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1. Introduction

A space is called *isocompact* if every closed countably compact subset is compact [2]. Among the classes of spaces having this property are the θ-refinable spaces [13] (and, hence, the Moore spaces), the spaces having G_δ-diagonal [3], and the symmetrizable spaces [10], to name a few.

In this paper, we discuss weak base properties generalizing symmetrizability of the type defined by Harley and Stephenson [6]. Further, we consider some generalizations of paracompactness of the type defined by Bacon [2] and their relations to the above mentioned base properties.

Our set theoretic notation will conform roughly to that used by Monk [9].

2. Weak Base Properties

Definition 2.1. Suppose X is a topological space and $B : \omega_o \times X \to \mathcal{P}X$. Consider the following conditions:

i) For each $n \in \omega_o$ and $x \in X$, $B(n+1,x) \subseteq B(n,x)$, and for each $x \in X$, $\bigcap_{n\in\omega_o} B(n,x) = \{x\}$.

ii) A subset $U \subseteq X$ is open if and only if, for each $x \in U$, there exists $n(x) \in \omega_o$ such that $B(n(x),x) \subseteq U$.

iii) If $F \subseteq X$ is closed and $x \notin F$, then there exists $n \in \omega_o$ such that for each $y \in B(n,x)\backslash\{x\}$, there exists $n(y) \in \omega_o$ such that $\{x,y\} \not\subseteq \bigcup_{f\in F} B(n(y),f)$.

iv) If $F \subseteq X$ is closed and $x \notin F$, then there exists $n \in \omega_o$ such that for each $y \in B(n,x)\backslash\{x\}$, there exists $n(y) \in \omega_o$ such that $\{x,y\} \cap \bigcup_{f\in F} B(n(y),f) = \emptyset$.

v) If $F \subseteq X$ is closed and $x \notin F$, then there exists $n \in \omega_o$ such that $x \notin \bigcup_{f\in F} B(n,f)$.

A. A space X which has a function B satisfying i) and ii) is called *weakly first countable* [1].

B. A space X which has a function B satisfying i), ii) and iii) is called an $\mathcal{F}$*-space* [6].

C. A space X which has a function B satisfying i), ii) and iv) is called an $\mathcal{F}_s$*-space* (or *strong* $\mathcal{F}$*-space*).

D. As seen in 2.2 below, a space X which has a function B satisfying i), ii) and v) is a *symmetrizable* space.

The following is due to Harley and Stephenson and a proof is given in [6].

Theorem 2.2. A space X *is symmetrizable if and only if it has a function* B *satisfying i), ii) and v) of 2.1.*

Corollary 2.2.1. Every $\mathcal{F}_s$*-space in which the isolated points form an* F_σ *set is symmetrizable.*

Proof. Suppose X is an $\mathcal{F}_s$ with function $B : \omega_o \times X \to \mathcal{P}X$ satisfying i), ii) and iv) of 2.1. Let I be the set of isolated points of X, and suppose $I = \bigcup_{n\in\omega_o} F_n$, where F_n is closed for each $n \in \omega_o$.

Suppose $x \in X\backslash I$. We may inductively choose, for each $k \in \omega_o$, $n_k(x) \in \omega_o$ such that 1) $n_k(x) \geq k$, 2) $B(n_k(x),x) \cap F_k = \emptyset$ and 3) $n_k(x) \geq n_{k-1}(x)$ whenever $k-1 \in \omega_o$.

Define $B^* : \omega_o \times X \to \mathcal{P}X$ as follows: if $x \in I$, let $B^*(k,x) = \{x\}$, for each $k \in \omega_o$; if $x \in X\backslash I$, let $B^*(k,x) = B(n_k(x),x)$, for each $k \in \omega_o$.

We shall show B* satisfies i), ii) and v) of 2.1. Since $B(n_k(x),x) = B^*(k,x) \subseteq B(k,x)$ for $x \in X\backslash I$, i) and ii) are clearly satisfied. Suppose $x \notin F$. If $x \in X\backslash I$, then there exists $k_x \in \omega_o$ such that $x \notin \bigcup_{f\in F} B(k_x,f) \supseteq \bigcup_{f\in F} B(n_{k_x}(f),f) = \bigcup_{f\in F} B^*(k_x,f)$. If $x \in I$, then there exists $n \in \omega_o$ such that $x \in F_n$, thus $x \notin \bigcup_{y\neq x} B^*(n,y)$.

