TOPOLOGY PROCEEDINGS

Volume 1, 1976

Pages 17–28

http://topology.auburn.edu/tp/

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by

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Topology Proceedings

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ISSN:	0146-4124

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1. Introduction

Consider the following conditions which a continuum M may satisfy.

(*) Each subcontinuum of M is a monotone retract of M.

(**) (Pointed version) Each subcontinuum of M which contains a fixed point p is a monotone retract of M.

It is easy to verify that dendrites satisfy both conditions (see [10], Theorem 2.1). The second author has proved that if M is a dendroid, then each of (*) and (**) implies that M is a dendrite ([10], Theorem 2.3, and [12], Theorem 3). More recently, the authors have obtained the same conclusion for arbitrary metric continua satisfying (*), and for arcwise connected metric continua satisfying (**) [6].

In particular, (*) and (**) are equivalent for arcwise connected continua. However, they are not equivalent in general since the familiar "sin 1/x curve" satisfies (**).

Thus it is natural to ask for a characterization of continua satisfying (**). The main purpose of this paper is to provide such a characterization.

Theorem. A continuum M *satisfies* (**) *if and only if*

(a) M *is smooth at* p, *and*

(b) *for each subcontinuum* N *of* M *containing* p, N *is accessible and the components of* M − N *form a null family.*

In this result "smoothness" refers to the concept introduced by the first author in [4]. A more general definition of "smoothness" has been studied by T. Maćkowiak [14].

It is also shown that a metric continuum M satisfying (**) becomes a dendrite under the canonical monotone decomposition $\mathcal{D}$ of smooth continua defined in [4]. Thus condition (*) is recovered in the decomposition space $M/\mathcal{D}$.

2. Definitions and Preliminary Remarks

A *continuum* is a compact connected Hausdorff space. The reader is referred to [7] for basic properties of continua and undefined terms.

A subcontinuum N of a continuum M is called a *monotone retract* of M if there exists a mapping $r: M \to N$ which is both monotone and a retraction.

Let X be a subset of a continuum M. A point $x \in X$ is said to be *accessible* from a point $y \in M - X$ if there exists a subcontinuum H such that $y \in H$ and $H \cap X = \{x\}$. If some point of X is accessible from some point of M - X, then X is called *accessible*.

A collection $\mathcal{C}$ of subsets of a continuum M will be called a *null family* if each convergent net C_n of elements of $\mathcal{C}$ which is not eventually constant has a degenerate limit.

A continuum M is *irreducible* from the point p to the point q if no proper subcontinuum of M contains p and q. If, in addition, no proper connected subset of M contains p and q, then M is called an *arc* (sometimes generalized arc or ordered continuum).

The continuum M is *hereditarily unicoherent at* p if for each pair of subcontinua H and K containing p, $H \cap K$ is connected; or equivalently, if for each q in $M - \{p\}$, there is a unique subcontinuum, denoted by pq, which is irreducible from p to q. If M is hereditarily unicoherent at p and for each convergent net q_n, $\lim q_n = q$ implies that the net of subcontinua pq_n converges to pq, then M is said to be *smooth at* p [4].

A *tree (dendrite)* is a locally connected, hereditarily unicoherent (metric) continuum. A *generalized tree (smooth dendroid)* is an arcwise connected, hereditarily unicoherent, smooth (metric) continuum.

Let M be a continuum which is hereditarily unicoherent at the point p. The *weak cutpoint order* on M with respect to p will be denoted by $\leq$ (i.e., $x \leq y$ if $px \subseteq py$). For each $x \in M$ the set $D(x) = \{y \in M: py = px\}$ is the *level set* of x relative to $\leq$. The collection $\mathcal{D}$ of all level sets forms a decomposition (not necessarily upper semicontinuous) of M. Let $\phi: M \to M/\mathcal{D}$ denote the natural mapping where $M/\mathcal{D}$ is given the quotient topology. Observe that for any subcontinuum N of M which contains p, $\phi^{-1}(\phi(N)) = N$. We now list, for reference, some of the basic facts concerning the decomposition $\mathcal{D}$.

