# A COLLECTIONWISE NORMAL WEAKLY θ-REFINABLE DOWKER SPACE WHICH IS NEITHER IRREDUCIBLE NOR REALCOMPACT

by

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## 1. Introduction

The concept of irreducibility was first used by Arens and Dugundji [1]. Wicke and Worrell [8] introduced θ-refinability as a generalization of paracompactness and observed that θ-refinable spaces are irreducible. Lutzer [6] introduced weak θ-refinable and Boone [2] raised the question: Is every weakly θ-refinable space irreducible? An answer to this question is given by van Douwen and Wicke [3] who construct without any unusual set theoretic assumptions a regular weakly θ-refinable space which is neither irreducible nor normal. Our construction is the result of trying to find a normal weakly θ-refinable space which is not irreducible. We have so far been unable to do so without ♣. We are grateful to Professor Robert L. Blair for asking if our space is realcompact. Gardner [4] has shown that if X is a normal weakly θ-refinable countably paracompact space such that the cardinality of each discrete subspace is of measure zero then X is realcompact. Our example shows that countable paracompactness cannot be dropped from the hypothesis in this statement.

I would like to thank my research supervisor, Professor Jack W. Rogers, Jr., for many helpful discussions.

## 2. Construction

Axiom ♣ is assumed, N denotes the set of positive integers, $\omega_1$ denotes the set of countable ordinals with their usual order topology and $\Lambda$ denotes the set of limit ordinals in $\omega_1$. If H is a collection of sets then H* denotes the union of all of the

elements of H.  A collection M of two-element subsets of $\omega_1$ is

*increasing* provided that for each two sets in M, each element of

one of these two sets precedes each element of the other set.

If B is an infinite set then the expression *A is almost all of*

*B* means that A is a subset of B and that there do not exist

infinitely many elements of B which are not in A.  For each n

in N, L(n) denotes $\omega_1 \times \{n\}$ and L(n) is referred to as *level n.*

If m and n are in N then *level m is below level n* only if m is

less than n.  If p and q are two points in $\omega_1 \times$ N then *p is below*

*q* only if the level which contains p is below the level which

contains q and *p precedes q* only if the first coordinate of p

precedes the first coordinate of q.

   *Lemma 1.  There is a function T such that (1) the domain of*

*T is $\Lambda$, (2) if $\lambda$ is in $\Lambda$ then T($\lambda$) is an infinite subset of the*

*predecessors of $\lambda$ and T($\lambda$) has only $\lambda$ as a limit point in $\omega_1$ and*

*(3) if M is an uncountable increasing collection of two-element*

*subsets of $\omega_1$ then there is a $\delta$ in $\Lambda$ such that if X is almost*

*all of T($\delta$) then some set in M is a subset of X.*

   *Proof of Lemma 1.*  Let f denote a one-to-one function whose

domain is $\omega_1$ and whose range is the set of all two-element sub-

sets of $\omega_1$.  Using ♣ let C denote a function such that (1) the

domain of C is $\Lambda$, (2) if $\lambda$ is in $\Lambda$ then C($\lambda$) is an infinite

subset of the predecessors of $\lambda$ and C($\lambda$) has only $\lambda$ as a limit

point in $\omega_1$ and (3) if B is an uncountable subset of $\omega_1$ then

for some $\lambda$ in $\Lambda$, C($\lambda$) is a subset of B.  Let $\Gamma$ denote the set

of all $\lambda$ in $\Lambda$ such that f(C($\lambda$)) is increasing.  That $\Gamma$ is un-

countable follows from the fact that if $\alpha$ belongs to $\omega_1$ then

there is an uncountable subset H of $\omega_1$ such that f(H) is in-

creasing and each element of H follows $\alpha$.  Let g denote the

function with domain $\Gamma$ such that for each $\lambda$ in $\Gamma$, g($\lambda$) is the

first limit point of f(C($\lambda$))* in $\omega_1$.  Further, for each $\lambda$ in $\Gamma$,

using the fact that $f(C(\lambda))$ is increasing, let $C'(\lambda)$ denote a

subset of $C(\lambda)$ such that $f(C'(\lambda))*$ is an infinite subset of the

predecessors of $g(\lambda)$ which has only $g(\lambda)$ as a limit point and

let $K(\lambda)$ denote the subset of $\Gamma$ to which a point $\alpha$ belongs only

if $g(\alpha)$ is $g(\lambda)$. Define $\Gamma'$ to be the set of all $\alpha$ in $\Gamma$ such

that if $\beta$ precedes $\alpha$ in $\Gamma$ then $g(\beta)$ is not $g(\alpha)$. Restricted to

$\Gamma'$, g is one-to-one.

