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1. Introduction

The n-dimensional homology group $H_n(X, \mathfrak{A})$ of a compactum X over an Abelian group $\mathfrak{A}$ is understood here always in the sense of Vietoris (see, for instance [3], p. 36). Thus the elements of a such group are homology classes (γ) in X, where γ is a true n-dimensional cycle in X with coefficients belonging to $\mathfrak{A}$. For simplicity, we shall write "*cycle*" instead of "*true cycle*" and we may assume that X is a subset of the Hilbert cube Q.

Let X' be another compactum (lying in Q) and let $\underline{f}:X \to X'$ be a fundamental sequence (concerning this notion, and also other notions of the theory of shape, see [3]). It is well known ([3], p. 70) that $\underline{f}$ induces a homomorphism

$$\underline{f}_n : H_n(X, \mathfrak{A}) \to H_n(X', \mathfrak{A})$$

assigning covariantly to every homology class $(\gamma) \in H_n(X, \mathfrak{A})$ a homology class $\underline{f}_n((\gamma)) \in H_n(X', \mathfrak{A})$.

By a *power domain* (Z, α_k) one understands (compare [3], p. 91) a system consisting of a set Z and of a family of functions $\alpha_k: Z \to Z$ assigned to indices $k = 0, \pm1, \pm2, \cdots$ and such that $\alpha_1(z) = z$ and $\alpha_k\alpha_m(z) = \alpha_{km}(z)$ for every $z \in Z$ and $k, m = 0, \pm1, \pm2, \cdots$

By a *homomorphism* of (Z, α_k) into another power domain (Z', α'_k) one understands a function $\phi: Z \to Z'$ such that

$$\phi\alpha_k(z) = \alpha'_k\phi(z) \text{ for every } z \in Z \text{ and } k = 0, \pm1, \pm2, \cdots$$

Two power domains (Z, α_k), (Z', α'_k) are said to be *isomorphic* if there exists a one-to-one homomorphism ϕ of (Z, α_k) onto (Z', α'_k). It is clear that then the function $\psi = \phi^{-1}: Z' \to Z$ is also a homomorphism. If we assume only that there exist two homomorphisms ϕ of (Z,α_k) into (Z',α'_k) and ψ of (Z',α'_k) into (Z,α_k) such that $\phi\psi$ is the identity, then we say that the power domain (Z,α_k) r-*dominates* the power domain (Z',α'_k).

2. Cancellable Cycles

Let Ω be a family of n-dimensional cycles in a compactum X over an Abelian group $\mathfrak{A}$ and let m be a non-negative integer. An m-dimensional cycle γ in X over $\mathfrak{A}$ is said to be *cancellable* rel. Ω provided there exists a fundamental sequence $\underline{f}:X \to X$ such that:

(2.1) $\underline{f}_n((\omega)) = (\omega)$ for every cycle $\omega \in \Omega$,

(2.2) $\underline{f}_m((\gamma)) = 0$.

Then we say that $\underline{f}$ realizes the cancellation of γ rel. Ω.

It is clear that the cancellability of γ rel. Ω depends only on the homology class (γ) of γ and on the collection (Ω) of the homology classes of cycles belonging to Ω. Consequently we may speak about homology classes *cancellable* rel. (Ω).

Observe that if γ is an m-dimensional cycle in X over $\mathfrak{A}$, cancellable rel. Ω, then for every integer k the cycle $k.\gamma$ is also cancellable rel. Ω. Consequently the collection Z of all m-dimensional homology classes in X over $\mathfrak{A}$, cancellable rel. (Ω), is a power domain (Z,α_k), where the function α_k is defined by the formula $\alpha_k((\gamma)) = k(\gamma)$. Let us denote this power domain by $\Gamma_m(X,\mathfrak{A},(\Omega))$. In special case when $(\Omega) = H_n(X,\mathfrak{A})$, we shall write $\Gamma_m(X,\mathfrak{A},n)$ instead of $\Gamma_m(X,\mathfrak{A},(\Omega))$. In the case when $\mathfrak{A} = \mathfrak{N}$ is the group of integers, we shall write $\Gamma_m(X,n)$ instead of $\Gamma_m(X,\mathfrak{N},n)$.

