The quasitopological fundamental group and the first shape map

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Introduction

Joint with Paul Fabel.

- J. Brazas, P. Fabel, *Thick Spanier groups and the first shape map*, To appear in Rocky Mountain J. Math.
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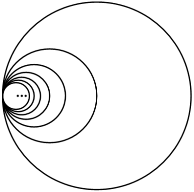
- Natural topologies on homotopical invariants provide (wild) geometric models for objects in topological algebra.

The Hawaiian earring $\mathbb{H}$



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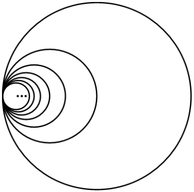

The homomorphisms $\pi_1(\mathbb{H},0) \to \pi_1\left(\bigvee_{i=1}^{n} S^1, 0\right) = F(x_1, ..., x_n)$ induce a canonical homomorphism

$$\Psi : \pi_1(\mathbb{H},0) \to \varprojlim_{n} F(x_1, ..., x_n)$$

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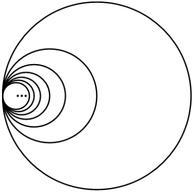

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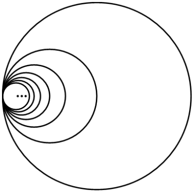

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Theorem (Griffiths, Morgan, Morrison): $\ker\Psi = 1$ so Ψ is injective. An element in $\pi_1(\mathbb{H},0) = Im(\Psi)$ is a sequence $(w_1, w_2, ...)$ where $w_n \in F(x_1,...,x_n)$ and for every fixed generator x_i the number of times x_i appears in w_n is eventually constant.

The Čech expansion

Choose a finite open cover $\mathscr{U}_n$ of X consisting of path connected open balls U with $diam(U) < \frac{1}{n}$ such that $\mathscr{U}_{n+1} \succcurlyeq \mathscr{U}_n$ (refinement).

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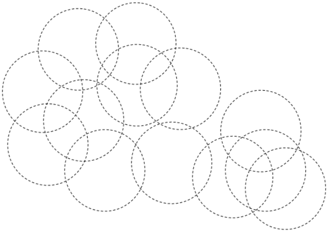

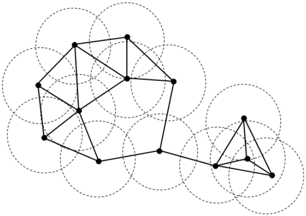

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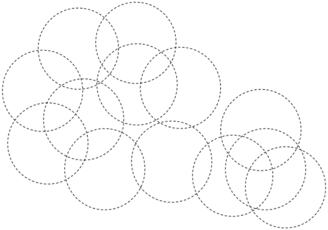

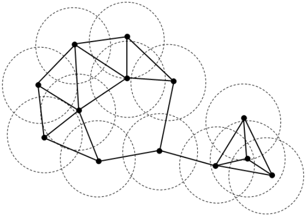

Refinement gives an inverse sequence of polyhedra

$$\cdots \longrightarrow X_{n+1} \xrightarrow[p_{n+1,n}]{} X_n \xrightarrow[p_{n,n-1}]{} \cdots \longrightarrow X_2 \xrightarrow[p_{2,1}]{} X_1$$

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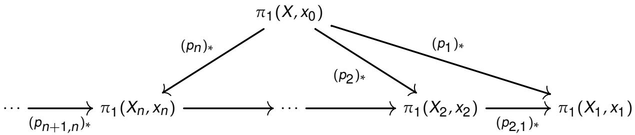

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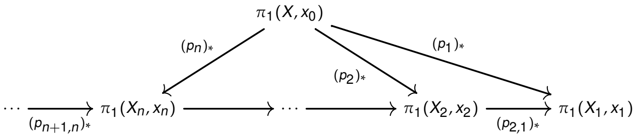

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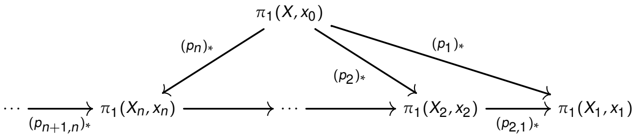

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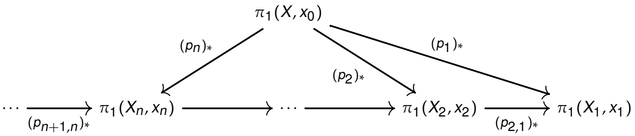

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The quasitopological fundamental group

The quasitopological fundamental group $\pi_1^{qtop}(X, x_0)$ is the usual fundamental group endowed with the quotient topology w.r.t. $\Omega(X, x_0) \to \pi_1(X, x_0)$, $\alpha \to [\alpha]$.