For each $\lambda$ in $\Gamma'$, $K(\lambda)$ is a countable set. To see this

suppose that $\lambda$ is in $\Gamma'$. Denote by P the set of all two ele-

ment subsets of the set of all predecessors of $g(\lambda)$ and let $\beta$

denote the first element in $\omega_1$ which is preceded by each point

in $f^{-1}(P)$. Suppose that $\beta$ precedes $\gamma$ in $\Gamma'$. Then almost all

of $C'(\gamma)$ is preceded by $\beta$ and at most a finite number of two-

element subsets of P are in $f(C'(\gamma))$. Consequently $f(C'(\gamma))*$

contains at most finitely many predecessors of $g(\lambda)$ and $\gamma$ is

not in $K(\lambda)$. It has been shown that $\beta$ cannot precede any

point in $K(\lambda)$ and this establishes that $K(\lambda)$ is a countable

set.

Using this result there is a function C" defined on $\Gamma$ and

there is a function Q defined on $\Gamma'$ such that for each $\lambda$ in $\Gamma'$

and $\alpha$ in $K(\lambda)$, $C"(\alpha)$ is almost all of $C'(\alpha)$ and $Q(\lambda)$ is the

set

$$\{f(C"(\alpha))* \mid \alpha \in K(\lambda)\}*$$

which has only $g(\lambda)$ as a limit point in $\omega_1$.

Now denote by T a function domain $\Lambda$ such that (1) if for

some $\lambda$ in $\Lambda$, $\delta$ is $g(\lambda)$, then $T(\delta)$ is $Q(\lambda)$ and (2) if $\delta$ is in

$\Lambda$ but not in $g(\Lambda)$, then $T(\delta)$ is an arbitrary infinite subset of

the set of predecessors of $\delta$ which has only $\delta$ as a limit point

in $\omega_1$.

T satisfies statements (1) and (2) of this lemma. To see

that T satisfies statement (3) of this lemma, suppose that M

is an uncountable increasing collection of two-element subsets

of $\omega_1$. There is an $\alpha$ in $\Lambda$ such that $C(\alpha)$ is a subset of $f^{-1}(M)$. It follows that $f(C(\alpha))$ is an infinite subset of M which must be increasing since M is. Thus $\alpha$ is in $\Gamma$ and there is a $\lambda$ in $\Gamma'$ such that $\alpha$ is in $K(\lambda)$. Let $\delta$ denote $g(\lambda)$ and suppose that X is almost all of $T(\delta)$. $T(\delta)$ is $Q(\lambda)$ so X contains almost all of $f(C''(\alpha))$* and since $f(C''(\alpha))$ is an infinite increasing sub-set of M, X contains both elements of infinitely many two-element sets in M. Hence T satisfies statement (3) of this lemma, and Lemma 1 is proved.

Let T denote a function which satisfies statements (1), (2) and (3) of Lemma 1.

*Lemma 2. There is a function D such that (1) the domain of D is $\Lambda \times \Lambda$, (2) if each of $\alpha$ and $\gamma$ is in $\Lambda$ then $D(\alpha,\gamma)$ is almost all of $T(\alpha)$, (3) if $\alpha$ precedes $\beta$ in $\Lambda$ and $\beta$ precedes $\gamma$ in $\Lambda$ then no two of $D(\alpha,\gamma)$, $D(\beta,\gamma)$ and $D(\gamma,\gamma)$ intersect and (4) if $\gamma$ precedes $\delta$ in $\Lambda$ then $\gamma$ precedes each point in $D(\delta,\gamma)$.*