Hence, by 2.2, X is symmetrizable.

Corollary 2.2.2. Every perfect $\mathcal{F}_s$-space is symmetrizable.

In view of the observation above, one might well ask if the difference between iv) and v), or, in fact, between iii) and iv), of 2.1 is merely one of semantics. To dispel any such concern, we cite the following examples.

Example 2.3.

a) The Sorgenfrey line [11] is an $\mathcal{F}$-space but fails to be and $\mathcal{F}_s$-space.

b) The Michael line [8] is an $\mathcal{F}_s$-space but fails to be symmetrizable.

To demonstrate that $\mathcal{F}$-spaces (and, hence, $\mathcal{F}_s$-spaces and symmetrizable spaces) are isocompact, we state, without proofs, Theorems 2.4, 2.5, and 2.6. All are due to Harley and Stephenson [6].

Theorem 2.4. If X is an $\mathcal{F}$-space, then X is $\aleph_1$-compact if and only if X is Lindelöf.

Theorem 2.5. If X is an $\mathcal{F}$-space, $A \subseteq X$, and A is either open or closed, then A is an $\mathcal{F}$-space.

Theorem 2.6. Every $\mathcal{F}$-space is isocompact.

Further, it follows easily from 2.4 and 2.5, that under the assumption that closed sets are G_δ sets, we may extend 2.4 to include the hereditary versions of these properties.

3. Covering Properties

Definition 3.1. For $\kappa \in$ Card, and $\mathcal{U}$ and $\mathcal{V}$ collections of subsets of a space X, we say $\mathcal{V}$ is κ *-weakly cushioned* in $\mathcal{U}$ if and only if there exists $f : \mathcal{V} \to \mathcal{U}$ such that if $\mathcal{G} \subseteq \mathcal{V}$ with

$|\mathcal{G}| \leq \kappa$ and $x : \mathcal{G} \to \cup\mathcal{G}$ with $x(G) \in G$, for each $G \in \mathcal{G}$, then $\overline{\{x(G) : G \in \mathcal{G}\}} \subseteq \cup f\mathcal{G}$.

We say a space X has *property* κL if and only if for every open cover $\mathcal{U}$ of X there is a sequence $\langle \mathcal{D}_n : n \in \omega_o \rangle$ such that $\cup_{n\in\omega_o} \mathcal{D}_n$ is a covering of X, and, for each $n \in \omega_o$, $\mathcal{D}_n$ is κ-weakly cushioned in $\omega\mathcal{U}$. ($\omega\mathcal{U}$ is the set of unions of countable subcollections of $\mathcal{U}$.)

For the case $\kappa = \aleph_o$, we have property L as defined by Bacon [2].

In view of Michael's characterization of paracompactness in terms of cushioned refinements [7], these properties seem to be natural generalizations of paracompactness.

Among the classes of spaces which satisfy property κL, for every $\kappa \in$ Card, are the θ-refinable spaces, the semistratifiable spaces, the semimetric spaces, and the regular σ-spaces, to name a few.

Remark.

1) If $\alpha \leq \beta \in$ Card, then $\beta L \Rightarrow \alpha L$.
2) If X is a space with countable tightness, then, for each $\kappa \in$ Card, X has κL if and only if X has L.
3) If X is a space which satisfies κL and A is an F_σ-subset of X, then A satisfies κL.

The following result is due to Bacon [2].

Theorem 3.2. Every space which satisfies L *is isocompact.*

The question of which of the isocompact spaces have the stronger property that $\aleph_1$-compact and Lindelöf are equivalent has been of great interest. This interest is heightened by the example of Wicke [12] of a T_1, $\aleph_1$-compact, hereditarily weakly θ-refinable, scattered space which is not metaLindelöf, and the subsequent example of van Douwen and Wicke [5] which has

the above properties and is locally compact, separable and submetrizable, in addition.

