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If $\{X_n: n \in \omega\}$ is a family of spaces, $\Box_{n\in\omega} X_n$, called the box product of those spaces, denotes the Cartesian product of the sets with the topology generated by all sets of the form $\Pi_{n\in\omega} G_n$, where G_n need only be open in each factor space X_n. If $X_n = X \; \forall \; n \in \omega$, we denote $\Box_{n\in\omega} X_n$ by $\Box^{\omega}X$.

Box products have generated considerable interest during the past ten years, as first as "counter-example producing machines," later, as mathematical objects in their own right.[1] Yet, except for a few surprising counter-examples there have been no non-trivial absolute results. As corollaries to more general results, M. E. Rudin and K. Kunen have proved that if the Continuum Hypothesis (CH) is assumed, then $\Box^{\omega}(\omega_1+1)$ is paracompact; however, in [6,8] they question what occurs when CH is false. Kunen [6] has proved that if Martin's Axiom (MA) is assumed, then $\Box_{n\in\omega} X_n$ is paracompact whenever each X_n is compact first countable; however, as stated in [2], the really interesting case occurs when $\Box^{\omega}(\omega+1)$ when both CH and MA + $\neg$CH fail, as they do in the "random real" models of Solovay [10]. We prove:

Theorem 1: If $\Box^{\omega}(\alpha+1)$ *is paracompact* $\forall \; \alpha < \omega_1$, *then* $\Box^{\omega}(\omega_1+1)$ *is paracompact.*

Theorem 2: If there exists a λ*-scale in* ${}^{\omega}\omega$, *then* $\Box^{\omega}(\omega+1)$ *is paracompact.*

Suppose that for each $n \in \omega$ X_n is a set, then for each

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[1] "Box Products" is the title of Chapter X of [9] where all the results attributed by this author to others may be found, if not referenced here.

$x \in \prod_{n\in\omega} X_n$,

$$\overline{x} = \{y \in \prod_{n\in\omega} X_n : \exists\, m \in \omega \ni n > m \Rightarrow y(n) = x(n)\}$$

defines an equivalence relation on X and the ensuing quotient set is denoted by $\nabla_{n\in\omega} X_n$ and called the *reduced Frechét product* [6]. If $\nabla_{n\in\omega} X_n$ is given the quotient topology from $\square_{n\in\omega} X_n$, then G_δ-sets are open; therefore, $\nabla_{n\in\omega} X_n$ is paracompact if, and only if, every open cover has a pairwise-disjoint open refinement. Kunen first observed [6] that when each X_n is compact, $\square_{n\in\omega} X_n$ is paracompact if, and only if, $\nabla_{n\in\omega} X_n$ is paracompact.

Proof of Theorem 1:

We suppose $\mathcal{F}$ is a basic open covering of $\nabla^\omega(\omega_1+1)$. For each $\alpha < \omega_1$ and $A \subseteq \omega$ define

$$A(\alpha)(n) = \begin{cases} [\alpha+1, \omega_1] & \text{if} \quad n \in A \\ [0, \alpha] & \text{if} \quad n \notin A, \end{cases}$$

$A(\alpha) = \prod_{n\in\omega} A(\alpha)(n)$, and $\overline{A(\alpha)} = \{\overline{x} : x \in A(\alpha)\}$. The sets $\overline{A(\alpha)}$ are clopen and form a partition of $\nabla^\omega(\omega_1+1)$ since $\overline{A(\alpha)} \neq \overline{B(\alpha)}$ iff $(A - B) \cup (B - A)$ is infinite.

We construct for each $\alpha < \omega_1$ a collection $\mathcal{F}(\alpha)$ satisfying

(1) $G \in \mathcal{F}(\alpha) \Rightarrow G$ is clopen and contained in a member of $\mathcal{F}$,

(2) $\bigcup \mathcal{F}(\alpha)$ is clopen and $\mathcal{F}(\alpha)$ is a pairwise disjoint collection,

(3) $\beta < \alpha < \omega_1 \Rightarrow \mathcal{F}(\beta) \subseteq \mathcal{F}(\alpha)$,

(4) $\bigcup\{\mathcal{F}(\alpha) : \alpha < \omega_1\}$ is a cover of $\nabla^\omega(\omega_1+1)$.

