Small Perturbations of z^n

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Summer Conference on Topology and its Applications
Nipissing University
July 24, 2013

Joint work with Bob Devaney

Basic Notions: Complex Dynamics

In Complex Dynamics, we consider the behavior of points under iteration of a holomorphic function.

In this setting, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ will be a rational map.

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Definitions

- For a point $z \in \hat{\mathbb{C}}$, the sequence $(z, f(z), f^2(z), \ldots)$ is called the *orbit* of z under f.
- If $z = f^n(z)$ for some n, with n minimal, then we say z is *periodic*, with *period* n.
- In this case, the complex number $\lambda = (f^n)'(z)$ is called the *multiplier* of z.

Julia and Fatou Sets

For a periodic point z, we say z is:

- *attracting* if $|\lambda| < 1$
- *superattracting* if $\lambda = 0$
- *repelling* if $|\lambda| > 1$
- *indifferent* if $|\lambda| = 1$

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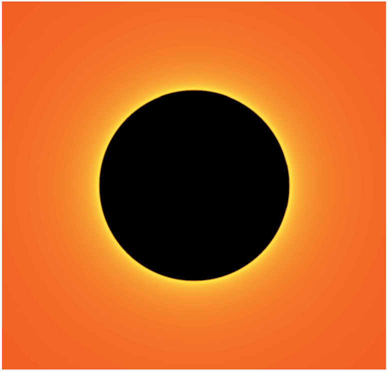
This gives a natural partition of the Riemann sphere:

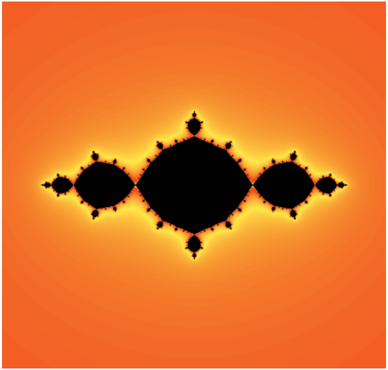
- The *Julia set*, $J(f)$, is the closure of the set of repelling periodic points.
  - Dynamics of f on the Julia set are "chaotic."
- The *Fatou set* is the complement of the Julia set.
  - Dynamics of f on the Fatou set are "stable."

Examples: Quadratic Polynomials



$f(z) = z^2$

"The Unit Circle"



$f(z) = z^2 - 1$

"The Basilica"

The Julia set is the boundary between the black and orange regions.

Introduction: Perturbed Polynomials

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For $\lambda = 0$ this map is the complex polynomial $z \mapsto z^n$.

When $\lambda \neq 0$ we have replaced the superattracting fixed point at the origin with a pole.

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- These map to two critical values at $\pm 2\sqrt{\lambda}$

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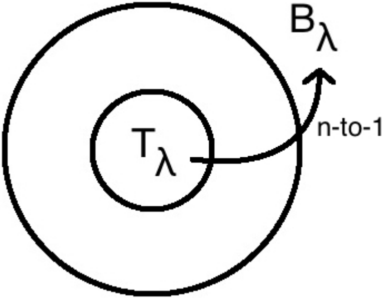