To explore this question for the properties κL, $\kappa \in \text{Card}$, we first prove the following basic lemma, and we will use the notation of the lemma throughout the remainder of this section.

Lemma. *If $\mathcal{U}$ is a non-empty collection of non-empty open subsets of X, and $\mathcal{D}$ is $\aleph_0$-weakly cushioned in $\omega\mathcal{U}$ via $f : \mathcal{D} \to \omega\mathcal{U}$, with no countable subcollection of $\mathcal{U}$ covering $\cup \mathcal{D}$, then there exist functions $a : \omega_1 \to \cup \mathcal{D}$, $D : \omega_1 \to \mathcal{D}$, and $U : \omega_1 \to$ {the open subsets of X} such that the following are true:*

1) for each $\alpha \in \omega_1$, $a_\alpha \in \cup\mathcal{D} \setminus \cup_{\beta<\alpha} fD_\beta$

2) for each $\alpha \in \omega_1$, $a_\alpha \in D_\alpha$

3) $U_\alpha \cap \{a_\beta : \beta < \omega_1\} = \{a_\alpha\}$

4) $\cup_{\alpha<\omega_1} U_\alpha = \cup_{\alpha<\omega_1} fD_\alpha$

Proof. Suppose $\mathcal{U}$, $\mathcal{D}$, and f are as in the hypothesis of the lemma. Since no countable subcollection of $\mathcal{U}$ covers $\cup \mathcal{D}$, choose $a_0 \in \cup \mathcal{D}$, and choose $D_0 \in \mathcal{D}$ such that $a_0 \in D_0$. Let $U_0 = fD_0$.

Suppose $\alpha < \omega_1$ and, for $\beta < \alpha$, we have chosen a_β, D_β such that $a_\beta \in D_\beta$ and $a_\beta \in \cup \mathcal{D} \setminus \cup_{\gamma<\beta} fD_\gamma$. Now $\cup_{\beta<\alpha} fD_\beta$ is the union of a countable subcollection of $\mathcal{U}$; hence, we choose $a_\alpha \in \cup \mathcal{D} \setminus \cup_{\beta<\alpha} fD_\beta$, and we choose $D_\alpha \in \mathcal{D}$ with $a_\alpha \in D_\alpha$. Let $U_\alpha = fD_\alpha \setminus \overline{\{a_\beta : \beta < \alpha\}}$.

Thus we inductively construct functions $a : \omega_1 \to \cup \mathcal{D}$, $D : \omega_1 \to \mathcal{D}$, $U : \omega_1 \to \mathcal{P}X$. Since, for each α, fD_α is open, we have that the range of U is contained in the open subsets of X. Clearly, we have 1) and 2) satisfied. Moreover, since $\mathcal{D}$ is $\aleph_0$-weakly cushioned in $\omega\mathcal{U}$ via f, we have, for each $\alpha < \omega_1$, $\overline{\{a_\beta : \beta < \alpha\}} \subseteq \cup_{\beta<\alpha} fD_\beta$. Hence, 3) and 4) are satisfied.

Theorem 3.3. *If $\kappa \geq \omega_1$ and X is a T_1-space satisfying*

κ L, *then* X *is* $\aleph_1$*-compact if and only if* X *is Lindelöf.*

Proof. Without further assumption Lindelöf implies $\aleph_1$-compact; hence, we consider the other implication.

Suppose $\mathcal{U}$ is an open cover of X with no countable subcover. Apply κ L to obtain a sequence $\langle \mathcal{D}_n : n \in \omega_o \rangle$ with $\mathcal{D}_n$ $\aleph_1$-weakly cushioned in $\omega\mathcal{U}$ via $f_n : \mathcal{D}_n \to \omega\mathcal{U}$, for each $n \in \omega_o$, and $\bigcup_{n\in\omega_o} \mathcal{D}_n$ covering X.

