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1. Introduction

The concept of irreducibility was first used by Arens and Dugundji [1]. Wicke and Worrell [8] introduced Θ-refinability as a generalization of paracompactness and observed that Θ-refinable spaces are irreducible. Lutzer [6] introduced weak Θ-refinable and Boone [2] raised the question: Is every weakly Θ-refinable space irreducible? An answer to this question is given by van Douwen and Wicke [3] who construct without any unusual set theoretic assumptions a regular weakly Θ-refinable space which is neither irreducible nor normal. Our construction is the result of trying to find a normal weakly Θ-refinable space which is not irreducible. We have so far been unable to do so without $\clubsuit$. We are grateful to Professor Robert L. Blair for asking if our space is realcompact. Gardner [4] has shown that if X is a normal weakly Θ-refinable countably paracompact space such that the cardinality of each discrete subspace is of measure zero then X is realcompact. Our example shows that countable paracompactness cannot be dropped from the hypothesis in this statement.

I would like to thank my research supervisor, Professor Jack W. Rogers, Jr., for many helpful discussions.

2. Construction

Axiom $\clubsuit$ is assumed, N denotes the set of positive integers, ω_1 denotes the set of countable ordinals with their usual order topology and Λ denotes the set of limit ordinals in ω_1. If H is a collection of sets then H* denotes the union of all of the elements of H. A collection M of two-element subsets of ω_1 is *increasing* provided that for each two sets in M, each element of one of these two sets precedes each element of the other set. If B is an infinite set then the expression *A is almost all of B* means that A is a subset of B and that there do not exist infinitely many elements of B which are not in A. For each n in N, L(n) denotes $\omega_1 \times \{n\}$ and L(n) is referred to as *level n*. If m and n are in N then *level m is below level n* only if m is less than n. If p and q are two points in $\omega_1 \times$ N then *p is below q* only if the level which contains p is below the level which contains q and *p precedes q* only if the first coordinate of p precedes the first coordinate of q.

Lemma 1. There is a function T *such that* (1) *the domain of* T *is* Λ, (2) *if* λ *is in* Λ *then* T(λ) *is an infinite subset of the predecessors of* λ *and* T(λ) *has only* λ *as a limit point in* ω_1 *and* (3) *if* M *is an uncountable increasing collection of two-element subsets of* ω_1 *then there is a* δ *in* Λ *such that if* X *is almost all of* T(δ) *then some set in* M *is a subset of* X.

Proof of Lemma 1. Let f denote a one-to-one function whose domain is ω_1 and whose range is the set of all two-element subsets of ω_1. Using ♣ let C denote a function such that (1) the domain of C is Λ, (2) if λ is in Λ then C(λ) is an infinite subset of the predecessors of λ and C(λ) has only λ as a limit point in ω_1 and (3) if B is an uncountable subset of ω_1 then for some λ in Λ, C(λ) is a subset of B. Let Γ denote the set of all λ in Λ such that f(C(λ)) is increasing. That Γ is uncountable follows from the fact that if α belongs to ω_1 then there is an uncountable subset H of ω_1 such that f(H) is increasing and each element of H follows α. Let g denote the function with domain Γ such that for each λ in Γ, g(λ) is the first limit point of f(C(λ))* in ω_1. Further, for each λ in Γ, using the fact that $f(C(\lambda))$ is increasing, let $C'(\lambda)$ denote a subset of $C(\lambda)$ such that $f(C'(\lambda))^*$ is an infinite subset of the predecessors of $g(\lambda)$ which has only $g(\lambda)$ as a limit point and let $K(\lambda)$ denote the subset of Γ to which a point α belongs only if $g(\alpha)$ is $g(\lambda)$. Define Γ' to be the set of all α in Γ such that if β precedes α in Γ then $g(\beta)$ is not $g(\alpha)$. Restricted to Γ', g is one-to-one.

For each λ in Γ', $K(\lambda)$ is a countable set. To see this suppose that λ is in Γ'. Denote by P the set of all two element subsets of the set of all predecessors of $g(\lambda)$ and let β denote the first element in ω_1 which is preceded by each point in $f^{-1}(P)$. Suppose that β precedes γ in Γ'. Then almost all of $C'(\gamma)$ is preceded by β and at most a finite number of two-element subsets of P are in $f(C'(\gamma))$. Consequently $f(C'(\gamma))^*$ contains at most finitely many predecessors of $g(\lambda)$ and γ is not in $K(\lambda)$. It has been shown that β cannot precede any point in $K(\lambda)$ and this establishes that $K(\lambda)$ is a countable set.