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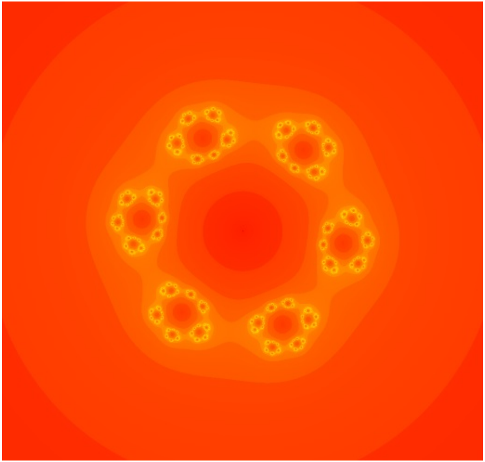
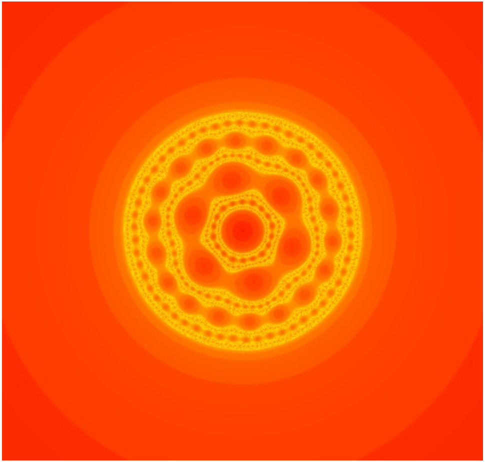
2. if v_λ lies in $T_\lambda \neq B_\lambda$, then $J(F_\lambda)$ is a Cantor set of concentric simple closed curves surrounding the origin;
3. in all other cases, $J(F_\lambda)$ is a connected set.
    - In particular, if $F_\lambda^j(v_\lambda) \in T_\lambda \neq B_\lambda$ for some $j \geq 1$, then $J(F_\lambda)$ is a *Sierpiński curve* (i.e., homeomorphic to the Sierpiński carpet). Sets of λ values where this occurs are called *Sierpiński holes*.

Example: Case 1, n=3



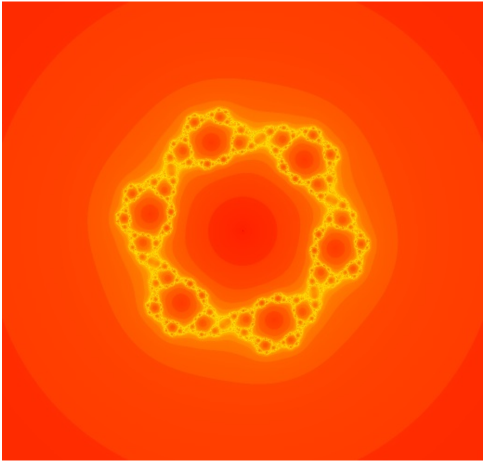
A Cantor set.

Example: Case 2, n=3



A Cantor set of simple closed curves.

Example: Case 3, n=3



A Sierpiński curve.

Small perturbations: $n \geq 3$

What happens when the parameter is very small?

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When $n \geq 3$, there exists a punctured neighborhood of $\lambda = 0$ for which Case 2 occurs, (the McMullen domain). The critical values both lie in the trap door and the Julia set is a Cantor set of simple closed curves.

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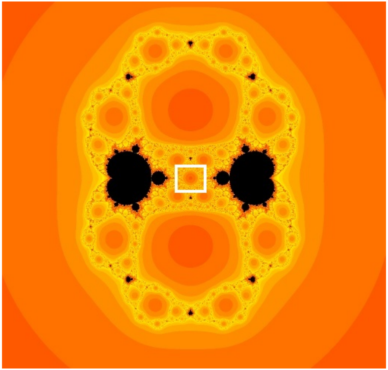

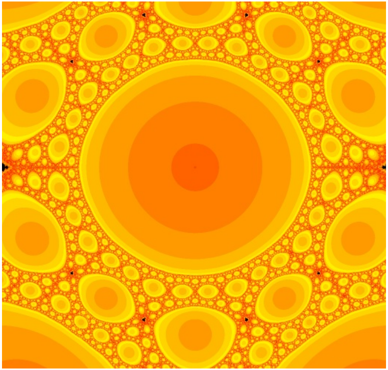

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Moreover, as a function of λ, the width of the largest annulus in the Fatou set is bounded away from zero (Devaney, Garijo; 2006).

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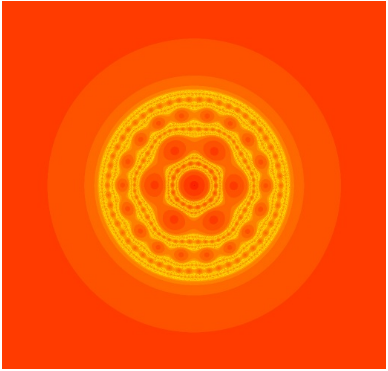
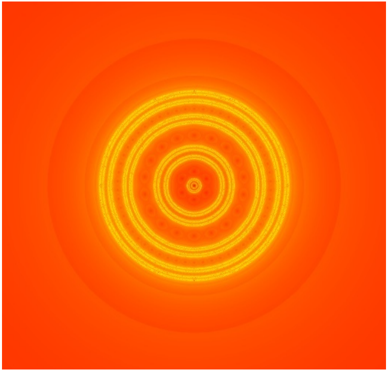


$n = 3, \lambda \approx -0.005$



$n = 3, \lambda \approx 10^{-6}$

There is always at least one "thick" annulus in the Fatou set as $\lambda \to 0$.