Since no countable subcollection of $\mathcal{U}$ covers X, there exists $n \in \omega_o$ such that no countable subcollection of $\mathcal{U}$ covers $\bigcup \mathcal{D}_n$. Apply lemma to $\mathcal{U}$, $\mathcal{D}_n$, f_n.

Suppose $x \in \overline{\{a_\alpha : \alpha < \omega_1\}} \setminus \{a_\alpha : \alpha < \omega_1\}$. Since $\mathcal{D}_n$ is actually $\aleph_1$-weakly cushioned in $^{\omega}\mathcal{U}$ by f_n, $x \in \bigcup_{\alpha<\omega_1} f_n D_\alpha = \bigcup_{\alpha<\omega_1} U_\alpha$. Choose $\alpha < \omega_1$ such that $x \in U_\alpha$. Let $U = U_\alpha \setminus \{a_\alpha\}$. Since $x \notin \{a_\alpha : \alpha < \omega_1\}$, U is open, $x \in U$ and $U \cap \{a_\alpha : \alpha < \omega_1\} = \emptyset$, which is impossible.

Thus $\{a_\alpha : \alpha < \omega_1\}$ is a closed, discrete subspace of cardinality $\aleph_1$, but this is impossible since X is $\aleph_1$-compact. Hence the result is proved.

Theorem 3.4.

a) If X *is a* T_1*-space with countable tightness which satisfies* L, *then* X *is* $\aleph_1$*-compact if and only if* X *is Lindelöf.*

b) If X *is a* T_1 *sequential space which satisfies* L, *then* X *is* $\aleph_1$*-compact if and only if* X *is Lindelöf.*

Proof. As noted previously, we need only show that $\aleph_1$-compact implies Lindelöf.

Suppose X is T_1, satisfies L, and is $\aleph_1$-compact, and $\mathcal{U}$ is an open cover of X with no countable subcover. Apply L to obtain a sequence $\langle \mathcal{D}_n : n \in \omega_o \rangle$ with $\mathcal{D}_n$ $\aleph_o$-weakly cushioned in $\omega\mathcal{U}$ via f_n, for each $n \in \omega_o$, and $\bigcup_{n\in\omega_o} \mathcal{D}_n$ covering X. There exists $n \in \omega_o$ such that no countable subcollection of

covers $\bigcup \mathcal{D}_n$. Apply lemma to $\mathcal{U}$, $\mathcal{D}_n$, f_n.

Suppose $B \subseteq \{a_\alpha : \alpha < \omega_1\}$ with $|B| \leq \aleph_o$ and $x \in \bar{B}$. If $|B| < \aleph_o$, $x \in B = \bar{B}$. If $|B| = \aleph_o$, let $B = \{a_{\alpha_k} : k \in \omega_o\}$, then $x \in \bar{B} \subseteq \bigcup_{k\in\omega_o} f_n D_{\alpha_k} \subseteq \bigcup_{\alpha<\omega_1} f_n D_\alpha \subseteq \bigcup_{\alpha<\omega_1} U_\alpha$. Choose $\alpha < \omega_1$ such that $x \in U_\alpha$. Since $U_\alpha \cap \{a_\beta : \beta < \omega_1\} = \{a_\alpha\}$, $x = a_\alpha$. Thus $\{a_\alpha : \alpha < \omega_1\}$ contains all its sequential limit points; in fact, it contains the closure of each of its countable subsets.

Hence, if X has countable tightness, or X is sequential, then $\{a_\alpha : \alpha < \omega_1\}$ is a closed discrete subset, contrary to X being $\aleph_1$-compact, and the result is proved.

Corollary 3.4.1. If X *is a* T_1*, first countable space satisfying* L*, then* X *is* $\aleph_1$*-compact if and only if* X *is Lindelöf.*

Theorem 3.5. If X *is hereditarily* $\aleph_1$*-compact and satisfies* L*, then* X *is Lindelöf.*

Proof. Suppose $\mathcal{U}$ is an open cover of X with no countable subcover. Apply L to obtain a sequence $\langle \mathcal{D}_n : n \in \omega_o \rangle$ with $\mathcal{D}_n$ $\aleph_o$-weakly cushioned in $\omega\mathcal{U}$ via f_n, for each $n \in \omega_o$, and $\bigcup_{n\in\omega_o} \mathcal{D}_n$ covering X. Choose $n \in \omega_o$ such that no countable subcollection of $\mathcal{U}$ covers $\bigcup \mathcal{D}_n$.

