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1. Introduction

The characterization of light open mappings given in [1] involves a sequence $\{C_i\}$ of closed coverings of X with various properties. Among these is the property that for each i, C_i is partitioned into collections C_k^i which are pairwise disjoint. In [2], we were concerned with a sequence $\{C_i\}$ of coverings of a compact metric space X where each C_k^i consists of 2^i pairwise disjoint closed sets with certain topological properties. These were necessary and sufficient for the existence of free action by a dyadic group on X. We defined dyadic polyhedra in [2] and used certain inverse systems of such polyhedra. Here, we generalize the concept to include p-adic polyhedra. We also show that these may be used to generate p-adic actions.

2. p-adic Polyhedra

We say that P is a *p-adic n-polyhedron*, where each of p and n is a positive integer, if and only if P is a polyhedron whose vertices can be partitioned into n+1 pairwise disjoint sets $V_1, V_2, \cdots, V_{n+1}$ consisting of either exactly p points or a singleton with not all sets being singletons such that (1) no two points in V_k are connected by an interval (1-simplex), (2) given two sets V_i and V_j such that a point in V_i is joined to a point in V_j by a 1-simplex, then each point of V_i is joined to each point of V_j by a 1-simplex, i.e., the join $V_i * V_j$ lies in P, and (3) if $\partial\alpha = (a_1, a_2, \cdots, a_{k+1})$ and $\partial\beta = (b_1, b_2, \cdots, b_{k+1})$ are the boundaries of two k-simplexes in P such that a_i and b_i belong to the same set V_i of the partitioning of V, then α lies in P iff β lies in P. Such a partitioning of V is said to be one *which defines* P.

Lemma 1. A p*-adic* n*-polygon admits a* <u>*natural*</u> *periodic homeomorphism of period* p.

Proof. By definition, the set of vertices V and P can be partitioned into sets $V_1, V_2, \cdots, V_{n+1}$ where at least one set, say V_1, contains exactly p points. Order these points as $x_0, x_1, \cdots x_{p-1}$. Define $h(x_s) = h(x_{s+1})$ with addition mod p. Now, define h similarly for each V_i which contains p points. If V_j is a singleton $\{x\}$, then let $h(x) = x$. Clearly, h can be extended linearly to all 1-simplexes in P. Suppose that h is defined on $\partial\alpha$ where $\alpha = (a_0, a_1, a_2)$ — a 2-simplex in P. Consider $\beta = (b_0, b_1, b_2)$ where $h(a_i) = b_i$. If $a_i \in V_i$, then V_0, V_1, and V_2 are *three* (different) members of the partitioning of V. Now, $b_i \in V_i$, consequently, β contains *three* vertices. Since the various 1-simplexes in $\partial\alpha$ are in P, it follows from (2) of the definition of P that $\partial\beta$ is in P. From (3) of the same definition, it follows that β lies in P. Now, h can be extended linearly taking α onto β. An induction yields that h can be extended linearly to all of P. It should be clear that h is a periodic homeomorphism with period p.

A p-adic n-polyhedron is *strongly connected* if and only if for $a \in V_i$ and $b \in V_j$, $i \neq j$, where $V_1, V_2, \cdots, V_{n+1}$ is a partitioning of the set V of vertices of P which defines P, then a is joined to b in P with a 1-simplex.

Lemma 2. Suppose that P *is a strongly connected* p*-adic* n*-polyhedron,* Q *is a strongly connected* p*-adic* m*-polyhedron and* ϕ *is a simplicial mapping of* P *onto* Q*. Furthermore, let* $V_1, V_2, \cdots, V_{n+1}$ *and* $U_1, U_2, \cdots, U_{m+1}$ *be partitions of the set of vertices of* P *and* Q*. If* $a, b \in V_i$*, then* $\phi(a), \phi(b) \in U_j$.

Proof. This is obvious since if $\phi(a) \in U_k$ and $\phi(b) \in U_j$, $k \neq j$, then a 1-simplex α joins $\phi(a)$ to $\phi(b)$ in Q. Since ϕ is simplicial, some simplex β in P with $a, b \in \alpha$ maps onto α

under ϕ. This implies that a is joined to b in P with a 1-simplex which contradicts the fact that P is a p-adic n-polyhedron.

3. Proper Inverse Systems of p-adic n_i-polyhedra

We say that an inverse system $\{P_i, \phi_i\}$ of p-adic n_i-polyhedra P_i with simplicial bonding maps $\phi_i: P_{i+1} \to P_i$ is a *proper inverse system* iff the partitionings of the vertices $\overline{V}_{k+1}$ and $\overline{V}_k$ defining P_{k+1} and P_k, respectively, are such that if V_i is an element of the partitioning of $\overline{V}_k$ and $a, b \in V_i$, then $\phi_k^{-1}(a) = \bigcup_{j=1}^{m} U_{s_j}$ and $\phi_k^{-1}(b) = \bigcup_{j=1}^{m} U_{t_j}$ such that (1) each of U_{s_j} and U_{t_j} for $j = 1, 2, \cdots, m$ is an element of the partitioning of $\overline{V}_{k+1}$ which defines P_{k+1} and (2) if V_i is not a singleton, then each of U_{s_j} and U_{t_j} is not a singleton (each consists of exactly p points). That is, if V_i is not a singleton, then $\phi^{-1}(a)$, for each $a \in V_i$, is the union of a *fixed* number m of elements of the "defining partition" of $\overline{V}_{k+1}$. We shall use this restriction to define p-adic actions on the inverse limit. Perhaps, a weaker restriction could be imposed, but an uncomplicated one has not come to our attention.

