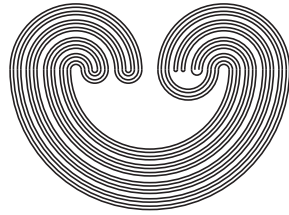

TOPOLOGY PROCEEDINGS



Volume 1, 1976

Pages 107–114

<http://topology.auburn.edu/tp/>

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ISSN: 0146-4124

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CONTINUA AND CONTINUOUS TRANSFORMATIONS

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This lecture¹ is designed to be a survey of some results, new and not so new, in Continua Theory with an emphasis on classification problems and special properties of continuous mappings. The selection of topics could, as usual, be highly subjective, reflecting the author's inclinations and tastes. Nevertheless, an effort has been made to select those themes which either seem to be significant or offer possibilities of further development.

1. Transformations

The continuity of a transformation is a condition which deals with closed subsets of the range space and the closedness of their preimages in the domain space. Studying continua, however, requires a combination of two properties: closedness and connectedness. A direct analogue of continuity here makes an obviously too restrictive condition, and the way to obtain interesting useful classes of transformations is only by somewhat relaxing the connectedness of the preimages of connected closed subsets. Recall that a topological space X is said to be *connected between* two subsets $A, B \subset X$ provided $X \neq U \cup V$, where $U, V \subset X$ are open, $U \cap V = \emptyset$ and $A \subset U$, $B \subset V$. Let $f: X \rightarrow Y$ be a continuous mapping of a topological space X onto a topological space Y . The mapping f is called *confluent* [5], *weakly confluent*, or *pseudo-confluent* [8] provided, for each connected closed non-empty set $C \subset Y$, the following conditions are satisfied, respectively:

¹The Third Annual Ralph B. Bennett Memorial Lecture delivered at the Auburn University Topology Conference on March 18, 1976.

(c) for each pair of points $x \in f^{-1}(C)$ and $y \in C$, the set $f^{-1}(C)$ is connected between $\{x\}$ and $f^{-1}(y)$;

(w) there exists a point $x_0 \in f^{-1}(C)$ such that, for each point $y \in C$, the set $f^{-1}(C)$ is connected between $\{x_0\}$ and $f^{-1}(y)$;

(p) for each pair of points $y, y' \in C$, the set $f^{-1}(C)$ is connected between $f^{-1}(y)$ and $f^{-1}(y')$.

Clearly, (c) implies (w), and (w) implies (p). If $f^{-1}(y)$ is compact for each $y \in Y$, then conditions (c), (w), (p) are equivalent to the following conditions, respectively:

(c') for each quasi-component Q of $f^{-1}(C)$, we have $C = f(Q)$;

(w') there exists a quasi-component Q of $f^{-1}(C)$ such that $C = f(Q)$;

(p') for each pair of points $y, y' \in C$, there exists a quasi-component Q of $f^{-1}(C)$ such that $y, y' \in f(Q)$.

Moreover, if X is compact, then "quasi-component" in (c'), (w'), (p') can be replaced by "component." By a *continuum* we understand a connected compact metric space.

Theorem 1 (Nadler [14]). *If X and Y are continua, then the space of confluent mappings of X onto Y with the uniform convergence topology is topologically complete.*

In [14], it is proved that the space of confluent mappings of X onto Y is a G_δ -subset of the space of weakly confluent mappings of X onto Y , and that the latter space is a closed subspace of the space of all continuous mappings of X onto Y with the uniform convergence topology. Also, the space of confluent mappings is not necessarily closed in the space of all continuous mappings. It is closed, however, if some conditions are imposed on Y , e.g., if Y is locally connected [9] or, more generally, if Y has Kelley's property κ [14]. The space of pseudo-confluent mappings of X onto Y is always a closed subspace of the space Y^X of all continuous mappings [9].