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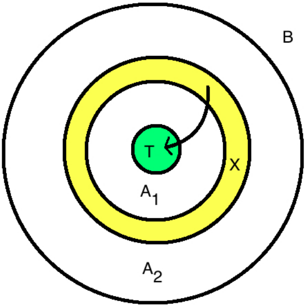

- If the critical values are to lie in the trap door, the Riemann-Hurwitz formula requires that the preimage of the trap door must be a single annulus X mapped $2n$-to-1 onto T_λ.
- The remaining annuli A_1 and A_2 are each mapped n-to-1 onto $A = A_1 \cup X \cup A_2$.

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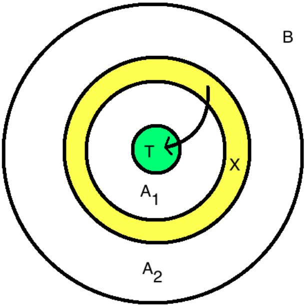

- An n-to-1 covering map expands the modulus of an annulus by a factor of n, so when $n = 2$ we have $\frac{1}{2}\mathrm{mod}\ A = \mathrm{mod}\ A_1 = \mathrm{mod}\ A_2$.
- But $\mathrm{mod}\ A = \mathrm{mod}\ A_1 + \mathrm{mod}\ X + \mathrm{mod}\ A_2$, so there is no room for X.

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Moreover, we have the following theorem:

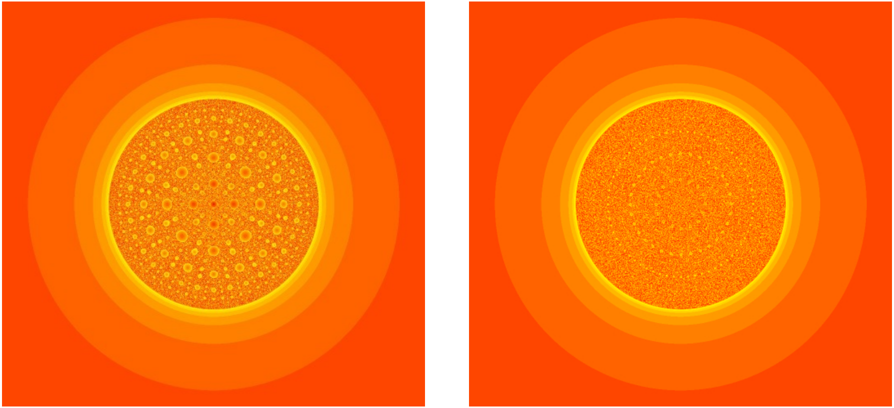
Theorem (Devaney, Garijo; 2006)

As λ approaches zero, the Julia set for the map

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2}$$

converges to the closed unit disk in the Hausdorff metric.

Small perturbations, n=2



As λ gets small, bounded components of the Fatou set shrink.

Small perturbations, n=2

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If the conclusion fails to hold, then for all ϵ sufficiently small, there exists a sequence λ_j converging to zero, and a corresponding sequence z_j in the closed disk such that $B_\epsilon(z_j)$ lies in the Fatou set of F_{λ_j} for all j.

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By compactness, there is a subsequence of the z_j converging to a point z_* in $\overline{\mathbb{D}}$, so we may assume wlog that $B_\epsilon(z_*)$ is in the Fatou set of F_{λ_j} for all j.

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Since $\lambda_j \to 0$, $F_{\lambda_j} \approx z^2$ for large j, so for large k, $F_{\lambda_j}^k$ wraps $B_\epsilon(z_*)$ around the origin, disconnecting the Julia set by forward invariance.

