (Full Sibling Collection)

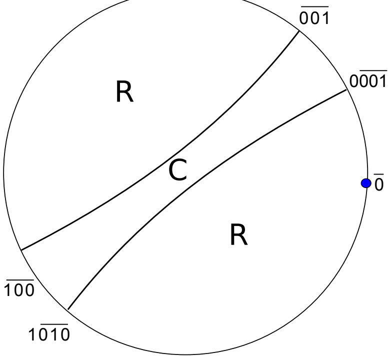
Let ℓ_1 be a chord with $\sigma_d(\ell_1) = \ell' \in \mathcal{L}$, ℓ' a non-degenerate chord. A collection $\{\ell_1, \ell_2, \dots, \ell_d\} \subset \mathcal{L}$ of pairwise disjoint chords such that $\sigma_d(\ell_i) = \ell'$ for all i is called a *full sibling collection*.

Definition (Sibling Portrait)

A collection of compatible (i.e., non-crossing) full sibling collections, each mapping to a single leaf, is called a *sibling portrait*.

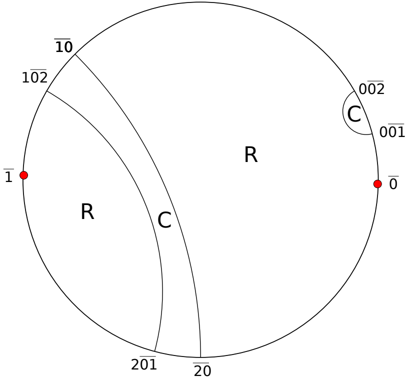
Sibling Portraits

Sibling portrait for σ_2



Rabbit

Sibling portrait for σ_3



Fat Ant

Sibling Invariant Lamination

Definition (Sibling Invariant)

A lamination $\mathcal{L}$ is said to be *sibling d-invariant* provided that:

1. (Forward Invariant) For every $\ell \in \mathcal{L}$, $\sigma_d(\ell) \in \mathcal{L}$.
2. (Backward Invariant) For every non-degenerate $\ell' \in \mathcal{L}$, there is a leaf $\ell \in \mathcal{L}$ such that $\sigma_d(\ell) = \ell'$.
3. (Sibling Invariant) For every $\ell_1 \in \mathcal{L}$ with $\sigma_d(\ell_1) = \ell'$, a non-degenerate leaf, there is a full sibling collection $\{\ell_1, \ell_2, \ldots, \ell_d\} \subset \mathcal{L}$ such that $\sigma_d(\ell_i) = \ell'$.

- A leaf maps as its endpoints map.
- Laminations corresponding to actual Julia sets satisfy an additional necessary condition: “No infnite polygons.”

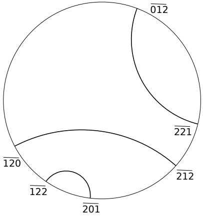
Sibling Invariant Lamination

Coming Attractions

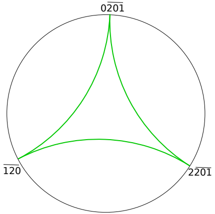
- Unicritical Polygons: Rotational, Rotation Return, and Identity Return – Brittany Burdette
- Bitransitive Identity Return Laminations – Thomas Sirna
- Fixed Point Portrait Laminations – Md. Abdul Aziz

Unicritical Example for σ_3: Identity Return: Helicopter

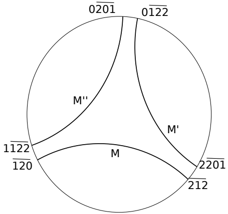
Forward Invariant Set

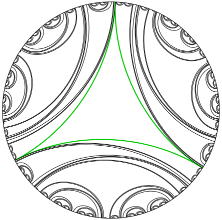


Critical Portrait

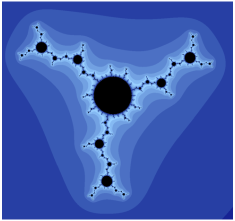


Sibling Portrait





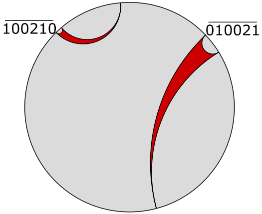
(Pullback) Lamination



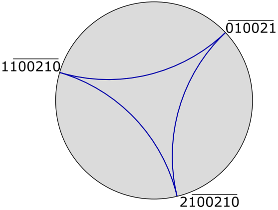
Julia Set (Helicopter)

Unicritical Example for σ_3: Rot. Ret.: Travelling Rabbit

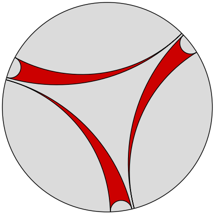
Forward Invariant Set

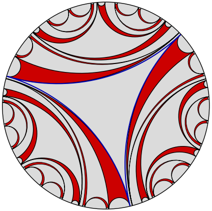


Critical Portrait

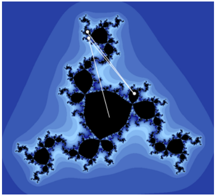


Sibling Portrait



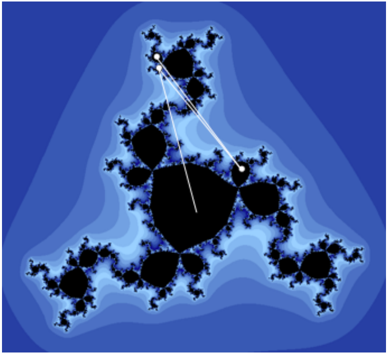


(Pullback) Lamination



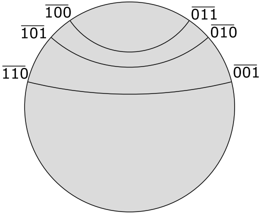
Julia Set (Travelling Rabbit)

Fancy Ninja Throwing Star?