Using this result there is a function C" defined on Γ and there is a function Q defined on Γ' such that for each λ in Γ' and α in $K(\lambda)$, $C''(\alpha)$ is almost all of $C'(\alpha)$ and $Q(\lambda)$ is the set

$$\{f(C''(\alpha))^* | \alpha \in K(\lambda)\}^*$$

which has only $g(\lambda)$ as a limit point in ω_1.

Now denote by T a function domain Λ such that (1) if for some λ in Λ, δ is $g(\lambda)$, then $T(\delta)$ is $Q(\lambda)$ and (2) if δ is in Λ but not in $g(\Lambda)$, then $T(\delta)$ is an arbitrary infinite subset of the set of predecessors of δ which has only δ as a limit point in ω_1.

T satisfies statements (1) and (2) of this lemma. To see that T satisfies statement (3) of this lemma, suppose that M is an uncountable increasing collection of two-element subsets of ω_1. There is an α in Λ such that $C(\alpha)$ is a subset of $f^{-1}(M)$. It follows that $f(C(\alpha))$ is an infinite subset of M which must be increasing since M is. Thus α is in Γ and there is a λ in Γ' such that α is in $K(\lambda)$. Let δ denote $g(\lambda)$ and suppose that X is almost all of $T(\delta)$. $T(\delta)$ is $Q(\lambda)$ so X contains almost all of $f(C''(\alpha))^*$ and since $f(C''(\alpha))$ is an infinite increasing subset of M, X contains both elements of infinitely many two-element sets in M. Hence T satisfies statement (3) of this lemma, and Lemma 1 is proved.

Let T denote a function which satisfies statements (1), (2) and (3) of Lemma 1.

Lemma 2. There is a function D *such that* (1) *the domain of* D *is* $\Lambda \times \Lambda$, (2) *if each of* α *and* γ *is in* Λ *then* $D(\alpha,\gamma)$ *is almost all of* $T(\alpha)$, (3) *if* α *precedes* β *in* Λ *and* β *precedes* γ *in* Λ *then no two of* $D(\alpha,\gamma)$, $D(\beta,\gamma)$ *and* $D(\gamma,\gamma)$ *intersect and* (4) *if* γ *precedes* δ *in* Λ *then* γ *precedes each point in* $D(\delta,\gamma)$.

Proof of Lemma 2. For each γ in Λ define $D(\gamma,\gamma)$ to be $T(\gamma)$ and let t denote a one-to-one function such that (1) the domain of t is some initial segment of N, (2) the range of t is the set of all points in Λ which are not preceded by γ and (3) $t(1)$ is γ. For each α preceding γ in Λ define $D(\alpha,\gamma)$ to be almost all of $T(\alpha)$ such that if n is in N and n is less than $t^{-1}(\alpha)$, then $D(\alpha,\gamma)$ does not intersect $D(t(n),\gamma)$. For each γ preceding δ in Λ define $D(\delta,\gamma)$ to be the set of all points in $T(\delta)$ which are preceded by γ. The function D which has just been defined satisfies statements (1), (2), (3) and (4) of the lemma, and Lemma 2 is proved.

Let D denote a function which satisfies statements (1) through (4) of Lemma 2.

For each γ in Λ and for each n in N, a *level n* γ*-region* is defined inductively as follows: R is a level 1 γ-region only if R is a degenerate subset of level 1; R is a level n+1 γ-region only if either R is a degenerate subset of level n+1 and the point in R does not have a first coordinate in Λ or each of the following four statements holds: (1) there is a point p in level n+1 and the first coordinate α of p is in Λ, (2) there is an X which is almost all of $D(\alpha,\gamma)$, (3) for each x in X there is a level n γ-region R_x which contains the point (x,n) and (4) R is the set to which a point q belongs if and only if q is p or for some x in X, q is in R_x.

Lemma 3. If each of γ *and* δ *is in* Λ, n *is in* N *and* R *is a level* n γ*-region then* (1) *there is only one point* p *of level* n *which is in* R, (2) *each point of* R *different from* p *lies in a level* n-1 γ*-region which is a subset of* R, (3) *each point of* R *different from* p *precedes* p *and is below* p *(from which it follows that* R *is countable) and* (4) *there is a level* n δ*-region which contains* p, *is a subset of* R *and is also a level* n γ*-region.*