Apply lemma to $\mathcal{U}$, $\mathcal{D}_n$, f_n. The set $\{a_\alpha : \alpha < \omega_1\}$ is not $\aleph_1$-compact in its subspace topology, and the result is proved.

Corollary 3.5.1. A space X *is hereditarily* $\aleph_1$*-compact and hereditarily satisfies* L *if and only if* X *is hereditarily Lindelöf.*

Theorem 3.6. A T_3*-space* X *is hereditarily Lindelöf if and only if* X *is perfect,* $\aleph_1$*-compact, and satisfies* L.

Proof. It is well known that a T_3 hereditarily Lindelöf space is perfect. Further, Lindelöf implies $\aleph_1$-compact and L, so the 'only if' is shown.

Suppose $A \subseteq X$ and $\mathcal{U}$ is an open collection covering A which

has no countable subcover of A. Since open subsets of X are F_σ sets, $\bigcup \mathcal{U}$ satisfies L. Choose a sequence $\langle \mathcal{D}_n : n \in \omega_o \rangle$ with $\mathcal{D}_n$ $\aleph_o$-weakly cushioned in $\omega\mathcal{U}$ via f_n, for each $n \in \omega_o$, and $\bigcup_{n\in\omega_o} \mathcal{D}_n$ covering $\bigcup \mathcal{U}$. Apply lemma to $\mathcal{U}$, $\mathcal{D}_n$, f_n in the space $\bigcup \mathcal{U}$.

Now, since open subsets are F_σ sets, $\bigcup_{\alpha<\omega_1} U_\alpha$ is $\aleph_1$-compact, thus, $\{a_\alpha : \alpha < \omega_1\}$ has a limit point in $\bigcup_{\alpha<\omega_1} U_\alpha$. However, since X is T_1, this is impossible.

4. Relations

We turn our attention now to the relationships which exist between the properties defined in sections 2 and 3.

Theorem 4.1. If X *is a symmetrizable space, then* X *satisfies* κL, *for every* $\kappa \in$ *Card.*

Proof. Suppose $\mathcal{U}$ is an open covering of X. Well order $\mathcal{U}$. For each $x \in X$, we define $F(x,\mathcal{U})$ to be the first element of $\mathcal{U}$ which contains x. For each $U \in \mathcal{U}$, let $p(U,\mathcal{U}) = \{x : U = F(x,\mathcal{U})\}$. For each $n \in \omega_o$, let $D_n = \{x : x \notin \bigcup_{z\in X\setminus F(x,\mathcal{U})} B(n,z)\}$, and let $\mathcal{D}_n = \{D_n \cap p(U,\mathcal{U}) : U \in \mathcal{U}\}$. Define $f_n : \mathcal{D}_n \to \mathcal{U}$ as follows:

1. if $D_n \cap p(U,\mathcal{U}) \neq \emptyset$, let $f_n(D_n \cap p(U,\mathcal{U})) = U$;
2. if $D_n \cap p(U,\mathcal{U}) = \emptyset$, let $f_n(D_n \cap p(U,\mathcal{U}))$ be the first element of $\mathcal{U}$.

Clearly, $\bigcup_{n\in\omega_o} \mathcal{D}_n$ is a covering of X.