(2.3) *Problem. Is it true that for every compactum* X, *for every* $m \neq n$, *for every Abelian group* $\mathfrak{A}$, *and for every* $(\Omega) \subset H_n(X,\mathfrak{A})$ *the power domain* $\Gamma_m(X,\mathfrak{A},(\Omega))$ *is a subgroup of the group* $H_m(X,\mathfrak{A})$?

3. Examples

In order to illustrate the sense of the cancellability, let us give some simple examples:

(3.1) *Example.* Let $X = X_1 \cup X_2$, where X_1, X_2 are compacta and $X_1 \cap X_2 \in AR$. Then every m-dimensional cycle γ in X_1 over any Abelian group $\mathfrak{A}$ is cancellable in X rel. each n-dimensional cycle ω in X_2 over $\mathfrak{A}$.

In fact, since $X_1 \cap X_2 \in AR$, there is a retraction $r: X_1 \to X_1 \cap X_2$. Setting $f(x) = r(x)$ for $x \in X_1$ and $f(x) = x$ for $x \in X_2$, we get a retraction $f: X \to X_2$. It is clear that f realizes the cancellation of γ rel. the family of all n-dimensional cycles in X_2 over $\mathfrak{A}$.

(3.2) *Example.* Let $X = X_1 \cup X_2$, where X_1 and X_2 are compacta, $m_1 = \dim X_1$ and $X_1 \cap X_2 \in AR$. If $m \leq m_1 < n$, then $\Gamma_m(X, \mathfrak{A}, n) \supset H_m(X_1, \mathfrak{A})$.

In order to show this, consider an n-dimensional cycle ω in X over $\mathfrak{A}$. Then there exists a cycle $\omega' \in (\omega)$ of the form

$$\omega' = \kappa_1 - \kappa_2 ,$$

where κ_ν is an infinite n-dimensional chain in X_ν over $\mathfrak{A}$, for $\nu = 1,2$. Since $X_1 \cap X_2 \in AR$, there exists in $X_1 \cap X_2$ an infinite n-dimensional chain μ over $\mathfrak{A}$ such that $\partial\mu = \partial\kappa_1 = \partial\kappa_2$. Then $\kappa_1 - \mu$ is an infinite n-dimensional chain in X_1 over $\mathfrak{A}$. Since $\dim X_1 = m_1 < n$, there exists in X_1 an infinite (n+1)-dimensional chain over $\mathfrak{A}$ having $\kappa_1 - \mu$ as its boundary. It follows that

$$\omega' = (\kappa_1 - \mu) - (\kappa_2 - \mu) \sim \mu - \kappa_2 \text{ in } X.$$

But $\mu - \kappa_2$ lies in X_2 and we infer by Example (3.1) that each n-dimensional cycle γ over $\mathfrak{A}$ is cancellable rel. $\mu - \kappa_2$, hence also cancellable relatively ω in X.

(3.3) *Example.* Let T be the surface of a torus. We may consider T as the Cartesian product of two circles S^1, that is every point $x \in T$ may be represented in the form (x_1,x_2), where $x_1,x_2 \in S^1$.

Consider a point $a \in S^1$ and let X_1 denote the circle consisting of all points of the form (a,x_2). Let γ be a 1-dimensional cycle in X_1 over the group of integers $\mathfrak{N}$, such that the homology class (γ) is a generator of the Betti group $H_1(X_1)$. Moreover, let ω be a 2-dimensional cycle in T over $\mathfrak{N}$ such that (ω) be a generator of the cyclic infinite Betti group $H_2(T)$. Let us show that γ is not cancellable rel. ω in T.

Consider a fundamental sequence $\underline{f}:T \to T$ such that

(3.4) $\underline{f}_2((\omega)) = (\omega)$.

Since $T \in ANR$, the fundamental sequence $\underline{f}$ is generated by a map $f:T \to T$ and we infer that $f(\omega)$ is a 2-dimensional cycle homologous to ω in T. Then f maps the oriented circle X_1 onto a loop in T.