Proof of Lemma 3. If in the statement of this lemma n is replaced by 1 then a true statement results. Suppose that the statement is true when n is replaced by n-1 and that n is greater than 1. Let each of γ and δ denote an element of Λ and let R denote a level n γ-region. The statement is clearly true if R is degenerate, and if R is not degenerate then there is some point p in level n whose first coordinate α is in Λ, there is an X which is almost all of $D(\alpha,\gamma)$, for each x in X there is a level n-1 γ-region R_x which contains the point (x,n-1) and R is the set to which a point q belongs if and only if q is p or for some x in X q is in R_x. If x is in X and q is a point in R_x then q is in level n-1 or in a level below level n-1. It follows that p is the only point of R in level n, that each point of R different from p lies in a level n-1 γ-region which is a subset of R and that each point in R different from p is below p. Suppose that q is a point of R different from p. Then for some x in X the first coordinate of q is x or precedes x. Since X is a subset of $D(\alpha,\gamma)$, x is in $T(\alpha)$ and x precedes α. It follows that q precedes p and that statements (1), (2) and (3) of the lemma are true.

Finally, define X' to be the common part of X and $D(\alpha,\delta)$ and for each x in X' let R'_x denote a level n-1 δ-region containing (x,n-1) which lies in R_x and which is also a level n-1 γ-region. Define R' to be the set to which a point q belongs if and only if q is p or for some x in X', q is in R'_x. X' is almost all of $D(\alpha,\gamma)$ and X' is almost all of $D(\alpha,\delta)$. It follows that R' is a subset of R which contains p and is both a level n γ-region and a level n δ-region. Statement (4) of this lemma is consequently true and Lemma 3 follows by induction.

If γ is in Λ, n is in N and R is a level n γ-region, then the *center* of R is the point p of R which is in level n and R is called a *γ-region centered at p* or simply a *γ-region* or *region*.

Lemma 4. The set of regions is a basis for the topology of a space S *on* $\omega_1 \times$ N.

Proof of Lemma 4. For each point p of S let B_p denote the set of all regions centered at p and notice that statements (1), (2) and (3) which follow hold true. (1) If p is in S and each of R and R' is in B_p, then there is an R" in B_p which is a subset both of R and R'. This follows by induction on the level of p, using Lemma 3. (2) If p is in S then p is in each region which is in B_p. (3) If p is in S and q is in a region R which is in B_p, then some region in B_q is a subset of R. These last two statements follow directly from definitions and Lemma 3. Lemma 4 follows from statements (1), (2), (3) and Theorem 4.5 of [9].

Lemma 5. If γ is in Λ and neither of two γ-regions contains the other's center and γ does not precede the first coordinate of the center of at least one of these two regions, then the two regions do not intersect.

Proof of Lemma 5. It follows from (3) and (4) in Lemma 2 that if α and β are two points in Λ and γ is a point in Λ which does not precede both α and β then $D(\alpha,\gamma)$ and $D(\beta,\gamma)$ do not intersect. Using this fact, the following claim is easily established by induction on the level containing p and q: If p and q are two points in the same level of S, γ is in Λ and γ does not precede both the first coordinate of p and the first coordinate of q, then no γ-region centered at p intersects any γ-region centered at q. Suppose that Lemma 5 holds for each two γ-regions whose centers are below level n but fails for some γ-region R_p with center p in level m less than or equal to n having first coordinate α and for some γ-region R_q with center q in level n having first coordinate β. Then R_p and R_q intersect but neither contains the center of the other. From the above claim, m is less than n. Now, (1) there is an X which is almost all of $D(\beta,\gamma)$ and (2) for each x in X there is a γ-region R_x centered at (x,n-1), such that R_q is the set to which a point r belongs if and only if r is q or, for some x in X, r is in R_x. Suppose that x is in X. By conditions (1) and (3) of Lemma 3, since x is not p and since x is not below p, x is not in R_p. p is not in R_x and γ does not precede at least one of α and x. Since x and q are both below level n it follows that R_x does not intersect R_p for each x in X. Thus R_q does not intersect R_p. This contradiction proves Lemma 5.

Lemma 6. S *is* T_1 *and there is a basis for* S *which consists of regions each of which is both open and closed in* S.

Proof of Lemma 6. The following claim follows directly by induction on the level of q: If p and q are two points in S and δ is in Λ then some δ-region centered at q does not contain p. Let B be the set to which a region R of S belongs if and only if there is a point p in S and there is a δ in ω_1 which does not precede the first coordinate of p and R is a δ-region centered at p. Lemma 4 and (4) of Lemma 3 ensures that B is a basis for S and it follows from the above claim and Lemma 5 that the regions in B are closed. Lemma 6 is proved.