Suppose $\mathcal{G} \subseteq \mathcal{D}_n$ and $x : \mathcal{G} \to \bigcup \mathcal{G}$ such that $x(G) \in G$, for each $G \in \mathcal{G}$. Suppose $x \notin \{x(g) : G \in \mathcal{G}\}$. If $x \notin \bigcup f_n \mathcal{G}$, then $B(n,x) \cap \{x(G) : G \in \mathcal{G}\} = \emptyset$. Suppose $x \in \bigcup f_n \mathcal{G}$. There is a first $G \in \mathcal{G}$ such that $x \in f_nG$, where $\mathcal{G}$ is given the order induced by $\mathcal{U}$. Let $\mathcal{F}(x,\mathcal{G})$ denote this element. Hence, we have $x \notin \bigcup\{f_nG : G < \mathcal{F}(x,\mathcal{G})\}$, so $B(n,x) \cap \{x(G) : G < \mathcal{F}(x,\mathcal{G})\} = \emptyset$. Now, if $G > \mathcal{F}(x,\mathcal{G})$, then $x(G) \in p(f_nG,\mathcal{U})$; thus $x(G) \notin f_n(\mathcal{F}(x,\mathcal{G}))$. However, $f_n(\mathcal{F}(x,\mathcal{G}))$ is open, so there

exists $k \in \omega_o$ such that $B(k,x) \subseteq f_n(\mathcal{F}(x,\mathcal{G}))$. Hence we have $B(k,x) \cap \{x(G) : G > \mathcal{F}(x,\mathcal{G})\}$ is empty. Finally, $x \notin \{x(G) : G \in \mathcal{G}\}$, and $\cap_{m\in\omega_o} B(m,x) = \{x\}$, so there exists $m \in \omega_o$ such that $x(\mathcal{F}(x,\mathcal{G})) \notin B(m,x)$. Let $N = \max\{n,k,m\}$, then we have $B(N,x) \cap \{x(G) : G \in \mathcal{G}\} = \emptyset$.

Hence we have that $\{x(G) : G \in \mathcal{G}\}$ is closed. Therefore, $\overline{\{x(G) : G \in \mathcal{G}\}} = \{x(G) : G \in \mathcal{G}\} \subseteq \cup f_n\,\mathcal{G}$. Since no reference is made to $|\mathcal{G}|$ in this argument, κL is satisfied, for every $\kappa \in Card$.

Theorem 4.2. Every $\mathcal{F}_s$-space satisfies κL, for every $\kappa \in Card$.

Proof. Suppose (X,τ) is an $\mathcal{F}_s$-space with $\mathcal{F}_s$-system B. Let $A = \{x : \{x\} \notin \tau\}$. Let $\mathcal{U}$ be an open covering of X.

Define $B^* : \omega_o \times X \to \mathcal{P}X$ as follows:

1. Suppose $x \in A$. Let $B^*(n,x) = B(n,x)$, for each $n \in \omega_o$.
2. Suppose $x \notin A$. For each $n \in \omega_o$, let $B^*(n,x)$ be the set $\{z \in A : x \in B^*(n,z)\} \cup \{x\}$.

Let $\sigma = \{U \subseteq X : x \in U \Rightarrow$ there is $n_x \in \omega_o$ such that $B^*(n_x,x) \subseteq U\}$. Clearly, σ is a topology on X and $\sigma \subseteq \tau$. We now show that (X,σ) is symmetrizable, by showing that i), ii) and v) of 2.1 are satisfied.

i) For each $x \in X$, $n \in \omega_o$, $B^*(n+1,x) \subseteq B^*(n,x)$. If $x \in A$, $B^*(n,x) = B(n,x)$ for each $n \in \omega_o$; hence, $\cap_{n\in\omega_o} B^*(n,x) = \{x\}$. For $x \notin A$, consider $z \neq x$. If $z \notin A$, $z \notin B^*(0,x)$. If $z \in A$, then there is $n \in \omega_o$ such that $x \notin B^*(n,z)$, but then $z \notin B^*(n,x)$. Hence, $\cap_{n\in\omega_o} B^*(n,x) = \{x\}$, for every $x \in X$.

ii) This is clear from the definition of σ.

v) Suppose $X\backslash F \in \sigma$ and $x \notin F$. Since $\sigma \subseteq \tau$, $X\backslash F \in \tau$. First, if $x \in A$, there exists $n \in \omega_o$ such that $B^*(n,x) \subseteq X\backslash F$ and $x \notin \cup_{z\in F} B(n,z)$. Therefore, $x \notin \cup_{z\in F} B^*(n,z)$.