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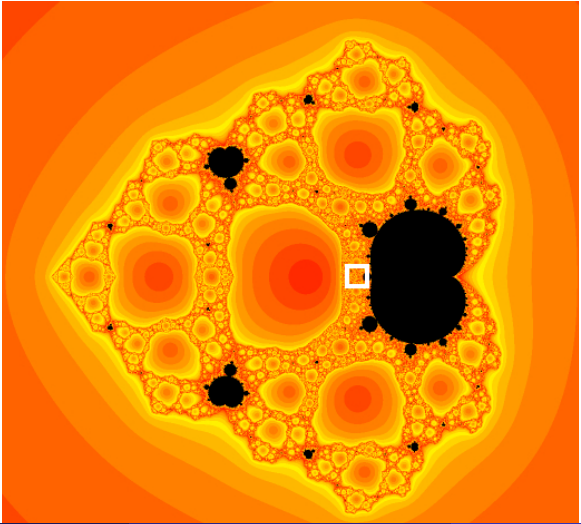

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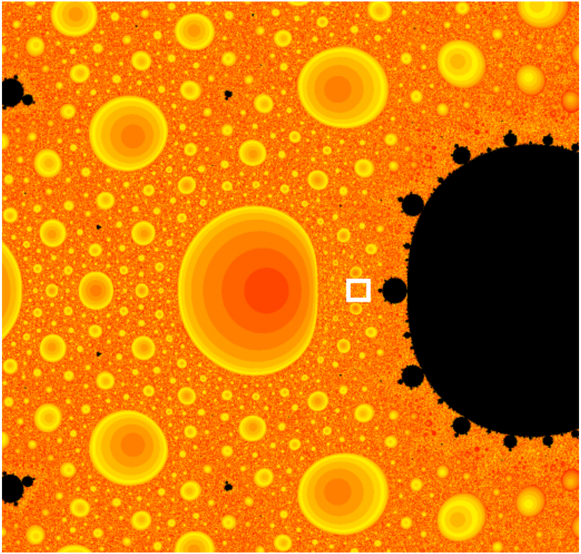

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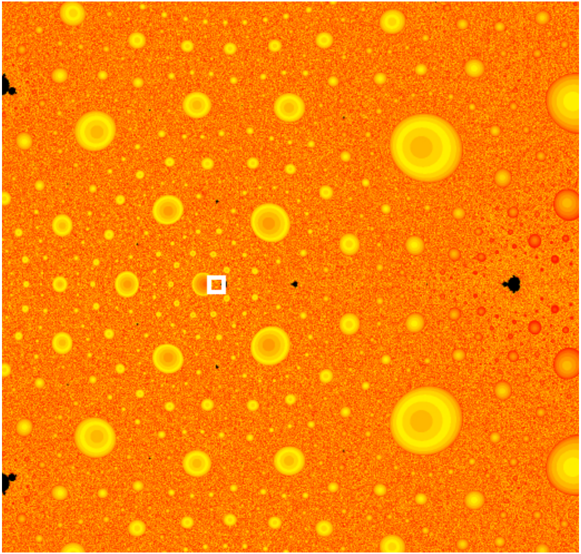

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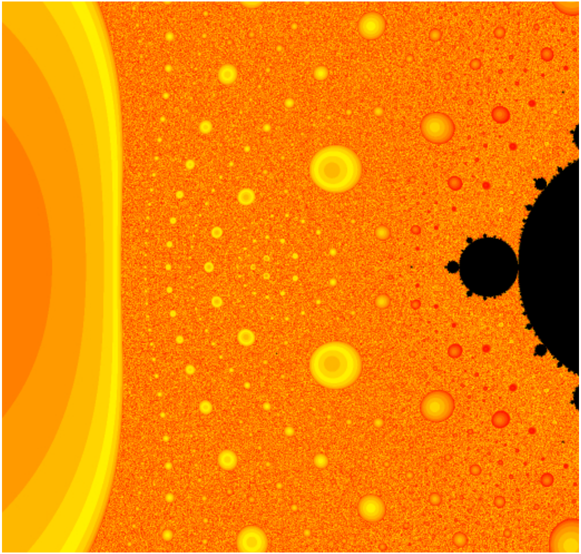