Lemma 7. If M *is an uncountable subset of some level of* S *then each higher level contains uncountably many limit points of* M. *Hence* S *is* ω_1*-compact.*

Proof of Lemma 7. This Lemma follows quickly from the following claim: If M is an uncountable subset of level n, and α is in ω_1, then M has a limit point in level n+1 whose first coordinate follows α. To prove this claim let M' denote the set of all points of M whose first coordinate follows α. By Lemma 1 there is a λ in Λ such that if X is almost all of $T(\lambda)$, then $X \times \{n\}$ contains a point of M'. Let p denote the point $(\lambda, n+1)$ and suppose that R is a region centered at p. Clearly the first coordinate of p follows α. R contains almost all of $T(\lambda) \times \{n\}$ and consequently R contains a point of M'. It follows that p is a limit point of M' and hence of M. The claim is proved.

Lemma 8. S *is weakly* Θ*-refinable but not irreducible.*

Proof of Lemma 8. Using Lemma 3 (1), each level L of S can be covered in a one-to-one fashion by a set of regions centered in L which refines any other open cover of L. It follows from this that S is weakly Θ-refinable. S is an uncountable space with a basis of countable regions. By Lemma 7 S is ω_1-compact. It follows that S cannot be irreducible and Lemma 8 is proved.

Lemma 9. S *is collectionwise normal.*

Proof of Lemma 9. Suppose for the sake of contradiction that H and K are two uncountable closed subsets of S which do not intersect. By Lemma 7 there is an n in N such that both H and K have uncountably many points in level n. There is an increasing uncountable set M of two-element subsets of ω_1 such that if α precedes β in one of these two-element subsets then (α,n) is in H and (β,n) is in K. By Lemma 1 there is a λ in Λ such that if X is almost all of $T(\lambda)\times\{n\}$, p is a limit point both of H and of K. This contradiction implies that uncountable closed sets in S intersect.

Suppose that H and K are two closed sets in S which do not intersect and suppose that H is countable. Let λ denote a point in Λ which follows the first coordinate of each point in H. For each point in H choose a λ-region centered there which does not intersect K and let O_H denote the open set which is their union. Similarly, for each point in K choose a λ-region centered there which does not intersect H and let O_K denote the open set which is their union. O_H contains H and O_K contains K. By Lemma 5, O_H does not intersect O_K. It follows that S is normal. By Lemma 7 S is ω_1-compact. Thus S is collectionwise normal and Lemma 9 is proved.

Lemma 10. S *is neither countably metacompact nor realcompact.*

Proof of Lemma 10. Let O be the countable open cover of S to which an open set V belongs only if for some level L in S, V is the union of all regions centered in L. Notice that each element of O intersects only a finite number of levels in S. Suppose for the sake of contradiction that C is a point-finite refinement of O which covers S. Then for each point p in the bottom level of S there is an n(p) in N such that if R is an open set in C which contains p then R does not intersect level

$n(p)$. There is a k in N and an uncountable subset M of the bottom level in S such that if p is in M then $n(p)$ is k. By Lemma 7 there is a limit point q of M in level k. There is an open set W in C which contains q. W contains a point of M and a point in level k. This contradiction proves that no refinement of O can be a point finite open cover of S. It follows that S is not countably metacompact.

Let F denote the set of all sets F_λ, where for each λ in Λ, F_λ is the set of all points in S whose first coordinate follows λ. By Lemma 3 (3) each element of F is a closed set. Since each element of F has a countable complement in S, each such element is a closed G_δ set. Let G be the set of all closed G_δ sets in S which contain an element of F. G is a z-filter on S. As already observed in the proof of Lemma 9, there do not exist two uncountable closed subsets of S which do not intersect. It follows that each uncountable closed G_δ subset of S is in G and that G is a z-ultrafilter on S. Since G has the countable intersection property, G is a real z-ultrafilter. But no point is common to the elements of G so G is not fixed and by 8.1 of [5], S is not realcompact. Lemma 10 is proved.

The following Theorem is a direct consequence of the above lemmas:

Theorem. S *is a collectionwise normal weakly* Θ*-refinable Dowker space which is neither irreducible nor realcompact.*

In [7] Proctor constructs a separable pseudonormal Moore space P which is the disjoint union of a countable set and a discrete set conumerous with ω_1. Let $M(1)$, $M(2)$, ... be pairwise disjoint countably infinite sets whose union M has no point in S. A separable space T having all the properties of S listed in the above Theorem can be constructed by adding the points of M to S in such a way that S is a subspace of T and for each n in N, the union of L(n) and M(n) is a copy of P.

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