Suppose $x \notin A$. There exists $n \in \omega_o$ such that $B^*(n,x) \subseteq X \backslash F$, but then x is not in $\bigcup_{z \in F} B^*(n,z)$.

Hence, (X,σ) is a symmetrizable space.

Suppose $U \in \tau$. For each $n \in \omega_o$, let $U_n = \{x \in A : B^*(n,x) \subseteq U$ and for every $y \in B(n,x) \backslash \{x\}$ there exists $n_y^x \in \omega_o$ such that $\{x,y\} \subseteq X \backslash \bigcup_{z \in U} B(n_y^x, z)\}$. Clearly, $\bigcup_{n \in \omega_o} U_n = U \cap A$. Let $L_n = \bigcup \{B(n,x) : x \in U_n\}$ and $I_n = \bigcup \{B^*(n_y^x, y) : y \in (X \backslash A) \cap B(n,x)$ for some $x \in U_n\}$. Let $V_n = L_n \cup I_n$ and $V_U = \bigcup_{n \in \omega_o} V_n$.

We shall show 1) $V_U \subseteq U$, 2) $V_U \cap A = U \cap A$, and 3) $V_U \in \sigma$.

1) Suppose $x \in V_U$. There exists $n \in \omega_o$ such that $x \in V_n$. Thus either there is $z \in U_n$ such that $x \in B(n,z) \subseteq U$, or there is $y \in (X \backslash A) \cap B(n,z)$ for some $z \in U_n$ with $x \in B^*(n_y^z, y)$. If $x \notin U$, $y \notin B^*(n_y^z, x)$, so $x \notin B^*(n_y^z, y)$. Therefore, $V_U \subseteq U$.

2) By 1), $V_U \cap A \subseteq U \cap A$. Further, since X is an $\mathcal{F}_s$-space, $\bigcup_{n \in \omega_o} U_n = U \cap A$, but $U_n \subseteq V_n$, for every $n \in \omega_o$. Thus $U \cap A = \bigcup_{n \in \omega_o} U_n \subseteq \bigcup_{n \in \omega_o} V_n \subseteq V_U$, and we have $U \cap A = V_U \cap A$.

3) Suppose $x \in V_U$. First, if $x \in V_U \cap A$, then there is $n \in \omega_o$ such that $x \in U_n$, by 2). Hence, $B^*(n,x) \subseteq V_n \subseteq V_U$. Now, suppose $x \in V_U \backslash A$. There is $n \in \omega_o$ such that $x \in V_n$. If $x \in I_n$, then there exists $z \in U_n$ such that $B^*(n_x^z, x) \subseteq V_n \subseteq V_U$. If $x \in L_n$, then there exists $z \in U_n$ such that $x \in B(n,z)$, so again there exists $n_x^z \in \omega_o$ such that $B^*(n_x^z, x) \subseteq V_n \subseteq V_U$. Thus $V_U \in \sigma$.

Now $\mathcal{V} = \{V_U : U \in \mathcal{U}$ and $U \cap A \neq \emptyset\}$ is an open cover of the symmetrizable space $\bigcup \mathcal{V}$. Hence, by 4.1, we may choose, for any $\kappa \in$ Card, a sequence $\langle \mathcal{D}_n : n \in \omega_o \backslash \{0\} \rangle$ such that $\bigcup_{n \in \omega_o \backslash \{0\}} \mathcal{D}_n$ covers $\bigcup \mathcal{V}$, and $\mathcal{D}_n$ is κ-weakly cushioned in $\omega \mathcal{V}$, for each $n \in \omega_o \backslash \{0\}$.

$\bigcup \mathcal{V} \in \sigma \subseteq \tau$, so $\bigcup \mathcal{V} \in \tau$. Let $\mathcal{D}_o = \{\{x\} : x \in X \backslash \bigcup \mathcal{V}\}$, then $\mathcal{D}_o$ is κ-weakly cushioned in $\omega \mathcal{U}$, since $X \backslash \bigcup \mathcal{V}$ is a closed

discrete subspace of (X,τ). Now $\omega\mathcal{V}$ refines $\omega\mathcal{U}$ and for $S \subseteq \bigcup \mathcal{V}$, $\overline{S}^{\sigma} \supseteq \overline{S}^{\tau}$, so $\mathcal{D}_n$ is κ-weakly cushioned in $\omega\mathcal{U}$, for each $n \in \omega_o$. Therefore (X,τ) satisfies κL, for each $\kappa \in$ Card.