Lemma 3. Suppose that $\{P_i, \phi_i\}$ *is a proper inverse system of* p-*adic* n_i*-polyhedra. Furthermore, for each* i, f_i *is the* <u>*natural*</u> *periodic homeomorphism of* P_i *onto itself of period* p (Lemma 1). *Then* $\phi_i f_{i+1} = \phi_i$ *and there is a periodic homeomorphism* g_k *of* P_k *onto itself of period* p^k *such that* $\phi_k g_k = g_{k-1}\phi_k$.

Proof. It should be clear from the definition of f_i and the fact that $\{P_i, \phi_i\}$ is proper that $\phi_i f_{i+1} = \phi_i$.

Now, the sets of vertices V and U of P_2 and P_1, respectively, are partitioned into sets $V_1, V_2, \cdots, V_{n_2+1}$ and $U_1, U_2, \cdots, U_{n_1+1}$ which define P_2 and P_1. If U_s is not a singleton, then $U_s = x_0, x_1, \cdots, x_{p-1}$ where $x_i \neq x_j$ for $i \neq j$. The elements of the partition of V can be labelled so that $\phi_1^{-1}(x_j) = \bigcup_{n=1}^{m} V_{jn}$, m fixed for all j. Also, each V_{jn} consists of exactly p points $\{x_{n0}^j, x_{n1}^j, \cdots, x_{p-1}^j\}$. We also say that the points in U_s and V_{jn} are labelled so that $f_1(x_i) = x_{i+1 \pmod p}$ and $f_2(x_{ni}^j) = x_{ni+1 \pmod p}^j$. Next, define $g_2(x_{ni}^j) = x_{ni}^{j+1}$ for $j < p - 1$, $g_2(x_{ni}^{p-1}) = x_{ni+1}^0$ for $i < p - 1$, and $g_2(x_{np-1}^{p-1}) = x_{n0}^0$. If U_s is a singleton $\{x\}$, then let $g_2(y) = f_2(y)$ for each $y \in \phi_1^{-1}(x)$. Extend g_1 linearly to the rest of P_2. It follows that $\phi_1 g_2 = f_1\phi_k$. Furthermore, g_2 is a periodic homeomorphism of P_k onto P_k of period p^2. Let $g_1 = f_1$.

Now, consider the partition $W_1, W_2, \cdots, W_{n_k}$ of the set of vertices W of P_3 which defines P_3. As before, if U_s is not a singleton, then $U_s = \{x_0, x_1, x_2, \cdots, x_{p-1}\}$ exactly p different points. Note that $(\phi_1\phi_2)^{-1}(x_i)$ is the union of a fixed number t of elements of the partition of W which defines P_3 each of which consists of exactly p points. In a manner similar to that used in defining g_2, we define g_3 which is a periodic homeomorphism of P_3 onto P_3 which has period p^3. Also, $\phi_3 g_3 = g_2\phi_3$. By induction, we define g_k which is a periodic homeomorphism of P_k onto P_k with period p^k such that $\phi_k g_k = g_{k-1}\phi_k$.

4. Inverse Limits of Proper Inverse Systems of p-adic Polyhedra Admit p-adic Actions

In this section, we provide a theorem which illustrates one use of p-adic polyhedra. It is easy to construct p-adic polyhedra and inverse systems of such polyhedra. See [2] for constructions of dyadic polyhedra and proper inverse systems. The techniques are applicable to constructions of p-adic polyhedra.

Theorem. Suppose that $\{P_i, \phi_i\}$ *is a proper inverse system of* p-*adic* n_i-*polyhedra. Furthermore, for each* i, t_i *is the* *natural periodic homeomorphism of* P_i *onto itself of period* p. *Then there is an action by a* p-*adic group* (homeomorphisms) *on the inverse limit* $X = \varprojlim P_i$.

Proof. By Lemma 3, there is a sequence $\{g_i\}$ such that for each i, g_i is a periodic homeomorphism of P_i onto P_i of period p^i and $\phi_i g_i = g_{i-1}\phi_i$. Thus, $G_p = \varprojlim G_i$ where G_i is the cyclic group of homeomorphisms generated by g_i (i.e., $g_i, g_i^2, \cdots, g_i^p, \cdots, g^{p^i} = id$). The homeomorphisms $\theta_i : G_i \to G_{i-1}$ for i > 1 are defined in the obvious manner by $\theta(g_i) = g_{i-1}$. That is, $\theta(g_i^k) = g_{i-1}^{k(\text{mod } p^{i-1})}$. Thus, G_p is a p-adic group. An action by G_p on X is given as follows: If $x = (x_1, x_2, \cdots)$ and $g \in G_p$ where $g = (a_1, a_2, \cdots)$, then $g(x) = (a_1(x_1), a_2(x_2), \cdots)$. This is, of course, the usual way that such actions are defined on inverse limits.

We proved in [2] that a necessary and sufficient condition that a dyadic group act (freely) on a compact metric space X is that there exist a sequence $\{C_i\}$ of coverings of X such that the (a) the inverse system $\{N(C_i), \phi_i\}$ where $N(C_i)$ is the nerve of C_i be a proper inverse system of (strict) dyadic n_i-polyhedra and (b) $X = \varprojlim P_i$. A similar theorem should hold true for p-adic actions on such spaces X. In fact, a proof should mimic the one for the dyadic case.

Questions. Is it possible to obtain an n-*manifold* M^n *as the inverse limit of a proper inverse system of* p-*adic* n_i-*polyhedra? Could such an inverse limit be an* ANR? *Just how "nice" can such an inverse limit be?*

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