(i) For each $x \in M$, D(x) is connected (see [9], Theorem 3, p. 210 for metric continua, and [2], Theorem 1.2 for the general case).

(ii) For each $x \in M$, D(x) has void interior in px (see [7], Theorem 3-44).

(iii) If M is smooth at p, then $\mathcal{D}$ is a monotone upper semicontinuous decomposition and $M/\mathcal{D}$ is a generalized tree which is smooth at D(p) (see [4], Theorem 5.2 and Theorem 4.1).

(iv) If $M/\mathcal{D}$ is a continuum which is smooth at D(p), then M is smooth at p (see [13], Theorem 3.1 for metric continua, and [11], Theorem 6.3 for the general case).

3. The Main Results

Throughout this section M will denote a continuum containing a fixed point p.

We shall prove

Theorem 1. *Each subcontinuum of* M *which contains* p *is a* *monotone retract of* M *(i.e.,* M *satisfies* (**)*) if and only if*

(a) M *is smooth at* p, *and*

(b) *for each subcontinuum* N *of* M *containing* p, N *is accessible and the components of* M − N *form a null family.*

Furthermore, if (**) *holds, then* M/$\mathfrak{D}$ *is a tree.*

We shall need several lemmas.

Lemma 1. *Let* M *be hereditarily unicoherent at* p, *and let* N *and* P *be subcontinua of* M *such that* $p \in N \subseteq P$. *If* $r: M \to N$ *is a monotone retraction, then* $r|P$ *is a monotone retraction.*

Proof. It suffices to show that $r^{-1}(x) \cap P$ is connected for each $x \in N$. If not, there exist disjoint closed sets A and B such that $r^{-1}(x) \cap P = A \cup B$ and $x \in A$. But this contradicts hereditary unicoherence at p since $(r^{-1}(x) \cup N) \cap P = (A \cup N) \cup B$ and $(A \cup N) \cap B = \emptyset$.

Lemma 2. *Let* M *be irreducible from* p *to* q. *If each subcontinuum of* M *which contains* p *is a monotone retract of* M, *then* M *is smooth at* p.

Proof. According to the Lemma of [6], M is hereditarily unicoherent at p. Thus, by (iv) of Section 2, it suffices to show that M/$\mathfrak{D}$ is a continuum which is smooth at D(p). We begin by showing that D(z) is closed for each z in M. First suppose that x and y belong to cl(D(z)) − D(z). By the hypothesis and Lemma 1, there is a monotone retraction $r: pz \to px \cup py$. By irreducibility $pz = (px \cup py) \cup r^{-1}(r(z))$. Thus $\{x,y\} \subseteq r(D(z)) = r(z)$ and $x = y$. In particular, $cl(D(z)) - D(z) = \{x\}$. Since D(z) is connected (by (i) of Section 2) and pz is irreducible, $pz = px \cup D(z)$. But this implies that D(z) has nonvoid interior in pz, contradicting (ii) of Section 2. Thus D(z) is closed. We now show that each element D(z) of $\mathfrak{D}$ distinct from D(p) and D(q) separates D(p) from D(q) in $M/\mathfrak{D}$. Let $z \in M - (D(p) \cup D(q))$ and let $r:M \to pz$ be a monotone retraction. By irreducibility $M = pz \cup r^{-1}(r(q))$ and $r(q) \in D(z)$. Since r(q) separates p from q in M, D(z) separates D(p) from D(q) in $M/\mathfrak{D}$. It follows that $M/\mathfrak{D}$ is an arc (e.g., [3], Theorem 2.1). Consequently $M/\mathfrak{D}$ is a continuum which is smooth at D(p).

A quite different (and somewhat longer) proof of Lemma 2 can be obtained by applying the characterization of smoothness for irreducible continua given by J. J. Charatonik in [1].

Example 2 in Section 4 shows that the converse of Lemma 2 is false.