In the above proof, we have shown that if (X,τ) is an $\mathcal{F}_s$-space, then there is a weaker topology σ on X such that (X,σ) is a symmetrizable space and a function $V : \tau \to \sigma$ such that for each $U \in \tau$, $V_U \subseteq U$ and $V_U \cap \{x : \{x\} \notin \tau\} = U \cap \{x : \{x\} \notin \tau\}$. The converse is not difficult, so we have the following.

Theorem 4.3. A space (X,τ) *is an* $\mathcal{F}_s$*-space if and only if there is a topology* σ *on* X *such that* (X,σ) *is a symmetrizable space and a mapping* $V : \tau \to \sigma$ *such that for each* $U \in \tau$, $V_U \subseteq U$ *and* $V_U \cap \{x : \{x\} \notin \tau\} = U \cap \{x : \{x\} \notin \tau\}$.

We now give an example showing that it is not true, in general, that all $\mathcal{F}$-spaces satisfy L.

Example 4.4. Let S be the set of one-to-one sequences of real numbers. Since $|S| = c$, choose $g : R \to S$, a bijection. (R denotes the real numbers) Let $X = R \times \{0,1\}$.

For $(x,1) \in R \times \{1\}$, we let $\{(x,1)\}$ be an open set. For $(x,0) \in R \times \{0\}$, we define $B^*(n,x) = \{(x,0)\} \cup \{(g(x)_k,1) : k \geq n\}$ and let $\{B^*(n,x) : n \in \omega_o\}$ be an open neighborhood base at $(x,0)$.

X is clearly an $\mathcal{F}$-space, for if we define $B : \omega_o \times X \to \mathcal{P}X$ by $B(n,(x,0)) = B^*(n,x)$ and $B(n,(x,1)) = \{(x,1)\}$, for each $x \in R$ and $n \in \omega_o$, then $x \notin \bigcup_{z \neq x} B(1,z)$ when x is a non-isolated point.

To see that X does not satisfy L, let $\mathcal{U} = \{B^*(1,x) : x \in R\} \cup \{\{(x,1)\} : x \in R\}$ and suppose $\langle \mathcal{D}_n : n \in \omega_o \rangle$ and $\langle f_n : n \in \omega_o \rangle$ are as in L. There exists $n \in \omega_o$ such that $|(\bigcup \mathcal{D}_n) \cap (R \times \{1\})| = c$. For each $D \in \mathcal{D}_n$, there is a countable subcollection of $\mathcal{U}$ covering D; hence, we may choose c points in

$R \times \{1\}$ from distinct elements of $\mathcal{D}_n$. In particular, we may choose countable sets $\{x_k : k \in \omega_o\} \subseteq R \times \{1\}$ and $\{D_k : k \in \omega_o\} \subseteq \mathcal{D}_n$ such that $x_k \in D_k$, for every $k \in \omega_o$ and if $i \neq j$, then $x_i \neq x_j$ and $D_i \neq D_j$. Now $|\overline{\{x_k : k \in \omega_o\}} \cap (R \times \{0\})| = c$; hence no countable subcollection of $\mathcal{U}$ may cover $\overline{\{x_k : k \in \omega_o\}}$. Thus X is an $\mathcal{F}$-space which does not satisfy L.

Remark. It was suggested to the author by E. K. van Douwen that a regular example of this type may be constructed using the techniques of [4]. This is, in fact, the case. With a small modification, example 1.2 of [4] is easily seen to supply such an example. The example above is a T_1 space but fails to be even T_2.

On the other side of the ledger, as one might expect, the results are all negative.

Example 4.5. The ordinal space $\omega_1 + 1$ is compact, but cannot be an $\mathcal{F}$-space since the open subset ω_1 is not even isocompact.

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