- Discrete iff X admits a universal covering (Fabel)
- $\pi_1^{qtop}(X, x_0)$ can fail to be a topological group, e.g. $\mathbb{H}$ (Fabel).
- $\pi_1^{qtop}(X, x_0)$ is a quasitopological group.
- A necessary intermediate for a group topology on $\pi_1(X, x_0)$ which has application to the general theory of topological groups, e.g. Every open subgroup of a free topological group is free topological (B).

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Topologizing π_1

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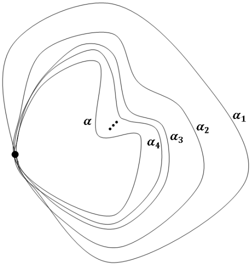
Guiding principle: If $\alpha_n \to \alpha$ in $\Omega(X, x_0)$, then $[\alpha_n] \to [\alpha]$ in $\pi_1^{qtop}(X, x_0)$.



Open subgroups and invariant separation

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Definition: A quasitopological group G is invariantly separated if whenever $g \neq h$, there is an open normal subgroup $N \subset G$ such that $gN \neq hN$.

Remark: G is invariantly separated $\Leftrightarrow \bigcap_{N \trianglelefteq G \text{ open}} N = 1.$

invariantly separated $\Rightarrow$ totally separated $\Rightarrow$ Hausdorff

Comparing the approaches

1. Shape theory $\Psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$,
2. Topological separation in $\pi_1^{qtop}(X, x_0)$.

Question: How much of $\pi_1(X, x_0)$ does each method retain (or forget)?

Comparing the approaches

1. Shape theory, $\Psi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$,
2. Classical covering maps $p : Y \to X$,
3. Topological separation in $\pi_1^{qtop}(X, x_0)$.

Question: How much of $\pi_1(X, x_0)$ does each method retain (or forget)?

Spanier groups

Definition:

The Spanier group of X with respect to $\mathscr{U}_n$ is the normal subgroup

$$\pi^{sp}(\mathscr{U}_n, x_0) = \langle [\alpha \cdot \gamma \cdot \alpha^{-}] | Im(\gamma) \subset U, U \in \mathscr{U}_n \rangle.$$

Remark: $\pi^{sp}(\mathscr{U}_{n+1}, x_0) \subset \pi^{sp}(\mathscr{U}_n, x_0)$, $n \geq 1$

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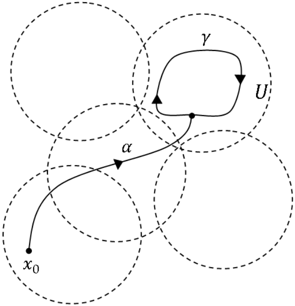

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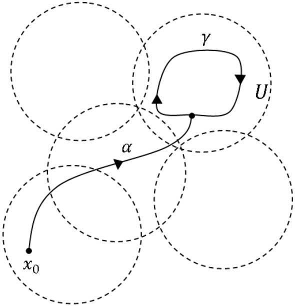

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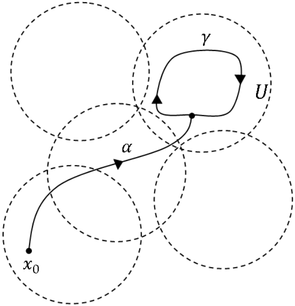

Spanier groups

Utility: Spanier groups provide a way to determine when (classical) covering maps exist.

Theorem (Spanier): Given $H \leq \pi_1(X, x_0)$,

$$\begin{array}{c}\text{there is a covering map}\\ p: Y \to X,\ p(y_0) = x_0\\ \text{such that } p_*(\pi_1(Y, y_0)) = H\end{array} \iff \pi^{sp}(\mathscr{U}_n, x_0) \subseteq H \text{ for some } n \geq 1$$

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Corollary: $\pi^{sp}(X, x_0)$ consists precisely of the homotopy classes $[\alpha] \in \pi_1(X, x_0)$ for which α lifts to a loop for every covering $p : (Y, y_0) \to (X, x_0)$, i.e.