*Proof of Lemma 2.* For each $\gamma$ in $\Lambda$ define $D(\gamma,\gamma)$ to be $T(\gamma)$ and let t denote a one-to-one function such that (1) the domain of t is some initial segment of N, (2) the range of t is the set of all points in $\Lambda$ which are not preceded by $\gamma$ and (3) $t(1)$ is $\gamma$. For each $\alpha$ preceding $\gamma$ in $\Lambda$ define $D(\alpha,\gamma)$ to be almost all of $T(\alpha)$ such that if n is in N and n is less than $t^{-1}(\alpha)$, then $D(\alpha,\gamma)$ does not intersect $D(t(n),\gamma)$. For each $\gamma$ preceding $\delta$ in $\Lambda$ define $D(\delta,\gamma)$ to be the set of all points in $T(\delta)$ which are preceded by $\gamma$. The function D which has just been defined satisfies statements (1), (2), (3) and (4) of the lemma, and Lemma 2 is proved.

Let D denote a function which satisfies statements (1) through (4) of Lemma 2.

For each $\gamma$ in $\Lambda$ and for each n in N, a *level n* $\gamma$-*region* is

defined inductively as follows:  R is a level 1 γ-region only
if R is a degenerate subset of level 1; R is a level n+1 γ-
region only if either R is a degenerate subset of level n+1 and
the point in R does not have a first coordinate in Λ or each of
the following four statements holds:  (1) there is a point p in
level n+1 and the first coordinate α of p is in Λ, (2) there is
an X which is almost all of D(α,γ), (3) for each x in X there
is a level n γ-region $R_x$ which contains the point (x,n) and
(4) R is the set to which a point q belongs if and only if q
is p or for some x in X, q is in $R_x$.

   *Lemma 3.  If each of γ and δ is in Λ, n is in N and R is a*
*level n γ-region then (1) there is only one point p of level*
*n which is in R, (2) each point of R different from p lies in a*
*level n-1 γ-region which is a subset of R, (3) each point of R*
*different from p precedes p and is below p (from which it fol-*
*lows that R is countable) and (4) there is a level n δ-region*
*which contains p, is a subset of R and is also a level n γ-region.*

   *Proof of Lemma 3.*  If in the statement of this lemma n is
replaced by 1 then a true statement results.  Suppose that the
statement is true when n is replaced by n-1 and that n is greater
than 1.  Let each of γ and δ denote an element of Λ and let R
denote a level n γ-region.  The statement is clearly true if R
is degenerate, and if R is not degenerate then there is some
point p in level n whose first coordinate α is in Λ, there is an
X which is almost all of D(α,γ), for each x in X there is a
level n-1 γ-region $R_x$ which contains the point (x,n-1) and R is
the set to which a point q belongs if and only if q is p or for
some x in X q is in $R_x$.  If x is in X and q is a point in $R_x$
then q is in level n-1 or in a level below level n-1.  It fol-
lows that p is the only point of R in level n, that each point
of R different from p lies in a level n-1 γ-region which is a

subset of R and that each point in R different from p is below
p.  Suppose that q is a point of R different from p.  Then for
some x in X the first coordinate of q is x or precedes x.  Since
X is a subset of $D(\alpha,\gamma)$, x is in $T(\alpha)$ and x precedes $\alpha$.  It
follows that q precedes p and that statements (1), (2) and (3)
of the lemma are true.

Finally, define X' to be the common part of X and $D(\alpha,\delta)$
and for each x in X' let $R'_x$ denote a level n-1 $\delta$-region contain-
ing (x,n-1) which lies in $R_x$ and which is also a level n-1 $\gamma$-
region.  Define R' to be the set to which a point q belongs if
and only if q is p or for some x in X', q is in $R'_x$.  X' is al-
most all of $D(\alpha,\gamma)$ and X' is almost all of $D(\alpha,\delta)$.  It follows
that R' is a subset of R which contains p and is both a level
n $\gamma$-region and a level n $\delta$-region.  Statement (4) of this
lemma is consequently true and Lemma 3 follows by induction.

If $\gamma$ is in $\Lambda$, n is in N and R is a level n $\gamma$-region, then
the *center* of R is the point p of R which is in level n and R
is called a $\gamma$-*region centered at p* or simply a $\gamma$-*region* or *region*.