Lemma 3. *Let* M *be hereditarily unicoherent at* p *and assume that* $M/\mathfrak{D}$ *is a tree. Let* N *be a subcontinuum of* M *containing* p *and let* C *be a component of* M - N. *Then*

(a) C *is open and continuumwise connected.*

(b) *At most one point of* N *is accessible from any point of* C.

(c) *If* $r:M \to N$ *is a monotone retraction, then* r(C) *is degenerate.*

Proof. Note that M is smooth at p by (iv) of Section 2.

(a) Using the facts that $\phi:M \to M/\mathfrak{D}$ is monotone and $\phi^{-1}(\phi(N)) = N$ (see (iii) of Section 2), it is easy to verify that $\phi^{-1}(\phi(C)) = C$. It follows that $\phi(C)$ is a component of $M/\mathfrak{D} - \phi(N)$. As a component of an open subset of a tree, $\phi(C)$ is open and arcwise connected. Thus $C = \phi^{-1}(\phi(C))$ is open and continuumwise connected.

(b) Suppose that x and y are distinct points of N which are accessible from points in C. Then there exist subcontinua X and Y of M such that $X \cap C \neq \emptyset \neq Y \cap C$, $X \cap N = \{x\}$, and $Y \cap N = \{y\}$. Applying (a), there exists a subcontinuum $K \subseteq C$ such that $X \cap K \neq \emptyset \neq Y \cap K$.

But then $(N \cup X \cup K) \cap (N \cup Y \cup K) = N \cup (K \cup (X \cap Y))$ which is a separation, contradicting hereditary unicoherence at p.

(c) If $x,y \in r(C)$, then $r^{-1}(x) \cap C \neq \emptyset \neq r^{-1}(y) \cap C$. Thus $x = y$ by (b).

We shall need the notion of aposyndesis due to F. B. Jones (see [8] for a discussion of the history of this concept). A continuum M is said to be *aposyndetic at* x *with respect to* y if there exists a subcontinuum K of M such that $x \in \text{int}(K) \subseteq K \subseteq M - \{y\}$. If for each pair of distinct points x and y of M, M is aposyndetic at x with respect to y (either one of the points with respect to the other), then M is said to be *aposyndetic (semi-aposyndetic)*.

In the next lemma we shall use the facts that every generalized tree is semi-aposyndetic, and that every aposyndetic generalized tree is a tree ([5], Theorem 3.5 and Corollary 2.1).

Lemma 4. *If* M *is smooth at* p *and for each subcontinuum* N *of* M *containing* p *the components of* M - N *form a null family, then* $M/\mathcal{D}$ *is a tree.*

Proof. Applying the hypothesis and the properties of ϕ discussed in Section 2, it is easy to verify that for each subcontinuum K of $M/\mathcal{D}$ which contains D(p), the components of $M/\mathcal{D} - K$ form a null family. Thus it suffices to assume that M is a generalized tree (i.e., $M = M/\mathcal{D}$), and prove that M is aposyndetic. Let x and y be distinct points of M. Since M is semi-aposyndetic, we can assume that there is a subcontinuum H of M such that $y \in \text{int}(H) \subseteq H \subseteq M - \{x\}$. If $x \leq y$, then M is aposyndetic at x with respect to y ([5], Corollary 3.6), and the proof is complete. Otherwise, $x \notin py \cup H$. Let $\mathcal{C}$ denote the components of $M - (py \cup H)$, and let C denote the member of $\mathcal{C}$ containing x. If $x \notin \text{int}(C)$, then there is a net x_n in

$M - (py \cup H \cup C)$ such that $\lim x_n = x$. Let C_n be the corresponding net in $\mathcal{C}$ (i.e., $x_n \in C_n$), and assume without loss of generality that C_n converges. Since $x_n \notin C$ for each n, C_n is not eventually constant. But $x \in \lim C_n$ and $(\lim C_n) \cap (py \cup H) \neq \emptyset$, which contradicts the assumption that $\mathcal{C}$ is a null family. Consequently, $x \in \text{int}(C) \subseteq \text{cl}(C) \subseteq M - \{y\}$, and M is aposyndetic.