Suppose that this loop is homologous to zero in T. Since the 1-dimensional fundamental group of T is Abelian (see, for instance [8], p. 149) this loop is homotopic in T to a constant and we infer that the map f is homotopic to a map f' by which the circle X_1 passes onto one point. It follows easily that f' may be represented in the form $\psi\phi$, where ϕ maps T onto a set A which is the union of a 2-dimensional sphere S^2 and of one of its diameters. But it is known (see [1], p. 526) that every map of S^2 into T is homotopic to a constant. Consequently the map ψ is homotopic to a map of A onto a 1-dimensional subset of T and we infer that $\psi\phi(\omega) \not\sim \omega$ in T, which contradicts (3.4). Hence the cycle γ is not cancellable rel. ω in T.

Remark. The fact considered in Example (3.3) is a direct consequence of the following theorem, due to A. Bojanowska [2]:

(3.5) *Theorem. Let* M_n *be a closed, compact and connected* *n-dimensional orientable manifold and let ω be an n-dimensional cycle in M_n such that (ω) is a generator of the Betti group $H_n(M_n)$ Then no m-dimensional cycle in M_n over any Abelian group $\mathfrak{A}$ is cancellable rel. ω in M_n.*

4. Shape Invariance of $\Gamma_m(X, \mathfrak{A}, (\Omega))$. Let us prove the following

(4.1) *Theorem. Let X, X' be compacta, $\mathfrak{A}$ be an Abelian group, $\underline{g}: X \to X'$ be a fundamental sequence and Ω be a collection of n-dimensional cycles in X over $\mathfrak{A}$. Then:*

(I) *If there exists a fundamental sequence $\hat{\underline{g}}: X' \to X$ such that $\hat{\underline{g}}\, \underline{g} \simeq \underline{i}_X$, then the homomorphism $\underline{g}_m / \Gamma_m(X, \mathfrak{A}, (\Omega))$ is a right inverse of the homomorphism $\hat{\underline{g}}_m / \Gamma_m(X', \mathfrak{A}, \underline{g}_n((\Omega)))$.*

(II) *If there exists a fundamental sequence $\hat{\underline{g}}: X' \to X$ such that $\hat{\underline{g}}\, \underline{g} \simeq \underline{i}_X$ and $\underline{g}\, \hat{\underline{g}} \simeq \underline{i}_{X'}$, then $\underline{g}_m / \Gamma_m(X, \mathfrak{A}, (\Omega))$ is an isomorphism.*

Proof. Assume that there exists a fundamental sequence $\hat{\underline{g}}: X' \to X$ such that $\hat{\underline{g}}\, \underline{g} \simeq \underline{i}_X$. The fundamental sequences $\underline{g}$ and $\hat{\underline{g}}$ induce homomorphisms:

$$\underline{g}_m : H_m(X, \mathfrak{A}) \to H_m(X', \mathfrak{A}), \quad \underline{g}_n : H_n(X, \mathfrak{A}) \to H_n(X', \mathfrak{A}),$$
$$\hat{\underline{g}}_m : H_m(X', \mathfrak{A}) \to H_m(X, \mathfrak{A}), \quad \hat{\underline{g}}_n : H_n(X', \mathfrak{A}) \to H_n(X, \mathfrak{A})$$

such that

$$\hat{\underline{g}}_m\, \underline{g}_m \simeq i_{H_m(X, \mathfrak{A})}, \quad \hat{\underline{g}}_n\, \underline{g}_n \simeq i_{H_n(X, \mathfrak{A})}.$$

Consider an m-dimensional cycle γ in X over $\mathfrak{A}$ such that $(\gamma) \in \Gamma_m(X, \mathfrak{A}, (\Omega))$ and let ω be a cycle belonging to Ω. Setting

(4.2) $\quad (\gamma') = \underline{g}_m((\gamma))$ and $(\omega') = \underline{g}_n((\omega)) \in \underline{g}_n((\Omega))$,

we infer by $\hat{\underline{g}}\, \underline{g} \simeq \underline{i}_X$ that

(4.3) $\quad \hat{\underline{g}}_m((\gamma')) = \hat{\underline{g}}_m \underline{g}_m((\gamma))$, $\hat{\underline{g}}_n((\omega')) = \hat{\underline{g}}_n \underline{g}_n((\omega)) = (\omega)$.

Since $\gamma \in \Gamma_m(X, \mathfrak{A}, \Omega)$ and $\omega \in \Omega$, there exists a fundamental sequence $\underline{f}:X \to X$ such that the homomorphisms $\underline{f}_m$ and $\underline{f}_n$ satisfy the conditions:

(4.4) $\underline{f}_m((\gamma)) = 0$ and $\underline{f}_n((\omega)) = (\omega)$, for every $\omega \in \Omega$.