$$\pi^{sp}(X, x_0) = \bigcap_{n \geq 1} \pi^{sp}(\mathscr{U}_n, x_0) = \bigcap_{p:(Y,y_0)\to(X,x_0)\text{ covering}} p_*(\pi_1(Y, y_0))$$

Thick Spanier groups

Definition: The thick Spanier group of X with respect to $\mathscr{U}_n$ is the normal subgroup

$$\Pi^{sp}(\mathscr{U}_n, x_0) = \langle [\alpha \cdot \gamma_1 \cdot \gamma_2 \cdot \alpha^-] | Im(\gamma_i) \subset U_i, U_i \in \mathscr{U}_n, i = 1,2 \rangle.$$

Note $\pi^{sp}(\mathscr{U}_n, x_0) \subseteq \Pi^{sp}(\mathscr{U}_n, x_0)$

$\Pi^{sp}(\mathscr{U}_m, x_0) \subseteq \pi^{sp}(\mathscr{U}_n, x_0)$ for large enough $m = m(n) \geq n$ by paracompactness

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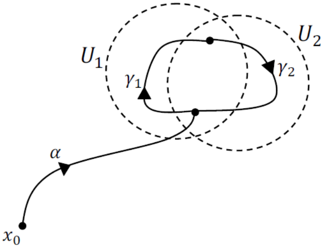

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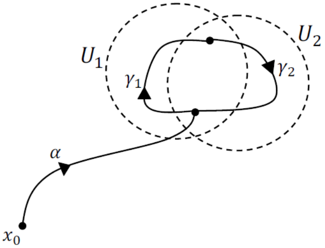

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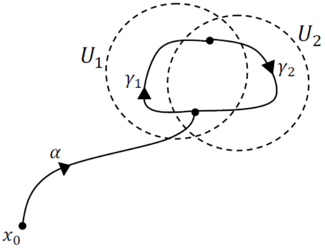

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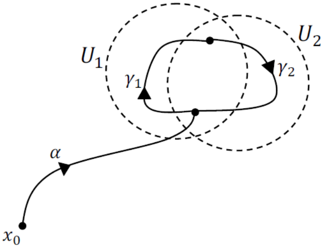

Thick Spanier groups

Theorem (B, Fabel): There is a level short exact sequence

$$1 \longrightarrow \Pi^{sp}(\mathscr{U}_n, x_0) \longrightarrow \pi_1(X, x_0) \xrightarrow{(p_n)_*} \pi_1(X_n, x_n) \longrightarrow 1$$

Applying $\varprojlim_n$ we obtain

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Theorem (B, Fabel): There is a level short exact sequence

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Corollary: If X is a Peano continuum, then X is π_1-shape injective $\Leftrightarrow$ $\pi_1^{qtop}(X, x_0)$ is invariantly separated.

Conclusion

The data of the fundamental group of a Peano continuum X retain by each of

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1. and 2. are exhausted but the topology of $\pi_1^{qtop}(X, x_0)$ is rarely generated by open normal subgroups.

Other data retained by $\pi_1^{qtop}(X, x_0)$

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Separation properties

$\pi_1^{qtop}(X, x_0)$	Interpretation
Invariantly separated	π_1-shape injective
Totally separated	$\Omega(X, x_0)$ is π_0-shape injective $\Psi_0 : \pi_1^{qtop}(X, x_0) = \pi_0(\Omega(X, x_0)) \to \check{\pi}_0(\Omega(X, x_0))$ is injective
0-dimensional	Ψ_0 is an embedding
T_3 (T_4)	?
T_2	?
T_0 (T_1)	Homotopically path-Hausdorff

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Example in cylindrical coordinates

The topology of $\pi_1^{qtop}(X, x_0)$ can topologically distinguish homotopy classes which are indistinguishable using shape/coverings.
Example (Conner, Meilstrup, Repovš, Zastrow, Željko):

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5. $\mathbb{S} = C \cup S \cup \bigcup_{d \in D} A_d$ is a Peano continuum such that $\ker \Psi \neq 1$ but $\pi_1^{qtop}(X, x_0)$ is T_1 (Fischer, Repovš, Virk, Zastrow)&(B, Fabel)

Open problems

Problem 1: If X is a Peano continuum and $\pi_1^{qtop}(X, x_0)$ is T_2, must $\pi_1^{qtop}(X, x_0)$ be invariantly separated (i.e. X π_1-shape injective)?

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Thank you!