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1. A *continuum* is a nondegenerate compact connected metric space. A continuum is *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected. A *dendroid* is an arcwise connected hereditarily unicoherent continuum; a *dendrite* is a locally connected dendroid.

Given two points x and y in a dendroid X there exists a unique arc, denoted [x,y], with noncut points x and y. For any fixed point $p \in X$ we define the *weak cut point partial order with respect to* p, $\leq_p$, by: $x \leq_p y$ if and only if $x \in [p,y]$. In case $x \leq_p y$ and $x \neq y$ we write $x <_p y$.

A dendroid X is *smooth* if for some $p \in X$, $\Gamma_p = \{(x,y) \in X \times X | x \leq_p y\}$ is a closed subset of $X \times X$. To emphasize the point p we will write "X is smooth at p."

It is well known (see, e.g. [4], p. 680) that if a dendroid X is smooth at p then X is locally connected at p. Moreover, a dendroid is a dendrite if and only if it is smooth at each of its points ([2], p. 298).

Let X and Y be dendroids and f a map (= continuous function) from X into Y. The map f is: (i) *monotone* if $f^{-1}(y) \subseteq X$ is connected for each $y \in Y$; (ii) $\leq_p$-*preserving* if $x \leq_p y$ in X implies $f(x) \leq_{f(p)} f(y)$ in Y. Finally, f is a *monotone* ($\leq_p$-*preserving*) *retraction* if $Y \subseteq X$ and f is a retraction which is monotone ($\leq_p$-preserving).

In [5] the following theorems were established.

Theorem A. A dendroid is smooth at p *if each subcontinuum of the form* $[p,x] \cup [p,y]$ *admits a* $\leq_p$-*preserving retraction.*

Theorem B. A dendroid is a dendrite if and only if each subcontinuum admits a monotone retraction.

It was also shown that the converse of Theorem A is not true. In this paper we alter slightly the hypotheses of Theorem A in order to obtain a characterization of smoothness in dendroids. This result is then used to sharpen the statement of Theorem B.

2. We first state, without proof, two properties of smooth dendroids. The reader is referred to [2] and [6] for the details.

Recall that a metric d on a dendroid X is *radially convex with respect to* $p \in X$ if $x <_p y$ implies $d(p,x) < d(p,y)$.

Lemma 1. A dendroid X *is smooth at* p *if and only if* X *has a metric which is radially convex with respect to* p.

If X is smooth at p and d is the metric of Lemma 1, one can show if $t < d(p,q)$ then there exists a unique $x \in [p,q]$ such that $d(p,x) = t$.

Lemma 2. If the dendroid X *is smooth at* p *then* X *has a basis of convex open sets.*

A set $U \subseteq X$ is *convex* if $x <_p y$ in U implies $[x,y] \subseteq U$.

Theorem 1. If the dendroid X *is smooth at* p *then each arc* $[p,q]$ *admits a* $\leq_p$*-preserving retraction.*

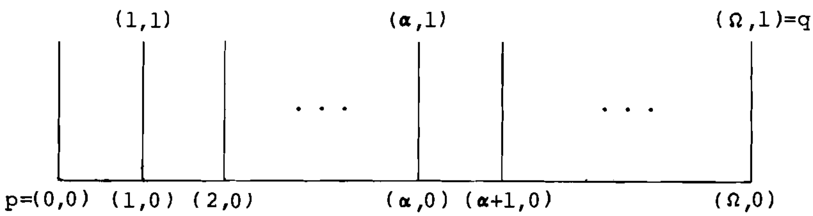
Proof. Let d be the metric of Lemma 1. Let $[p,q] \subseteq X$ be fixed, but arbitrary. Define the function $r : X \to [p,q]$ as follows:
If $d(p,x) < d(p,q)$ let $r(x)$ be the unique point $r(x) \in [p,q]$ satisfying $d(p,r(x)) = d(p,x)$; otherwise let $r(x) = q$. It is straightforward to verify r is a $\leq_p$-preserving retraction.