There is a first $\lambda \in \omega_1$ such that $\overline{\omega(\lambda)}$ is contained in an element of $\mathcal{F}$, let $\mathcal{F}(0) = \{\overline{\omega(\lambda)}\}$ and suppose that for $\alpha < \omega_1$ we have constructed $\mathcal{F}(\beta)\ \forall\ \beta < \alpha$ to satisfy (1), (2), and (3). If α is a limit ordinal, then let

$$\mathcal{F}(\alpha) = \bigcup\{\mathcal{F}(\beta) : \beta < \alpha\}.$$

If α is a non-limit ordinal, suppose $A \subseteq \omega$ and let

$$T(A) = \{\overline{y} \in \overline{A(\alpha)} : y^{-1}(\omega_1) = A\}.$$

Since $T(A)$ is homeomorphic to $\nabla^{\omega}(\alpha+1)$[2] we may find a pairwise disjoint basic open covering $\mathcal{S}(A)$ of $T(A)$ to satisfy

(i) $\overline{W} \in \mathcal{S}(A)$, $n, m \in A \Rightarrow \inf W(n) = \inf W(m)$ is a successor ordinal $> \alpha + 1$.

(ii) $\overline{W} \in \mathcal{S}(A) \Rightarrow \exists G \in \mathcal{F} \ni \overline{W} \subseteq G$.

By choosing only one representative A for each equivalence class $\overline{A(\alpha)}$, we let

$$\mathcal{F}(\alpha) = \mathcal{F}(\alpha-1) \cup \{\overline{W} - \cup \mathcal{F}(\alpha-1): \overline{W} \in \mathcal{S}(A),\ A \subseteq \omega\}.$$

In order to show $\mathcal{F}(\alpha)$ satisfies (1), (2), and (3) we need only show $\cup \mathcal{S}(A)$ is closed for each $A \subseteq \omega$. So we suppose

$$\overline{x} \in \overline{A(\alpha)} - \cup \mathcal{S}(A)$$

and $\overline{y} \in T(A)$ such that

$$y(n) = \begin{cases} x(n) & \text{if} \quad n \notin A \\ \omega_1 & \text{if} \quad n \in A. \end{cases}$$

Now choose $\overline{W} \in \mathcal{S}(A)$ such that $y \in W$ and define

$$V_x(n) = \begin{cases} W(n) & \text{if} \quad x(n) \in W(n) \\ [\alpha+1, \inf W(n)) & \text{if} \quad x(n) \notin W(n). \end{cases}$$

From (i) $\overline{x} \in \overline{V}_x \subseteq \overline{A(\alpha)}$; moreover, if $\overline{U} \in \mathcal{S}(A)$ and $\overline{U} \neq \overline{W}$, then we may assume

$$\left(\prod_{n\in\omega-A} U(n) \cap \left(\prod_{n\in\omega-A} W(n)\right)\right) = \emptyset.$$

Thus, $\overline{U} \cap \overline{V}_x = \emptyset$. Clearly, $\overline{A(\alpha)} - \cup \mathcal{S}(A)$ is open and our induction is completed.

To see (4) we observe that $\overline{x} \in \nabla^{\omega}(\omega_1+1) \Rightarrow$ either $\overline{x} = \overline{\omega}_1$ or $\exists$ a first $\alpha \ni$

$$\alpha > \sup\{x(n): x(n) \neq \omega_1\}.$$

In the first case $\overline{x} \in \cup \mathcal{F}(0)$, and in the second case $\overline{x} \in \cup \mathcal{F}(\alpha)$. Therefore, our proof is complete.

If λ is an ordinal, a λ-*scale* in ${}^{\omega}\omega$ is an order-preserving injection $\Psi: \lambda \to {}^{\omega}\omega \ni$ given any $x \in {}^{\omega}\omega$ $\exists$ $\alpha < \lambda$ with $x(n) < \Psi(\alpha)(n)$ for all but finitely many $n \in \omega$. It should be clear that there

[2] $T(A)$ may actually be a singleton; however, this causes no disturbance.

can be no ω-scales in ${}^{\omega}\omega$; however, it is a fact, probably due to Hausdorff, that

$$\mathrm{CH} \Rightarrow \exists \text{ an } \omega_1\text{-scale in } {}^{\omega}\omega.$$

However, in the random real models for $\neg$CH, with the ground model "satisfying" CH, there is an ω_1-scale in ${}^{\omega}\omega$ [4]. Booth's theorem [9, pg. 40] says

$$\mathrm{MA} \Rightarrow \exists \text{ a } 2^{\omega}\text{-scale in } {}^{\omega}\omega.$$

In Cohen's original model for $\neg$CH there is no λ-scale in ${}^{\omega}\omega$. In [4] S. Hechler has shown that given cardinals λ and $\aleph$ and a model M of ZFC in which

$$\omega < \mathrm{cf}(\lambda) \leq \lambda \leq \min(2^{\omega}, \mathrm{cf}(\aleph))$$

then one can "extend" M to a model N in which $\aleph = 2^{\omega}$ and ${}^{\omega}\omega$ has a λ-scale.

van Douwen [1] and Hechler [3] have examined a number of topological cardinal functions which are implied by or are equivalent to the existence of a λ-scale. Kunen [5] proved

(a) $\exists\lambda$-scale in ${}^{\omega}\omega \Rightarrow \lambda \times \square^{\omega}(\omega+1)$ is not normal,

(b) $\exists 2^{\omega}$-scale in ${}^{\omega}\omega \Rightarrow \lambda \times \square^{\omega}(\omega+1)$ is normal for any ordinal λ such that $\mathrm{cf}(\lambda) \neq 2^{\omega}$.