Setting

$$\hat{\underline{f}} = \underline{g}\,\underline{f}\,\hat{\underline{g}}:X' \to X' ,$$

we infer by (4.2), (4.3) and (4.4) that

$$\hat{\underline{f}}_m((\gamma')) = \underline{g}_m \underline{f}_m \hat{\underline{g}}_m \underline{g}_m((\gamma)) = \underline{g}_m \underline{f}_m((\gamma)) = 0$$

and

$$\hat{\underline{f}}_n((\omega')) = \underline{g}_n \underline{f}_n \hat{\underline{g}}_n \underline{g}_n((\omega)) = \underline{g}_n \underline{f}_n((\omega)) = \underline{g}_n((\omega)) = (\omega').$$

Thus we have shown that the fundamental sequence $\hat{\underline{f}}$ realizes the cancellability of the cycle γ' relatively the n-dimensional cycle ω'. Hence the homomorphism $\underline{g}_m$ assigns to every element (γ) of $\Gamma_m(X,\mathfrak{A},(\Omega))$ an element (γ') of $\Gamma_m(X',\mathfrak{A},\underline{g}_n((\Omega)))$. Moreover the homomorphism $\hat{\underline{g}}_m$ assigns to $(\gamma') = \underline{g}_m((\gamma))$ the element (γ) of $\Gamma_m(X,\mathfrak{A},(\Omega))$. It follows that the power domain $\Gamma_m(X',\mathfrak{A},\underline{g}_n((\Omega)))$ r-dominates the power domain $\Gamma_m(X,\mathfrak{A},\Omega)$ and the proof of proposition (I) is finished.

If the hypotheses of (II) are satisfied, then one shows in the same way that the homomorphism $\hat{\underline{g}}_m$ assigns to each element (γ') of $\Gamma_m(X',\mathfrak{A},\underline{g}_n((\Omega)))$ an element (γ) of $\Gamma_m(X,\mathfrak{A},(\Omega))$ and that both relations

$$\hat{\underline{g}}_m \underline{g}_m = i_{H_m(X,\mathfrak{A})} \quad \text{and} \quad \underline{g}_m \hat{\underline{g}}_m = i_{H_m(X',\mathfrak{A})}$$

hold true. It follows that the power domain $\Gamma_m(X,\mathfrak{A},(\Omega))$ is isomorphic to the power domain $\Gamma_m(X',\mathfrak{A},\underline{g}_n((\Omega)))$. Thus the proof of (II) is finished and Theorem (4.1) is established.

(4.5) *Corollary. If* Sh(X) = Sh(X'), *then the power domain* $\Gamma_m(X,\mathfrak{A},n)$ *is isomorphic to the power domain* $\Gamma_m(X',\mathfrak{A},n)$.

In fact, the relation Sh(X) = Sh(X') implies that there exist two fundamental sequences $\underline{g}:X \to X'$ and $\hat{\underline{g}}:X' \to X$ satisfying the conditions:

$$\hat{\underline{g}}\,\underline{g} \simeq \underline{i}_X \quad \text{and} \quad \underline{g}\,\hat{\underline{g}} \simeq \underline{i}_{X'}.$$

Then the induced homomorphism $\underline{g}_n : H_n(X, \mathcal{A}) \to H_n(X', \mathcal{A})$ is an isomorphism. It follows that if Ω denotes the collection of all n-dimensional cycles in X over $\mathcal{A}$ and Ω' denotes the collection of all n-dimensional cycles in X' over $\mathcal{A}$, then $\underline{g}_n((\Omega)) = (\Omega')$ and consequently:

$$\Gamma_m(X, \mathcal{A}, (\Omega)) = \Gamma_m(X, \mathcal{A}, n) \quad \text{and}$$
$$\Gamma_m(X', \mathcal{A}, \underline{g}_n((\Omega))) = \Gamma_m(X', \mathcal{A}, n).$$

It suffices to apply Theorem (4.1), (II) in order to infer that $\Gamma_m(X, \mathcal{A}, n)$ is isomorphic to $\Gamma_m(X', \mathcal{A}, n)$.