Before proving the converse of Theorem 1 we remark that

Theorem 1 does not hold for generalized trees (the nonmetric analog of smooth dendroids). Let $[0,\Omega]$ denote the "extended long line" (see [3], p. 55). The continuum

$$X = \{(\alpha,t) \in [0,\Omega] \times [0,1] \mid \alpha \text{ is an ordinal or } t = 0\}$$

is a generalized tree which is smooth at $p = (0,0)$.



However, the (generalized) arc from p to $q = (\Omega,1)$ is not a retract of X. For if $r : X \to [p,q]$ were a retraction there would exist $\alpha < \Omega$ such that $r(\alpha,1) = (\Omega,t)$ for some $t \in [0,1]$. But then the nonmetric arc from $(\alpha,0)$ to $(\Omega,0)$ would be contained in the continuous image of the metric arc from $(\alpha,0)$ to $(\alpha,1)$.

The remaining theorems, as well as their proofs, generalize to the nonmetric setting.

Theorem 2. Let X *be a dendroid and let* $p \in X$. *If each arc* $[p,q] \subseteq X$ *admits a* $\leq_p$*-preserving retraction then* X *is smooth at* p.

Proof. We show Γ_p is closed. Let (x_n,y_n) be a sequence in Γ_p converging to (x,y). Let $r : X \to [p,x]$ be a $\leq_p$-preserving retraction. If $(x,y) \notin \Gamma_p$ then either $y <_p x$ or x and y are not related (with respect to $\leq_p$).

Suppose $y <_p x$. Then $p,x,y \in [p,x]$. Thus $(r(x_n),r(y_n))$ is a sequence in $\Gamma_p \cap ([p,x] \times [p,x])$ converging to $(r(x),r(y)) = (x,y)$. Since the dendroid $[p,x]$ is smooth at p, we infer $(x,y) \in \Gamma_p$. This contradicts the assumption that $y <_p x$.

Suppose x and y are not related. Define z by the equation

$[p,z] = [p,x] \cap [p,y]$ and note that $z <_p x$. Since $z \in [x,y] \subset \mathrm{Li}[x_n,y_n]$ ([1], Corollary 1, p. 7) there exist points $z_n \in [x_n,y_n]$ (i.e., $x_n \lneq_p z_n \lneq_p y_n$) converging to z. Applying the above argument to the sequences z_n and x_n we infer $(x,z) \in \Gamma_p$ which is a contradiction.

Thus, $(x,y) \in \Gamma_p$ and X is smooth at p.

Theorem 3. Let X *be a dendroid and let* $p \in X$. *Then* X *is a dendrite if and only if each arc* $[p,q]$ *admits a monotone retraction.*

Proof. The necessity follows from Theorem B. To prove the sufficiency we first note that X is smooth at p by Theorem 2 and [2], p. 309.

Assume X is not locally connected at x_0. Then there exists a convex open neighborhood U of x_0 such that $p \notin \mathrm{cl}\ U$ and cl U contains no connected neighborhood of x_0. Let C denote the component of cl U which contains x_0. Then $x_0 \notin \mathrm{int}\ C$. Hence there exists a sequence $x_n \in \mathrm{cl}\ U - C$ converging to x_0.

Define the sequence z_n by the equation $[p,z_n] = [p,x_n] \cap [p,x_0]$. Passing to a subsequence, if necessary, assume $\lim_{n\to\infty} z_n = z$ for some $z \in X$.

Since X is smooth at p and $z_n \le x_n$ it follows that $z \le_p x_0$. Note also that $z \in X - U$. For if $z \in U$ then $z_n \in U$ for some n. Since U is convex $[z_n,x_n] \cup [z_n,x_0] \subset U$; in particular, x_n and x_0 lie in the same component of the closure of U.

Now let $r : X \to [p,x_0]$ be a monotone retraction. Since x_n, $r(x_n) \in r^{-1}(r(x_n))$ and r is monotone it follows that $[x_n,r(x_n)] \subseteq r^{-1}(r(x_n))$. In particular, $z_n \in r^{-1}(r(x_n))$. Consequently

$$z = r(z) = \lim_{n\to\infty} r(z_n) = \lim_{n\to\infty} r(x_n) = r(x_0) = x_0 \in U.$$

This contradiction proves the theorem.

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