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In 1960 R. H. Bing [2, p. 228] asked, "Does each homogeneous circle-like continuum other than a solenoid contain a pseudo-arc?" The primary purpose of this paper is to answer Bing's question in the affirmative.

Using a theorem of E. G. Effros [4, Theorem 2.1] involving topological transformation groups, the author [7, Lemma 4] proved the following:

Lemma 1. Let M *be a homogeneous continuum with metric* ρ. *Suppose* ε *is a given positive number and* x *is a point of* M. *Then* x *belongs to an open subset* W *of* M *having the following property. For each pair* y, z *of points of* W, *there exists a homeomorphism* h *of* M *onto* M *such that* $h(y) = z$ *and* $\rho(v, h(v)) < \varepsilon$ *for all* v *belonging to* M.

Our next lemma is similar to Theorem 1 of [7].

Lemma 2. Let M *be a homogeneous hereditarily unicoherent circle-like continuum that is not a solenoid. If* A *is a decomposable subcontinuum of* M, *then* A *contains a homogeneous indecomposable continuum.*

Proof. Since A is decomposable, there exist proper subcontinua B and C of A such that $A = B \cup C$. Let b and c be points of B - C and C - B respectively. Let E be a continuum in A that is irreducible between b and c.

Since M is atriodic and hereditarily unicoherent, one can show (using Lemma 1) that E does not have an indecomposable subcontinuum with nonvoid interior (relative to E) [7, p. 38 (paragraph 4)]. Hence E is a continuum of type A' in the sense of

E. S. Thomas [7, p. 36]. Thus E has a unique minimal admissible decomposition $\mathcal{D}$, each of whose elements has void interior. (The existence of $\mathcal{D}$ also follows from [14, Theorem 3, p. 216].) Let k: E $\to$ [0,1] be the quotient map associated with $\mathcal{D}$.

There exists a number s $(0 < s < 1)$ such that $k^{-1}(s)$ is not degenerate; for otherwise, E would contain an arc [16, Theorem 21, p. 29] and M would be a solenoid [2, Theorem 9, p. 228]. Let Y denote the continuum $k^{-1}(s)$.

Let p and q be distinct points of Y. We shall prove that Y is a homogeneous subcontinuum of A by establishing the existence of a homeomorphism of Y onto itself that takes p to q.

Let r and t be numbers such that $0 < r < s < t < 1$. Define ε to be $\rho(k^{-1}[[r,t]], k^{-1}(0) \cup k^{-1}(1))$.

Let $\mathcal{W}$ be an open cover of Y such that for each $W \in \mathcal{W}$, if $y, z \in W$, then there exists a homeomorphism h of M onto M such that $h(y) = z$ and $\rho(v, h(v)) < \varepsilon$ for all $v \in M$ (Lemma 1). Since Y is a continuum, there exists a finite sequence $\{W_i\}_{i=1}^n$ of elements of $\mathcal{W}$ such that $q \in W_1$, $p \in W_n$, and $W_i \cap W_{i+1} \neq \emptyset$ for $1 \leq i < n$.

Choose $\{p_i\}_{i=0}^n$ such that $p_0 = q$, $p_n = p$, and $p_i \in W_i \cap W_{i+1}$ for $0 < i < n$. For each i $(1 \leq i \leq n)$, let h_i be a homeomorphism of M onto M such that $h_i(p_i) = p_{i-1}$ and $\rho(v, h_i(v)) < \varepsilon$ for all $v \in M$.

Each h_i maps Y onto itself [7, p. 39 (paragraphs 4-6)]. It follows that $h_1 h_2 \cdots h_n | Y$ is a homeomorphism of Y onto Y that takes p to q. Hence Y is homogeneous.

Since Y is a homogeneous hereditarily unicoherent continuum, Y is indecomposable [6, Theorem 1] [12, Theorem 1].

Theorem. Suppose M *is a homogeneous circle-like continuum and* M *is not a solenoid. Then* M *contains a pseudo-arc.*

Proof. We consider three cases.

Case 1. If M is hereditarily indecomposable, then M is a pseudo-arc [5, Theorem 2] [8, Corollary 2] [15, Theorem 2].

Case 2. If M is planar and not hereditarily indecomposable, then M is decomposable [9, Theorem 1]. Hence M is a circle of homogeneous nonseparating plane continua [13, Theorem 2]. Since each proper subcontinuum of M is chainable, M is a circle of pseudo-arcs [1] [3].

Case 3. If M is not planar and not hereditarily indecomposable, then M is indecomposable [11, Theorem 8] and contains a decomposable continuum A. Since M is an indecomposable circle-like continuum, M is hereditarily unicoherent. It follows from Lemma 2 that A contains a homogeneous indecomposable continuum Y. Since Y is a proper subcontinuum of M, it is chainable. Hence Y is a pseudo-arc [1] and our proof is complete.

Recently the author [10] proved that every homogeneous continuum having only arcs for proper subcontinua is a solenoid, answering in the affirmative another question of Bing [2, p. 219]. Still unanswered is Bing's question [2, p. 210]. "Is there a homogeneous tree-like continuum that contains an arc?"

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