Lemma 5. *Let* M *be smooth at* p *and assume that for each subcontinuum* N *of* M *containing* p, N *is accessible and the components of* M - N *form a null family. If* N *is a subcontinuum of* M *containing* p *and* C *is a component of* M - N, *then* $N \cap \text{cl}(C)$ *is degenerate.*

Proof. Suppose that N and C are as in the hypothesis and that $N \cap \text{cl}(C)$ is nondegenerate. According to Lemma 4, $M/\mathcal{D}$ is a tree and thus Lemma 3 applies. Consequently C is open and M - C is a subcontinuum of M containing p. By hypothesis and Lemma 3(b), there is a unique point $x \in M - C$ which is accessible from each point of C. Observe that $x \in N \cap \text{cl}(C)$. Let H be a nondegenerate subcontinuum of M such that $N \cap H = \{x\}$ and $H - \{x\} \subseteq C$. Let $y \in N \cap \text{cl}(C)$ such that $y \neq x$, and let y_n be a net in C - H converging to y. Arguing as above, we conclude that for each n there is a unique point $z_n \in N \cup H$ which is accessible from y_n. Since C is continuumwise connected (Lemma 3(a)) and M is hereditarily unicoherent at p it follows easily that $z_n \in H - N$ for each n. Let C_n denote the component of $M - (H \cup N)$ which contains y_n. Passing to a subnet if necessary, assume that C_n converges to a continuum C_o. Since $y \in C_o$ and $C_o \cap H \neq \emptyset$, the net C_n must be eventually constant. It follows that z_n is eventually constant, say $z_n = z_o$ for sufficiently large n. Thus $z_o \in py_n$ for sufficiently large n; and by smoothness $z_o \in py \subseteq N$. Thus $z_o \in N$ and $z_o \in H - N$ which is a contradiction.

Proof of Theorem 1. (Only if) By the Lemma in [6], M is hereditarily unicoherent at p; and by Lemma 1 and 2, each irreducible subcontinuum of the form px is smooth at p. To show that M is smooth at p it suffices to prove that $\leq$ is closed in $M \times M$ ([5], Theorem 3.1). Let (x_n, y_n) be a net in $\leq$ converging to (x,y). Let $r: M \to px \cup py$ be a monotone retraction. By [4], Theorem 4.1, r preserves order, and hence $r(x_n) \leq r(y_n)$ for each n. Since px and py are smooth at p, so is $px \cup py$; and consequently $x = r(x) \leq r(y) = y$. Thus (x,y) belongs to $\leq$ and M is smooth at p.

Let N be any subcontinuum of M containing p, and let $r: M \to N$ be a monotone retraction. If $x \in M - N$, then $r^{-1}(r(x)) \cap N = \{r(x)\}$, so N is accessible.

We next show that $M/\mathfrak{D}$ is a tree. By [12], Theorem 3, it suffices to show that each arc of the form D(p)D(x) in $M/\mathfrak{D}$ is a monotone retract of $M/\mathfrak{D}$. Let $r: M \to px$ be a monotone retraction. Since r preserves order, it is easy to verify that the induced map $r^*: M/\mathfrak{D} \to D(p)D(x)$ defined by $r^*(D(y)) = D(r(y))$ for each $y \in M$ is a monotone retraction.

Finally, let N be a subcontinuum of M containing p and let $\mathcal{C}$ denote the components of M - N. Assume that C_n is a net of elements of $\mathcal{C}$ which is not eventually constant and converges to a subcontinuum C. Since each C_n is open by Lemma 3, it follows that $C \subseteq N$. Let $r: M \to N$ be a monotone retraction. Then, by Lemma 3, $r(C_n)$ is degenerate for each n. Hence $C = r(C) = \lim r(C_n)$ is degenerate; i.e., $\mathcal{C}$ forms a null family.