2. Classifications

The best known is a classification of continua by means of ϵ -mappings. Let $\epsilon > 0$ be a number. An ϵ -mapping is a continuous mapping whose point-inverses all have diameters less than ϵ . Given a class Σ of topological spaces, a class of continua, called Σ -like continua, is introduced by taking all continua X such that, for each number $\epsilon > 0$, there exists an ϵ -mapping of X onto an element of Σ . If Σ is the class of all trees (that is, spaces homeomorphic to 1-dimensional acyclic connected polyhedra) or the class of all arcs, we get *tree-like* continua or *arc-like* continua, respectively. The easiest example of a tree which is not an arc is, of course, a *simple triod*, i.e., the union of three arcs having a common end-point and pairwise disjoint outside that point. Then arcs are exactly those trees which do not contain any simple triods. It turns out, however, that the situation is quite different if arcs and trees are generalized to arc-like continua and tree-like continua, respectively. By a *trioid* we mean a continuum which is the union of three continua having a proper subcontinuum as their common part and pairwise disjoint outside that subcontinuum. A continuum is called *atriodic* provided it does not contain any triod.

Theorem 2 (Ingram [4]). There exists an uncountable collection of pairwise disjoint atriodic tree-like continua on the plane such that none of them is arc-like.²

Theorem 3 (McLean [12]). Each confluent image of a tree-like continuum is tree-like.

Problem 1. Is each confluent image of an arc-like

²Professor W. T. Ingram in his talk presented at this conference gives some further intriguing properties of the same collection of continua.

continuum arc-like? [6]

In connection with Problem 1, we mention another unsolved problem concerning products of mappings. If $f_i: X_i \rightarrow Y_i$ are mappings ($i = 1, 2$), we write $f_1 \times f_2$ to denote the mapping of $X_1 \times X_2$ into $Y_1 \times Y_2$ defined by

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)) \quad (x_i \in X_i; i = 1, 2).$$

Problem 2. Suppose $f_i: X_i \rightarrow Y_i$ are confluent mappings and X_i are arc-like continua ($i = 1, 2$). Is the product $f_1 \times f_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ confluent?

The assumption that X_i are arc-like is essential in Problem 2. Maćkowiak [10] has an example of a confluent mapping of a (non-arc-like) continuum whose product with the identity mapping of an arc is not pseudo-confluent. We say that a continuum is *rational* provided it admits a basis of open sets with countable boundaries.

Problem 3. Is each confluent image of a rational continuum rational?³ [7]

Weakly confluent mappings can destroy the tree-likeness, even of an arc. Without some local conditions like rationality, they can also destroy the atriodicity. For instance, by [3], each dendrite is a weakly confluent image of any continuum which contains a copy of the product of an arc and the Cantor set. A continuum is called *acyclic* provided each continuous mapping of it into the circle is homotopic to a constant mapping.

³Professor E. D. Tymchatyn in a letter to the author, dated March 23, 1976, outlines a construction of a rational continuum on the plane which seems to be a good candidate for a counterexample to Problem 3. His example is very much non-atriodic and non-acyclic. Perhaps then, confluent mappings would preserve the rationality of continua if the continua were assumed to be acyclic or, say, arc-like as in Problem 2.

Theorem 4 (Cook; see [8]). Each pseudo-confluent image of a 1-dimensional acyclic continuum is at most 1-dimensional.

Theorem 5 (Lelek [5]). Each confluent image of an acyclic continuum is acyclic.

There are two more methods of classification of continua. The first method introduces classes of continua by means of the existence of continuous mappings of certain objects. The second method does it by means of the existence of some mappings onto certain objects. The class of locally connected continua is characterized by the first method, namely, these continua coincide with continuous images of the arc. Continuous images of the pseudo-arc have also been investigated. We quote a few theorems which have some bearing on the second method. We say that a continuum is *Suslinian* provided each collection of pairwise disjoint non-degenerate subcontinua of it is countable. Obviously, each rational continuum is Suslinian, but there exist Suslinian continua that are not rational.

Theorem 6 (Cook and Lelek [3]). A continuum X is atriodic and Suslinian if and only if no mapping of X onto a simple triod is weakly confluent.

Theorem 7 (Rogers [15]). Each indecomposable continuum can be mapped continuously onto Knaster's indecomposable continuum K .