*Lemma 4.  The set of regions is a basis for the topology
of a space S on $\omega_1 \times N$.*

*Proof of Lemma 4.*  For each point p of S let $B_p$ denote the
set of all regions centered at p and notice that statements (1),
(2) and (3) which follow hold true.  (1) If p is in S and each
of R and R' is in $B_p$, then there is an R" in $B_p$ which is a sub-
set both of R and R'.  This follows by induction on the level
of p, using Lemma 3.  (2) If p is in S then p is in each region
which is in $B_p$.  (3) If p is in S and q is in a region R which
is in $B_p$, then some region in $B_q$ is a subset of R.  These last
two statements follow directly from definitions and Lemma 3.
Lemma 4 follows from statements (1), (2), (3) and Theorem 4.5
of [9].

*Lemma 5.  If $\gamma$ is in $\Lambda$ and neither of two $\gamma$-regions con-tains the other's center and $\gamma$ does not precede the first co-ordinate of the center of at least one of these two regions, then the two regions do not intersect.*

*Proof of Lemma 5.*  It follows from (3) and (4) in Lemma 2 that if $\alpha$ and $\beta$ are two points in $\Lambda$ and $\gamma$ is a point in $\Lambda$ which does not precede both $\alpha$ and $\beta$ then $D(\alpha,\gamma)$ and $D(\beta,\gamma)$ do not intersect.  Using this fact, the following claim is easily established by induction on the level containing p and q:  If p and q are two points in the same level of S, $\gamma$ is in $\Lambda$ and $\gamma$ does not precede both the first coordinate of p and the first coordinate of q, then no $\gamma$-region centered at p intersects any $\gamma$-region centered at q.  Suppose that Lemma 5 holds for each two $\gamma$-regions whose centers are below level n but fails for some $\gamma$-region $R_p$ with center p in level m less than or equal to n having first coordinate $\alpha$ and for some $\gamma$-region $R_q$ with center q in level n having first coordinate $\beta$.  Then $R_p$ and $R_q$ intersect but neither contains the center of the other.  From the above claim, m is less than n.  Now, (1) there is an X which is almost all of $D(\beta,\gamma)$ and (2) for each x in X there is a $\gamma$-region $R_x$ centered at (x,n-1), such that $R_q$ is the set to which a point r belongs if and only if r is q or, for some x in X, r is in $R_x$. Suppose that x is in X.  By conditions (1) and (3) of Lemma 3, since x is not p and since x is not below p, x is not in $R_p$. p is not in $R_x$ and $\gamma$ does not precede at least one of $\alpha$ and x. Since x and q are both below level n it follows that $R_x$ does not intersect $R_p$ for each x in X.  Thus $R_q$ does not intersect $R_p$.  This contradiction proves Lemma 5.

*Lemma 6.  S is $T_1$ and there is a basis for S which consists of regions each of which is both open and closed in S.*

*Proof of Lemma 6.*  The following claim follows directly by

induction on the level of q:  If p and q are two points in S and
$\delta$ is in $\Lambda$ then some $\delta$-region centered at q does not contain p.
Let B be the set to which a region R of S belongs if and only
if there is a point p in S and there is a $\delta$ in $\omega_1$ which does
not precede the first coordinate of p and R is a $\delta$-region cen-
tered at p.  Lemma 4 and (4) of Lemma 3 ensures that B is a
basis for S and it follows from the above claim and Lemma 5
that the regions in B are closed.  Lemma 6 is proved.

*Lemma 7.  If M is an uncountable subset of some level of
S then each higher level contains uncountably many limit points
of M.  Hence S is $\omega_1$-compact.*

*Proof of Lemma 7.*  This Lemma follows quickly from the
following claim:  If M is an uncountable subset of level n, and
$\alpha$ is in $\omega_1$, then M has a limit point in level n+1 whose first
coordinate follows $\alpha$.  To prove this claim let M' denote the
set of all points of M whose first coordinate follows $\alpha$.  By
Lemma 1 there is a $\lambda$ in $\Lambda$ such that if X is almost all of T($\lambda$),
then X × {n} contains a point of M'.  Let p denote the point
($\lambda$,n+1) and suppose that R is a region centered at p.  Clearly
the first coordinate of p follows $\alpha$.  R contains almost all of
T($\lambda$)×{n} and consequently R contains a point of M'.  It follows
that p is a limit point of M' and hence of M.  The claim is
proved.