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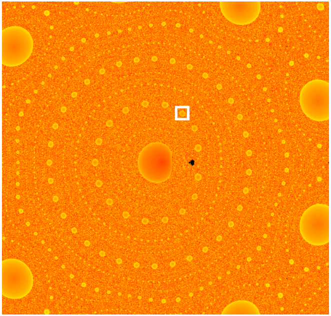
We can see many "rings" of Sierpiński holes alternating with what appear to be baby Mandelbrot sets.



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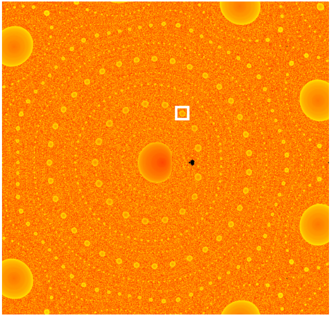

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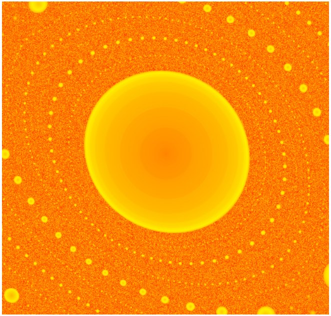

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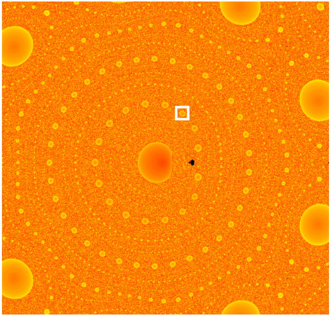

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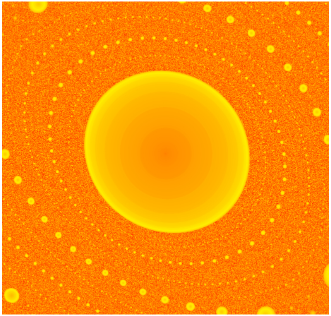

Our goal is ultimately to describe this structure completely.

Mandelpiński Necklaces

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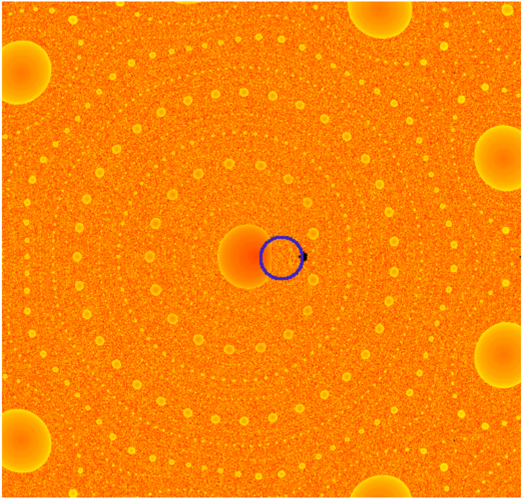
The k^{th} Mandelpiński necklace in the dynamical plane is a simple closed curve of parameters for which the critical values lie on C_{-k}.

Example: A Mandelpiński Necklace for n=2



Sub-necklaces

In the dynamical plane, the sector of points whose arguments lie between two adjacent critical points is mapped 1-1 onto the complement of the rays extending from the critical values to infinity.

This sector therefore contains a preimage of the trap door, as well as of all C_{-k} such that v_λ lies outside C_{-k}.

Thus the preimage of the trap door is surrounded by infinitely many simple closed curves that map onto these C_{-k}'s.

This recursively yields a structure of sub-rings and sub-sub-rings in the dynamical plane that appears to be replicated in parameter space.

Can we explicitly show that this structure persists in the parameter plane at each level? (Work in progress).

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- What formula gives the number of Sierpiński holes in each sub-necklace (and sub-sub-necklace, etc.)?
- Can we prove the existence of all baby Mandelbrot sets in each necklace?

Thanks

Thank you for your attention.