Recall [7] that a space Y is λ-*metrizable* for an ordinal λ, $\mathrm{cf}(\lambda) > \omega$, whenever each $y \in Y$ has a local base $\{B(y,\alpha): \alpha < \lambda\}$ satisfying

(i) $\beta < \alpha \Rightarrow B(y,\alpha) \subseteq B(y,\beta)$

(ii) $y \in B(z,\alpha) \Rightarrow z \in B(y,\alpha)$

(iii) $y \in B(z,\alpha) \Rightarrow B(y,\alpha) \subseteq B(z,\alpha)$.

It is well known that λ-metrizable spaces are paracompact.

Our original proof of Theorem 2, presented during this conference, was similar to the proof of Theorem 1 and made use of:

If there is a λ-scale in ${}^{\omega}\omega$, then the intersection of less than $\mathrm{cf}(\lambda)$ open sets of $\nabla^{\omega}(\omega+1)$ is open.

We give thanks to Brian Scott who has provided us with the "if" part of the Lemma from which our theorem 2 is immediate.

Proof of Theorem 2:

Lemma: Let λ be a regular cardinal. Then $\nabla^{\omega}(\omega+1)$ is λ-metrizable if, and only if, there is a λ-scale in ${}^{\omega}\omega$.

Proof: Suppose $\{B_\alpha: \alpha < \lambda\}$ is a well-ordered decreasing local base at $\overline{\omega}$. It is easy to find

$$\{G_\alpha: \alpha < \lambda\} \subseteq \{B_\alpha: \alpha < \lambda\} \text{ and } \{x_\alpha: \alpha < \lambda\} \subseteq {}^{\omega}\omega.$$

such that whenever $\alpha < \beta < \lambda$,

$$G_\beta \subseteq \overline{\prod_{n\in\omega} [x_\beta(n), \omega]} \subseteq G_\alpha, \text{ and } \{G_\alpha: \alpha < \lambda\} \text{ is a local base at } \overline{\omega}.$$

If $\Psi(\alpha) = x_\alpha$, then $\Psi: \lambda \to {}^{\omega}\omega$ is a λ-scale in ${}^{\omega}\omega$.

Conversely, suppose $\Psi: \lambda \to {}^{\omega}\omega$ is a λ-scale in ${}^{\omega}\omega$. For each $\overline{x} \in \nabla^{\omega}(\omega+1)$, let $d(\overline{x},\overline{x}) = \lambda$, and if $\overline{y} \neq \overline{x}$, let

$$d(\overline{x},\overline{y}) = \inf\{\alpha < \lambda: |\{n \in \omega: \inf(x(n),y(n)) \leq \Psi(\alpha)(n) \text{ and } x(n) \neq y(n)\}| = \omega\}.$$

We see that $d: \nabla^{\omega}(\omega+1) \times \nabla^{\omega}(\omega+1) \to \lambda + 1$ satisfies the criterion of [7, Theorem 4.8(B)], and hence $\nabla^{\omega}(\omega+1)$ is λ-metrizable.

The previous lemma establishes that the λ-metrizability of $\nabla^{\omega}(\omega+1)$ is independent of the axioms of ZFC whenever $cf(\lambda) > \omega$. In answer to one of the questions we presented at this conference, Eric van Douwen has recently shown[3] that $\nabla^{\omega}(\omega+1)$ in the previous lemma may be replaced by $\nabla_{n\in\omega} X_n$, whenever each X_n is a compact metrizable space. In answer to another of our questions, Judith Roitman has proved:

In a model of set theory which is an iterated CCC extension of length λ, $cf(\lambda) > \omega \Rightarrow \nabla_{n\in\omega} X_n$ is paracompact if each X_n is regular and separable. Furthermore, if λ is regular and $\lambda \geq 2^{\omega}$ in the ground model, then $\nabla_{n\in\omega} X_n$ is paracompact whenever each X_n

[3]Presented at the Ohio University Conference on Topology, May 1976.

is compact first countable.

The following questions are outstanding:

1. Is $\square^{\omega}(\omega+1)$ always paracompact or normal?
2. Is $\square^{\omega_1}(\omega+1)$ normal in any model of ZFC?
3. Can there be a normal non-paracompact box product of compact spaces?
4. Is the box product of countably many compact linearly ordered topological spaces paracompact?

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