*Lemma 8.  S is weakly $\theta$-refinable but not irreducible.*

*Proof of Lemma 8.*  Using Lemma 3 (1), each level L of S
can be covered in a one-to-one fashion by a set of regions
centered in L which refines any other open cover of L.  It
follows from this that S is weakly $\theta$-refinable.  S is an un-
countable space with a basis of countable regions.  By Lemma 7
S is $\omega_1$-compact.  It follows that S cannot be irreducible and
Lemma 8 is proved.

*Lemma 9.  S is collectionwise normal.*

*Proof of Lemma 9.*  Suppose for the sake of contradiction that H and K are two uncountable closed subsets of S which do not intersect.  By Lemma 7 there is an n in N such that both H and K have uncountably many points in level n.  There is an increasing uncountable set M of two-element subsets of $\omega_1$ such that if $\alpha$ precedes $\beta$ in one of these two-element subsets then $(\alpha,n)$ is in H and $(\beta,n)$ is in K.  By Lemma 1 there is a $\lambda$ in $\Lambda$ such that if X is almost all of $T(\lambda) \times \{n\}$, p is a limit point both of H and of K.  This contradiction implies that uncountable closed sets in S intersect.

Suppose that H and K are two closed sets in S which do not intersect and suppose that H is countable.  Let $\lambda$ denote a point in $\Lambda$ which follows the first coordinate of each point in H.  For each point in H choose a $\lambda$-region centered there which does not intersect K and let $O_H$ denote the open set which is their union.  Similarly, for each point in K choose a $\lambda$-region centered there which does not intersect H and let $O_K$ denote the open set which is their union.  $O_H$ contains H and $O_K$ contains K.  By Lemma 5, $O_H$ does not intersect $O_K$.  It follows that S is normal.  By Lemma 7 S is $\omega_1$-compact.  Thus S is collectionwise normal and Lemma 9 is proved.

*Lemma 10.  S is neither countably metacompact nor realcompact.*

*Proof of Lemma 10.*  Let O be the countable open cover of S to which an open set V belongs only if for some level L in S, V is the union of all regions centered in L.  Notice that each element of O intersects only a finite number of levels in S.  Suppose for the sake of contradiction that C is a point-finite refinement of O which covers S.  Then for each point p in the bottom level of S there is an n(p) in N such that if R is an open set in C which contains p then R does not intersect level

n(p). There is a k in N and an uncountable subset M of the
bottom level in S such that if p is in M then n(p) is k.  By
Lemma 7 there is a limit point q of M in level k.  There is an
open set W in C which contains q.  W contains a point of M and
a point in level k.  This contradiction proves that no refine-
ment of O can be a point finite open cover of S.  It follows
that S is not countably metacompact.

Let F denote the set of all sets $F_\lambda$, where for each $\lambda$ in
$\Lambda$, $F_\lambda$ is the set of all points in S whose first coordinate
follows $\lambda$.  By Lemma 3 (3) each element of F is a closed set.
Since each element of F has a countable complement in S, each
such element is a closed $G_\delta$ set.  Let G be the set of all closed
$G_\delta$ sets in S which contain an element of F.  G is a z-filter on
S.  As already observed in the proof of Lemma 9, there do not
exist two uncountable closed subsets of S which do not inter-
sect.  It follows that each uncountable closed $G_\delta$ subset of S
is in G and that G is a z-ultrafilter on S.  Since G has the
countable intersection property, G is a real z-ultrafilter.  But
no point is common to the elements of G so G is not fixed and by
8.1 of [5], S is not realcompact.  Lemma 10 is proved.

The following Theorem is a direct consequence of the above
lemmas:

*Theorem.  S is a collectionwise normal weakly $\Theta$-refinable
Dowker space which is neither irreducible nor realcompact.*

In [7] Proctor constructs a separable pseudonormal Moore
space P which is the disjoint union of a countable set and a
discrete set conumerous with $\omega_1$.  Let M(1), M(2), ... be pair-
wise disjoint countably infinite sets whose union M has no
point in S.  A separable space T having all the properties of S
listed in the above Theorem can be constructed by adding the
points of M to S in such a way that S is a subspace of T and

for each n in N, the union of L(n) and M(n) is a copy of P.

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Atlanta Junior College

Atlanta, Georgia 30310