Thus the power domain $\Gamma_m(X, \mathcal{A}, n)$ is a shape invariant of X.

5. Application

The shape invariance of $\Gamma_m(X, \mathcal{A}, n)$ allows us to prove the following

(5.1) *Theorem. Let* X *be a continuum satisfying the following conditions:*

1° X *is movable,*

2° *The number* $n = Fd(X)$ *is finite,*

3° *The Betti group* $H_n(X)$ *is a cyclic infinite group,*

4° X *is approximatively* 1*-connected,*

5° $\Gamma_m(X, n) = 0$, *for* $m < n$.

Then for every point $a \in X$, *the shape* Sh(X,a) *is simple.*

We shall use in the proof of this theorem the following, well-known proposition:

(5.2) *If* X *is a movable, approximatively* 1*-connected continuum with* $Fd(X) \leq n$ *and if the Betti groups* $H_m(X)$ *vanish for* $m = 1, 2, \cdots, n$, *then* Sh(X) *is trivial.*

In order to see it, consider a point $a \in X$. It follows by the well-known modified Hurewicz theorem (due to K. Kuperberg [6], p. 26) that the fundamental groups $\underline{\pi}_m(X,a)$ vanish for

$m = 1,2,\cdots,n$. Using the modified theorem of Whitehead, transferred into theory of shape by M. Moszyńska ([7], p. 260), see also J. Keesling ([5], p. 248), we infer that setting

$$f_k(x) = a \text{ for every point } x \in Q \text{ and for } k = 1,2,\cdots,$$

one obtains a fundamental sequence $\underline{f} = \{f_k, X, (a)\}$ which is a fundamental equivalence. Hence $Sh(X) = Sh(a)$, i.e. $Sh(X)$ is trivial.

Proof of Theorem (5.1). Otherwise there would exist two continua X_1, X_2 with non-trivial shapes such that

$$X_1 \cap X_2 = (a) \quad \text{and} \quad Sh(X) = Sh(X_1 \cup X_2).$$

It is clear that (X_1,a) and (X_2,a) are retracts of (X,a) and we infer by 1°, 2° and 4° that X_1, X_2 are movable, $Fd(X_1), Fd(X_2) \leq n$ and that X_1 and X_2 are approximatively 1-connected. Moreover 3° implies that at least one of the groups $H_n(X_1), H_n(X_2)$ vanishes. We may assume that $H_n(X_1) = 0$.

It is clear that every cycle lying in X_1 is cancellable rel. the family of all cycles lying in X_2 and that every n-dimensional cycle ω, with integers as coefficients, lying in X is homologous (in X) to the sum $\omega_1 + \omega_2$ of two cycles lying in X_1 and X_2 respectively. Since $H_n(X_1) = 0$, we infer that $\omega_1 \sim 0$ in X_1 and consequently ω is homologous to a cycle lying in X_2. It follows that every m-dimensional cycle γ lying in X_1, with $m < n$, is cancellable relatively every n-dimensional cycle lying in X and we infer by 5° that $\gamma \sim 0$ in X. Since X_1 is a retract of X, it is also $\gamma \sim 0$ in X_1. Thus $H_m(X_1) = 0$ for $m = 1,2,\cdots,n$. It follows by (5.2) that $Sh(X_1)$ is trivial, contrary to our supposition. Thus the proof of Theorem (5.1) is finished.

Using Theorems (3.5) and (4.1), we get the following

(5.3) *Corollary. If* M_n *is a closed, compact, connected,* 1-*connected* n-*dimensional manifold and if* $a \in M_n$, *then* $Sh(M_n, a)$

is simple.

In order to obtain this corollary from Theorem (5.1), let us notice that $X = M_n$ satisfies the conditions 1°, 2°, 3°, 4°. Also condition 5° is satisfied, because of Theorem (3.5).

The problem if the hypothesis that M_n be 1-connected is essential for Corollary (5.3) remains open.[1] Let us only mention, that in the case n=2 this hypothesis may be omitted, as it has been shown recently by A. Kadlof [4].

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[1] R. Sher has informed me, in an oral communication, that the hypothesis that the manifold M_n is 1-connected is superfluous. This result will appear in a joint paper of R. Sher and J. Hollingsworth.