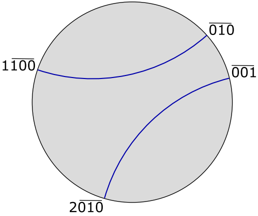


Bitransitive Example for σ_3: Identity Return Leaf

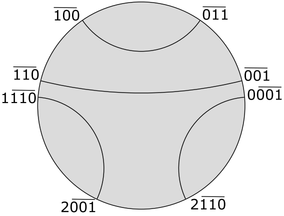
Forward Invariant Set

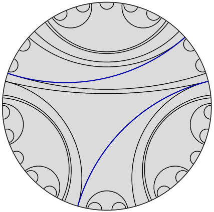


Critical Portrait

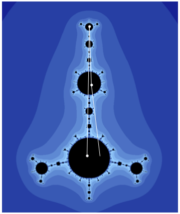


Sibling Portrait





(Pullback) Lamination



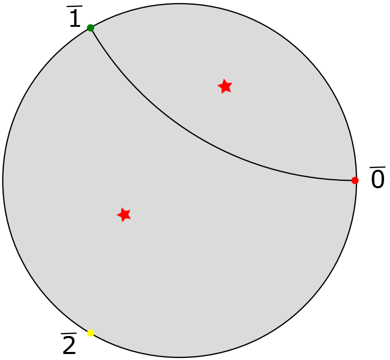
Julia Set (Identity Return Leaf)

Example: A Fixed Point Portrait for σ_4

- In order to correspond to a degree *d* Julia set, a lamination must have *d* fixed objects, counted with multiplicity.

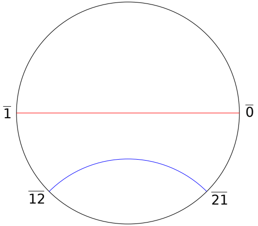
Definition (Fixed Point Portrait)

A (possibly empty) collection of leaves joining the fixed points of σ_d together with sufficient invariant gaps constitutes the *fixed point portrait* for a *d*-lamination.

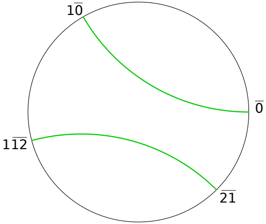


Bicritical Example for σ_3: Fixed Point Portrait: Scepter

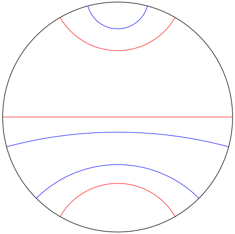
Fixed Point Portrait

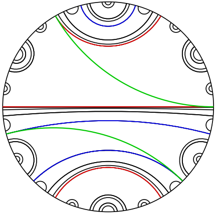


Critical Portrait

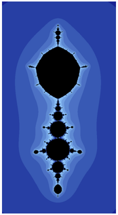


Sibling Portrait





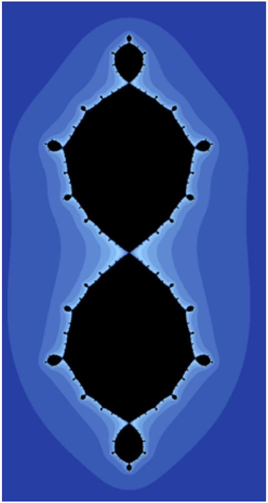
Lamination

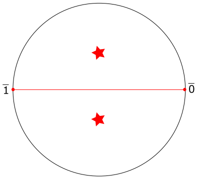


Julia Set (Scepter)

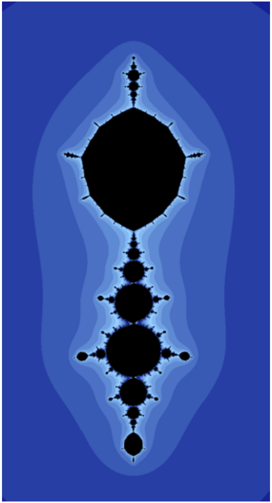
"Same" Fixed Point Portrait

Cubic Bug



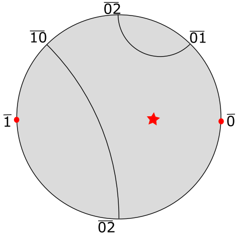


Scepter

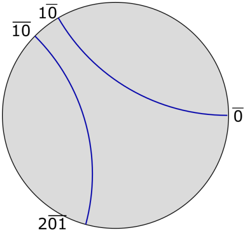


Bicritical Example for σ_3: Fixed Point Portrait – Fat Ant

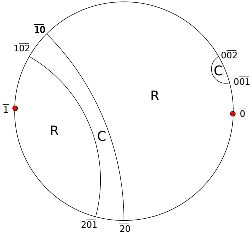
Forward Invariant Set and FPP

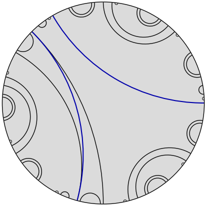


Critical Portrait



Sibling Portrait





Lamination

Julia Set (Fat Ant)

The End

Thanks!

Questions?