Theorem 8 (Bellamy [1]). Each hereditarily indecomposable continuum can be mapped continuously onto the pseudo-arc.

The continuum K in Theorem 7 is a well-known standard example of an indecomposable arc-like continuum. One component of K is a one-to-one continuous image of a half-open segment of

the real line, and the remaining composants are one-to-one continuous images of an open segment. A class of indecomposable arc-like continua is described in [15] with the property that each of its elements is a continuous image of each indecomposable continuum, and K belongs to this class. Again, a class of arc-like continua is described in [1] with the property that each of its elements is a continuous image of each hereditarily indecomposable continuum, and the pseudo-arc belongs to it.

3. Three results

The following results are particularly interesting because of the techniques used in their proofs. By a λ -dendroid we understand a hereditarily decomposable tree-like continuum.

Theorem 9 (Mañka [11]). *If X is a λ -dendroid and $F: X \rightarrow X$ is an upper semicontinuous continuum-valued function, then F has a fixed point, i.e., there exists a point $x_0 \in X$ such that $x_0 \in F(x_0)$.*

The proof of Theorem 9 makes use extensively of the theory of irreducible continua.

Theorem 10 (Mohler [13]). *If X is a 1-arcwise connected continuum and $h: X \rightarrow X$ is a homeomorphism, then h has a fixed point.*

In this theorem, measure-theoretic techniques are used in the proof.

Returning to Knaster's indecomposable continuum K , there is still an open question of whether the "invisible" composants of K are homeomorphic to one another. (The "invisible" composants of K are those which are one-to-one continuous images of an open segment.) A partial solution is given by Bellamy [2] who provides a classification of composants of K and shows, among other

things, that, with an exception of two composants, all other ones are homeomorphic with some composants different from them, under powers of a standard homeomorphism of K onto itself. Confluent mappings and some group-theoretic techniques are used in [2].

References

- [1] D. P. Bellamy, *Mapping hereditarily indecomposable continua onto a pseudo-arc*, Topology Conference, Virginia Polytech. Inst. and State Univ., Blacksburg, Va., 1973, 6-14; Lecture Notes in Math., vol. 375, Springer-Verlag, Berlin, 1974.
- [2] _____, *Homeomorphisms of composants*, to appear.
- [3] H. Cook and A. Lelek, *Weakly confluent mappings and atriodic Suslinian curves*, to appear.
- [4] W. T. Ingram, *An uncountable collection of mutually exclusive planar atriodic tree-like continua with positive span*, Fund. Math. 85 (1974), 73-78.
- [5] A. Lelek, *On confluent mappings*, Colloq. Math. 15 (1966), 223-233.
- [6] _____, *Some problems concerning curves*, Colloq. Math. 23 (1971), 93-98.
- [7] _____, *Several problems of continua theory*, Topology Conference, Univ. of North Carolina, Charlotte, N.C., 1974, 325-329; Studies in Topology, Academic Press, New York, 1975.
- [8] _____ and E. D. Tymchatyn, *Pseudo-confluent mappings and a classification of continua*, Canad. J. Math. 27 (1975), 1336-1348.
- [9] T. Maćkowiak, *On the subsets of confluent and related mappings of the space Y^X* , to appear.
- [10] _____, *The product of confluent and locally confluent mappings*, to appear.
- [11] R. Mañka, *Association and fixed points*, Fund. Math. 91(1976), 105-121.
- [12] T. B. McLean, *Confluent images of tree-like curves are tree-like*, Duke Math. J. 39 (1972), 465-473.
- [13] L. Mohler, *The fixed point property for homeomorphisms of 1-arcwise connected continua*, Proc. Amer. Math. Soc. 52 (1975), 451-456.
- [14] S. B. Nadler, Jr., *Concerning completeness of the space of*

confluent mappings, Houston J. Math. 2 (1976), 561-580.

- [15] J. W. Rogers, Jr., *On mapping indecomposable continua onto certain chainable indecomposable continua*, Proc. Amer. Math. Soc. 25 (1970), 449-456.

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