(If) Let N be an subcontinuum of M which contains p. We must define a monotone retraction $r: M \to N$. For each $x \in M - N$, denote by C(x) the component of M - N containing x. Define $r: M \to N$ to be the unique retraction such that for each $x \in M - N$, $\{r(x)\} = N \cap cl(c(x))$. Note that r is a well-defined function by Lemma 5. Since point inverses of r are clearly connected, it remains only to show that r is continuous. If not, there exists an open set U in the relative topology on N such that $r^{-1}(U)$ is not open in M. Let $z \in r^{-1}(U) - \text{int}(r^{-1}(U))$. Applying Lemma 4 and Lemma 3, it follows that C(x) is open for each x, and thus $z \in U \subseteq N$. Consequently, there is a net z_n in M - N such that $z_n \notin r^{-1}(U)$ for each n, $\lim z_n = z$, and $(N \cap \text{cl}(C(z_n))) \cap U = \emptyset$ for each n. Without loss of generality, assume that the net $C(z_n)$ converges to a continuum C. The net $C(z_n)$ is not eventually constant; for otherwise $z_n \in r^{-1}(z) \subseteq r^{-1}(U)$ for sufficiently large n. But C contains z and meets N - U, contradicting the assumption that the components of M - N form a null family. Thus r is continuous.

Corollary 1. *Let* M *be a generalized tree which is smooth at* p. *Then* M *is a tree if and only if for each subcontinuum* N *of* M *containing* p, *the components of* M - N *form a null family.*

Proof. (Only if) If M is a tree then each subcontinuum of M is a monotone retract of M ([10], Theorem 2.1), and Theorem 1 applies.

(If) By Lemma 4, $M/\mathfrak{D} = M$ is a tree.

Corollary 2. *Let* M *be a continuum which is irreducible about a finite set. Each subcontinuum of* M *which contains* p *is a monotone retract of* M *(i.e.,* M *satisfies* (**)*) if and only if*

(a) M *is smooth at* p, *and*

(b) *each subcontinuum of* M *containing* p *is accessible.*

Furthermore, if (**) *holds, then* $M/\mathfrak{D}$ *is a finite tree.*

Proof. If N is a subcontinuum of M which contains p, then the components of M - N form a finite, hence null, family. Now apply Theorem 1.

4. Examples

Corollary 2 shows that the "null family" condition in

Theorem 1 is superfluous for continua irreducible about finitely many points. The following example shows that this condition cannot be omitted in general, even if $M/\mathcal{D}$ is known to be a tree.

Example 1. Let M be the plane continuum defined by:

$$M = \{(x,y): \ y = 1 + \sin 1/x \text{ for } -1 \leq x < 0\}$$
$$\cup \{(x,y): \ x = 0 \text{ and } 0 \leq y \leq 2\}$$
$$\cup (\bigcup_{n=0}^{\infty} \{(x,y): \ y = nx \text{ for } 0 \leq y \leq 2\}).$$

Note that M is the union of a "simple harmonic fan" and a "sin 1/x curve." The "sin 1/x curve" is not a monotone retract of M; and $M/\mathcal{D}$ is a locally connected fan (i.e., a dendrite with only one ramification point).

The next example shows that the "accessibility" condition in Theorem 1 cannot be omitted even for irreducible continua.

Example 2. Let M be the plane continuum defined by:

$$M = \{(x,y): \ y = \sin 1/x \text{ for } -1 \leq x < 0 \text{ and } 0 < x \leq 1\}$$
$$\cup \{(x,y): \ x = 0 \text{ and } -1 \leq y \leq 1\}.$$

Note that M is the union of two "sin 1/x curves" with a common limit segment. Neither of the "sin 1/x curves" is a monotone retract of M. Thus M does not satisfy (**).

5. Concluding Remarks

Consider the following weak version of condition (**).

(***) Each subcontinuum of M which is irreducible between a fixed point p and some other point is a monotone retract of M.

If M is a dendroid, then (***) is equivalent to (**) by [12], Theorem 3.

Question. Are conditions (**) *and* (***) *equivalent for an arbitrary continuum* M?

We remark that it is possible to modify the proof of Theorem 1 to obtain an affirmative answer to this question in the special case when M is hereditarily unicoherent at the point p. Thus it suffices to determine whether a continuum M satisfying (***) must be hereditarily unicoherent at p.

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